CALCULUS
INTRODUCTION
Exam Questions

Question 1 (**)

Differentiate the following expression with respect to x

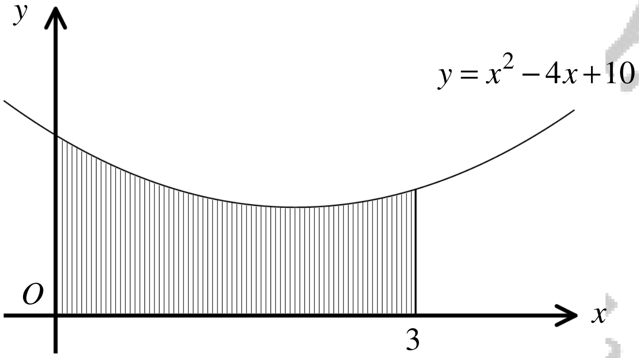
$$2x^3 - \sqrt{x} + \frac{x^2 + 2x}{x^2}, \quad x > 0 .$$

$\boxed{}$, $\boxed{6x^2 + \frac{1}{2\sqrt{x}} - \frac{2}{x^2}}$

$$\frac{d}{dx}\left[2x^3 - \sqrt{x} + \frac{x^2+2x}{x^2}\right] = \frac{d}{dx}\left[2x^3 - x^{\frac{1}{2}} + \frac{x^2}{x^2} + \frac{2x}{x^2}\right]$$

$$= \frac{d}{dx}\left[2x^3 - x^{\frac{1}{2}} + 1 + 2x^{-1}\right] = 6x^2 - \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-2}$$

Question 2 **(**)**

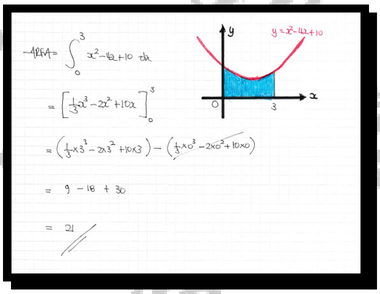


The figure above shows the curve with equation

$$y = x^2 - 4x + 10, \; x \in \mathbb{R}.$$

Find the area of the region, bounded by the curve the coordinate axes and the straight line with equation $x = 3$.

$\boxed{}$, $\boxed{\text{area} = 21}$



$$\text{AREA} = \int_0^3 x^2 - 4x + 10 \, dx$$

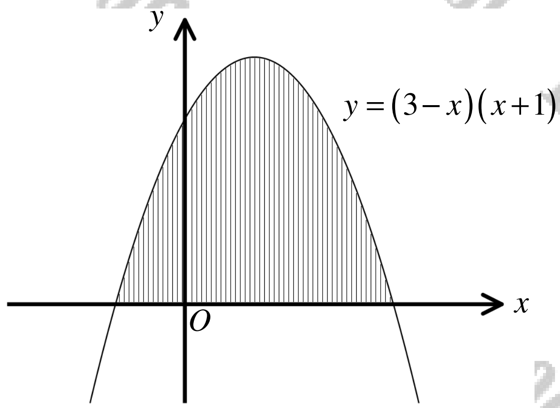
$$= \left[\frac{1}{3}x^3 - 2x^2 + 10x\right]_0^3$$

$$= \left(\frac{1}{3}\times 3^3 - 2\times 3^2 + 10\times 3\right) - \left(\frac{1}{3}\times 0^3 - 2\times 0^2 + 10\times 0\right)$$

$$= 9 - 18 + 30$$

$$= 21$$

Question 3 (**)

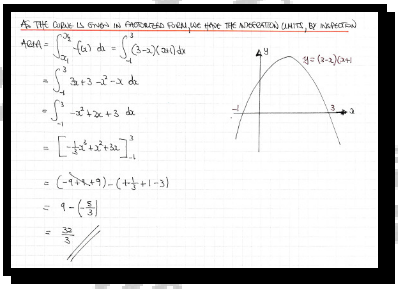


The figure above shows the curve with equation

$$y=(3-x)(x+1), \ x \in \mathbb{R}.$$

Find the exact area of the region, bounded by the curve and the x axis, shown shaded in the figure above.

$\boxed{}$, $\boxed{\text{area} = \frac{32}{3}}$



AS THE CURVE IS GIVEN IN FACTORIZED FORM, WE HAVE THE INTEGRATION LIMITS, BY INSPECTION

$\text{AREA} = \int_{x_1}^{x_2} f(x)\,dx = \int_{-1}^{3} (3-x)(x+1)\,dx$

$= \int_{-1}^{3} 3x+3-x^2-x\,dx$

$= \int_{-1}^{3} -x^2+2x+3\,dx$

$= \left[-\frac{1}{3}x^3+x^2+3x\right]_{-1}^{3}$

$= (-\cancel{9}+\cancel{9}+9)-(+\frac{1}{3}+1-3)$

$= 9-\left(-\frac{5}{3}\right)$

$= \frac{32}{3}$

Question 4 (**)

$$y = 2x^2 - \frac{6}{x^3} + 8x^3, \; x \neq 0.$$

Find a fully simplified expression for

$$\int y \, dx.$$

$\boxed{C1A}$, $\boxed{\frac{2}{3}x^3 + 3x^{-2} + 2x^4 + C}$

$$\int y \, dx = \int 2x^2 - 6x^{-3} + 8x^3 \, dx = \frac{2}{3}x^3 - \frac{6}{-2}x^{-2} + \frac{8}{4}x^4 + C$$
$$= \frac{2}{3}x^3 + 3x^{-2} + 2x^4 + C$$

Question 5 (**)

$$f(x) = 2x^2 - 7x + 1, \; x \in \mathbb{R}.$$

Find the coordinates of the point on the curve with equation $y = f(x)$, whose gradient is 5.

$\boxed{(3,-2)}$

$y = 2x^2 - 7x + 1$

$\frac{dy}{dx} = 4x - 7$

thus

$4x - 7 = 5$

$4x = 12$

$x = 3$

& $y = 2 \times 3^2 - 7 \times 3 + 1$

$y = 18 - 21 + 1$

$y = -2$

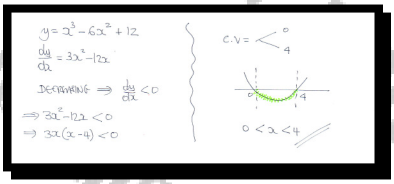
$\therefore (3, -2)$

Question 6 (**)

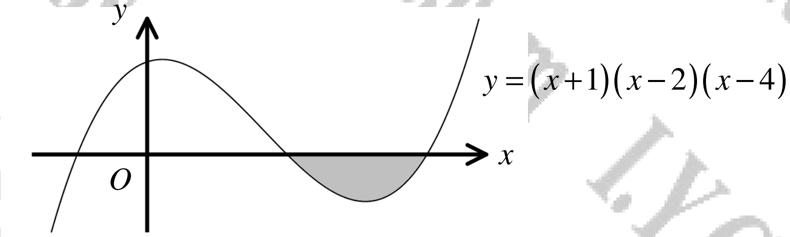
$$y = x^3 - 6x^2 + 12, \; x \in \mathbb{R}.$$

Find the range of values of x for which y is decreasing.

$\boxed{}$, $\boxed{0 < x < 4}$



Question 7 (**)



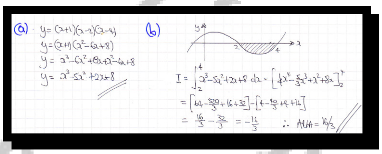
The figure above shows the curve with equation

$$y = (x+1)(x-2)(x-4), \; x \in \mathbb{R}.$$

a) Write the equation of the curve in the form $y = x^3 + ax^2 + bx + c$, where a, b and c are constants.

b) Find the exact area the shaded region.

$\boxed{}$, $\boxed{y = x^3 - 5x^2 + 2x + 8}$, $\boxed{\text{area} = \frac{16}{3}}$



Question 8 (**)

The curve C has equation

$$y = x^3 - x^2 - 5x + 2 .$$

Find the x coordinates of the points on C with gradient 3.

$$\boxed{x = -\frac{4}{3}, 2}$$

$y = x^3 - x^2 - 5x + 2$

$\frac{dy}{dx} = 3x^2 - 2x - 5$

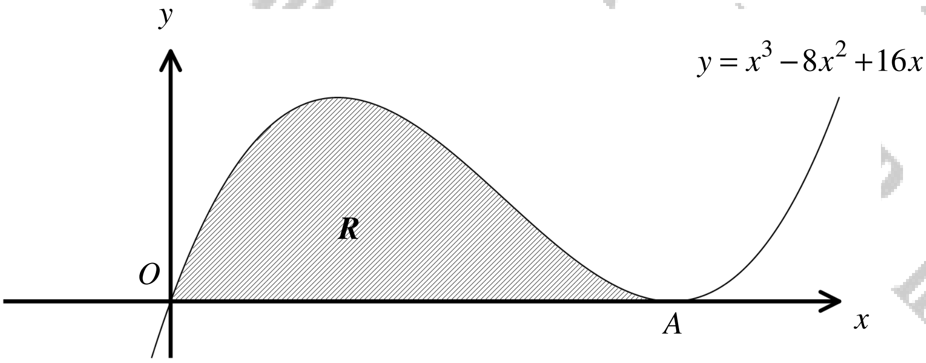
$3 = 3x^2 - 2x - 5$

$\Rightarrow 0 = 3x^2 - 2x - 8$

$\Rightarrow 0 = (3x+4)(x-2)$

$\therefore x = 2, -\frac{4}{3}$

Question 9 (**)



The figure above shows the cubic curve with equation

$$y = x^3 - 8x^2 + 16x, \ x \in \mathbb{R}.$$

The curve meets the x axis at the origin O and at the point A.

a) Show clearly that $x = 4$ at A.

The finite region R is bounded by the curve and the x axis.

b) Find the exact area of R.

$\boxed{}$, $\boxed{\dfrac{64}{3}}$

(a) $y = x^3 - 8x^2 + 16x$
$0 = x^3 - 8x^2 + 16x$
$0 = x(x^2 - 8x + 16)$
$0 = x(x-4)^2$
$\therefore x = 0 \leftarrow$ ORIGIN, $4 \leftarrow A$

(b) AREA $= \int_0^4 x^3 - 8x^2 + 16x \, dx$
$= \left[\frac{1}{4}x^4 - \frac{8}{3}x^3 + 8x^2\right]_0^4$
$= \left(\frac{1}{4}\times 4^4 - \frac{8}{3}\times 4^3 + 8\times 4^2\right) - (0)$
$= 64 - \frac{512}{3} + 128$
$= \frac{64}{3}$

Question 10 **(**)**

The point $P(1,3)$ lies on the curve with equation $y = f(x)$, whose gradient function is given by

$$f'(x) = 6x^2 - 4x, \ x \in \mathbb{R}.$$

Find an equation for $f(x)$.

$$\boxed{f(x) = 2x^3 - 2x^2 + 3}$$

$f(x) = \int 6x^2 - 4x \, dx$

$f(x) = 2x^3 - 2x^2 + C$

- when $x=1$ $y=3$

  $3 = 2 - 2 + C$

  $C = 3$

hence $f(x) = 2x^3 - 2x^2 + 3$

Question 11 **(**)**

The curve C has equation

$$y = x^5 - 6x^3 - 3x + 25.$$

Find an equation of the tangent to C at the point where $x = 2$.

$$\boxed{y = 5x - 7}$$

- $y = x^5 - 6x^3 - 3x + 25$

  when $x = 2$

  $\Rightarrow y = 2^5 - 6 \times 2^3 - 3 \times 2 + 25$

  $\Rightarrow y = 32 - 48 - 6 + 25$

  $\Rightarrow y = 57 - 54$

  $\Rightarrow y = 3$

  $\therefore (2,3)$

$\frac{dy}{dx} = 5x^4 - 18x^2 - 3$

$\left.\frac{dy}{dx}\right|_{x=2} = 5 \times 2^4 - 18 \times 2^2 - 3$

$= 80 - 72 - 3$

$= 5$

$y - y_0 = m(x - x_0)$

$y - 3 = 5(x - 2)$

$y - 3 = 5x - 10$

$y = 5x - 7$

Question 12 (**)

$$y = 3x^2 - 6\sqrt{x} - \frac{1}{x^2} + 4, \quad x > 0.$$

Find a fully simplified expression for

$$\int y \, dx.$$

$\boxed{4x + x^3 - 4x^{\frac{3}{2}} + x^{-1} + C}$

$$\int y \, dx = \int 3x^2 - 6x^{\frac{1}{2}} - x^{-2} + 4 \, dx = \frac{3}{3}x^3 - \frac{6}{\frac{3}{2}}x^{\frac{3}{2}} - \frac{1}{-1}x^{-1} + 4x + C$$
$$= x^3 - 4x^{\frac{3}{2}} + x^{-1} + 4x + C$$

Question 13 (**)

$$f(x) = -x^3 + 9x^2 - 15x - 13, \quad x \in \mathbb{R}.$$

a) Find the coordinates of the stationary points of $f(x)$.

b) Determine the nature of each of the two stationary points found in part (a).

c) Hence find the range of values of x for which $f(x)$ is decreasing.

$\boxed{\text{min at } (1,-20), \quad \text{max at } (5,12)}$, $\boxed{x < 1 \cup x > 5}$

(a) $f(x) = -x^3 + 9x^2 - 15x - 13$
$f'(x) = -3x^2 + 18x - 15$

• FOR STATIONARY POINTS $\boxed{f'(x) = 0}$

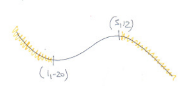
$-3x^2 + 18x - 15 = 0$
$x^2 - 6x + 5 = 0$
$(x-1)(x-5) = 0$
$x = \begin{cases} 1 \\ 5 \end{cases} \quad y = \begin{cases} -1 + 9 - 15 - 13 = \\ -125 + 225 - 75 - 13 \end{cases}$
$\therefore (1,-20)$ & $(5,12)$

(b) $f''(x) = -6x + 18$

$f''(1) = 12 > 0$
$\therefore (1, \ )$ IS A MIN
$f''(5) = -12 < 0$
$\therefore (5,12)$ IS A MAX

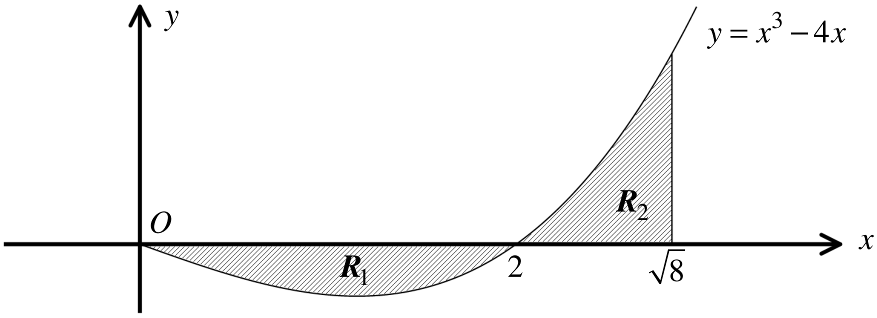
c) EITHER SOLVE $f'(x) < 0$

OR



$x < 1$ OR $x > 5$

Question 14 (**)



The figure above shows the cubic curve with equation

$$y = x^3 - 4x, \; x \geq 0.$$

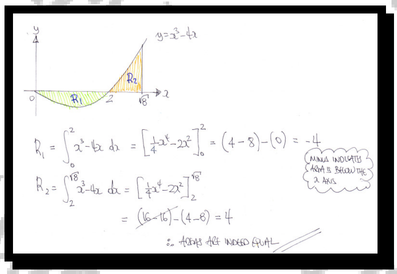
The curve meets the x axis at the origin O and at the point where $x = 2$.

The finite region R_1 is bounded by the curve and the x axis, for $0 \leq x \leq 2$.

The region R_2 is bounded by the curve and the x axis, for $2 \leq x \leq \sqrt{8}$.

Show that the area of R_1 is equal to the area of R_2.

$\boxed{}$, $\boxed{\text{proof}}$



Question 15 (**)

The curve C has equation

$$y = \frac{6}{x^2} + \frac{5x}{4} - 4, \quad x \neq 0.$$

a) Find an expression for $\frac{dy}{dx}$.

b) Determine an equation of the normal to the curve at the point where $x = 2$.

$\boxed{\frac{dy}{dx} = \frac{5}{4} - \frac{12}{x^3}}$, $\boxed{y = 4x - 8}$

(a) $y = \frac{6}{x^2} + \frac{5x}{4} - 4$

$y = 6x^{-2} + \frac{5}{4}x - 4$

$\frac{dy}{dx} = -12x^{-3} + \frac{5}{4}$

(or) $\frac{dy}{dx} = \frac{5}{4} - \frac{12}{x^3}$

(b) • when $x = 2$

$y = \frac{6}{2^2} + \frac{5 \times 2}{4} - 4$

$y = \frac{3}{2} + \frac{5}{2} - 4$

$y = 0$

$\therefore (2,0)$

• $\left.\frac{dy}{dx}\right|_{x=2} = \frac{5}{4} - \frac{12}{2^3} = \frac{5}{4} - \frac{12}{8}$

$= \frac{5}{4} - \frac{6}{4} = -\frac{1}{4}$

$\therefore$ NORMAL GRADIENT IS 4, (2,0)

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 0 = 4(x - 2)$

$\Rightarrow y = 4x - 8$

Question 16 (**)

$$f(x) = 6x + 9\sqrt{x} - \frac{4}{x^2}, \quad x > 0.$$

Find a fully simplified expression for

$$\int f(x)\, dx.$$

$\boxed{\quad}$, $\boxed{3x^2 + 6x^{\frac{3}{2}} + 4x^{-1} + C}$

$\int f(x)\,dx = \int 6x + 9x^{\frac{1}{2}} - 4x^{-2}\,dx = \frac{6}{2}x^2 + \frac{9}{\frac{3}{2}}x^{\frac{3}{2}} - \frac{4}{-1}x^{-1} + C$

$= 3x^2 + 6x^{\frac{3}{2}} + 4x^{-1} + C$

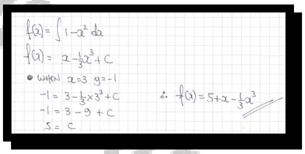
Question 17 (**)

The point $P(3,-1)$ lies on the curve with equation $y=f(x)$, whose gradient function is given by

$$f'(x)=1-x^2,\ x\in\mathbb{R}.$$

Find an equation for $f(x)$.

$\boxed{\quad}$, $\boxed{f(x)=5+x-\frac{1}{3}x^3}$


$f(x)=\int 1-x^2\,dx$

$f(x)=x-\frac{1}{3}x^3+C$

• WHEN $x=3$ $y=-1$

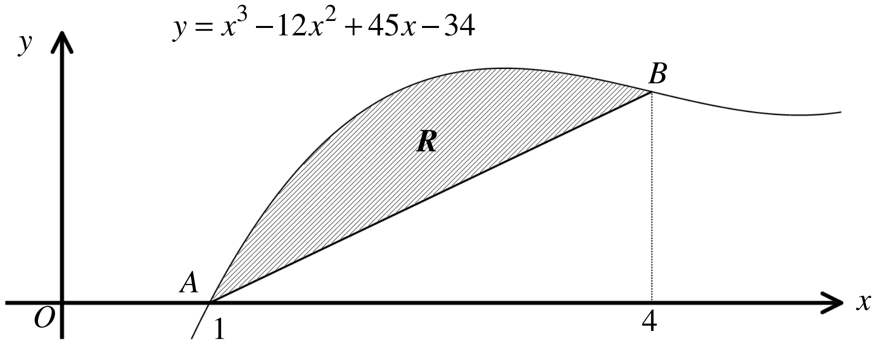
$-1=3-\frac{1}{3}\times 3^3+C$

$-1=3-9+C$

$5=C$

$\therefore f(x)=5+x-\frac{1}{3}x^3$

Question 18 (**)



The figure above shows the curve with equation

$$y = x^3 - 12x^2 + 45x - 34.$$

The points A and B lie on the curve, where $x = 1$ and $x = 4$, respectively.

The finite region R is bounded by the curve and the straight line segment AB.

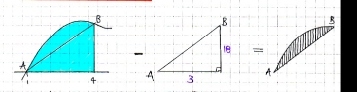
Show that the area of R, shown shaded in the figure, is exactly $\frac{81}{4}$.

$\boxed{}$, $\boxed{\text{proof}}$

START BY COMPUTING THE y CO-ORDS OF A & B

$y(1) = 1 - 12 + 45 - 34 = 0$ IE $A(1,0)$
$y(4) = 64 - 192 + 180 - 34 = 18$ IE $B(4,18)$

LOOKING AT THE DIAGRAM BELOW



$\int_1^4 x^3 - 12x^2 + 45x - 34 \, dx$ $\qquad$ $\frac{1}{2} \times 3 \times 18 = 27$

$= \left[\frac{1}{4}x^4 - 4x^3 + \frac{45}{2}x^2 - 34x\right]_1^4$

$= (64 - 256 + 360 + 136) - \left(\frac{1}{4} - 4 + \frac{45}{2} - 34\right)$

$= \frac{189}{4} = 47.25$

$\therefore$ REQUIRED AREA $= 47.25 - 27 = 20.25 = \frac{81}{4}$ // AS REQUIRED

Question 19 (**)

$$y = \frac{8}{x} + 3\sqrt{x}, \ x > 0.$$

Find the value of $\frac{dy}{dx}$ at the point where $x = 4$.

$\boxed{}$, $\boxed{\left.\frac{dy}{dx}\right|_{x=4} = \frac{1}{4}}$

REWRITE IN INDEX NOTATION

$$y = \frac{8}{x} + 3\sqrt{x}$$

$$y = 8x^{-1} + 3x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -8x^{-2} + \frac{3}{2}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{8}{x^2} + \frac{3}{2\sqrt{x}}$$

SUBSTITUTING $x=4$

$$\left.\frac{dy}{dx}\right|_{x=4} = -\frac{8}{4^2} + \frac{3}{2\sqrt{4}}$$

$$= -\frac{8}{16} + \frac{3}{4}$$

$$= -\frac{1}{2} + \frac{3}{4}$$

$$= \frac{1}{4}$$

Question 20 (**+)

The curve C has equation

$$y = -x^2(x+1), \; x \in \mathbb{R}.$$

The curve meets the coordinate axes at the origin O and at the point A.

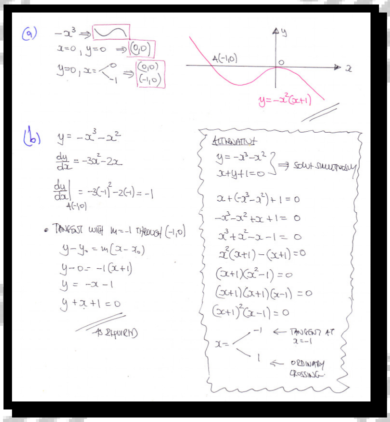
a) Sketch the graph of C, indicating clearly the coordinates of A.

b) Show that the straight line with equation

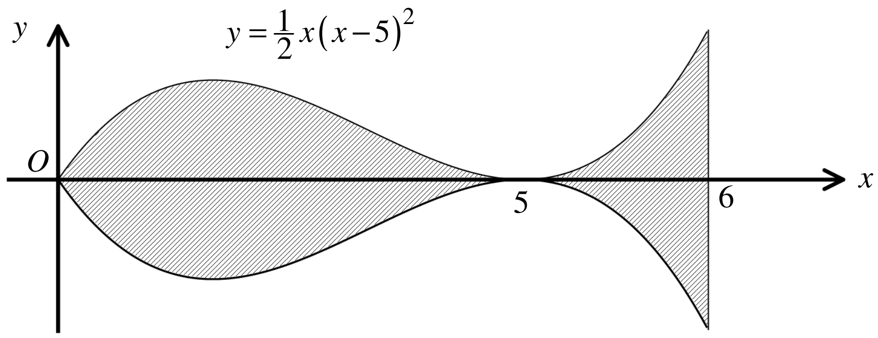
$$x + y + 1 = 0,$$

is a tangent to C at A.

$\boxed{}$, $\boxed{A(-1,0)}$



Question 21 (**+)



A fish logo is generated by the curve C with equation

$$y = \frac{1}{2}x(x-5)^2,\ 0 \le x \le 6,$$

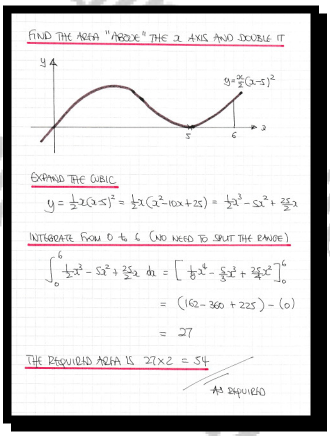
and its reflection in the x axis.

The curve C meets the x axis at the origin O and at the point $(5,0)$.

The finite region R is bounded by C, its reflection in the x axis and the straight line with equation $x = 6$.

Show that the area of R, shown shaded in the figure, is 54 square units.

$\boxed{}$, $\boxed{\text{proof}}$


FIND THE AREA "ABOVE" THE x AXIS AND DOUBLE IT

EXPAND THE CUBIC

$y = \frac{1}{2}x(x-5)^2 = \frac{1}{2}x(x^2-10x+25) = \frac{1}{2}x^3 - 5x^2 + \frac{25}{2}x$

INTEGRATE FROM 0 to 6 (NO NEED TO SPLIT THE RANGE)

$\int_0^6 \frac{1}{2}x^3 - 5x^2 + \frac{25}{2}x \, dx = \left[\frac{1}{8}x^4 - \frac{5}{3}x^3 + \frac{25}{4}x^2\right]_0^6$

$= (162 - 360 + 225) - (0)$

$= 27$

THE REQUIRED AREA IS $27 \times 2 = 54$

AS REQUIRED

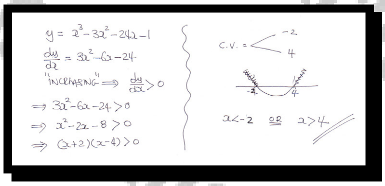
Question 22 (**+)

A curve C has equation

$$y = x^3 - 3x^2 - 24x - 1, \; x \in \mathbb{R}.$$

Find the range of values of x, for which y is increasing.

$\boxed{}$, $\boxed{x < -2 \; \cup \; x > 4}$



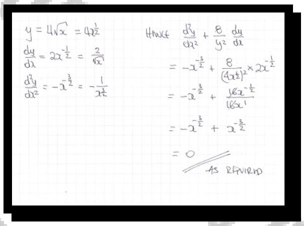
Question 23 (**+)

$$y = 4\sqrt{x}, \; x \in \mathbb{R}, \; x > 0.$$

Show clearly that

$$\frac{d^2y}{dx^2} + \frac{8}{y^2}\frac{dy}{dx} = 0.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 24 (**+)

$$y = x\left(6x - 5\sqrt{x}\right), \ x \ge 0 .$$

By showing all steps in the workings, find an expression for

$$\int y \, dx .$$

$\boxed{}$, $\boxed{2x^3 - 2x^{\frac{5}{2}} + C}$

$$\int y \, dx = \int x\left(6x - 5\sqrt{x}\right) dx = \int x\left(6x - 5x^{\frac{1}{2}}\right) dx$$
$$= \int 6x^2 - 5x^{\frac{3}{2}} \, dx = \frac{6}{3}x^3 - \frac{5}{\frac{5}{2}}x^{\frac{5}{2}} + C$$
$$= 2x^3 - 2x^{\frac{5}{2}} + C$$

Question 25 (**+)

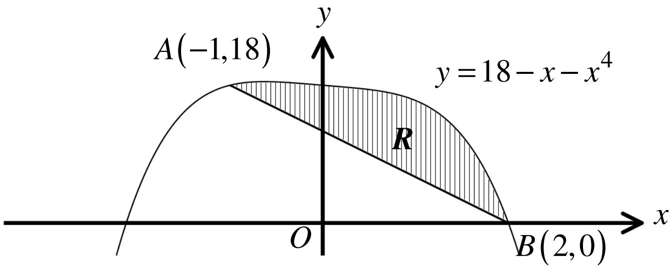
Find the value of the constant a if

$$\int_2^a a - 2x \ dx = -5 .$$

$\boxed{}$, $\boxed{a = \frac{9}{2}}$

$$\int_2^a a - 2x \, dx = -5 \qquad \Rightarrow 2a - 4 = 5$$
$$\Rightarrow \left[ax - x^2\right]_2^a = -5 \qquad \Rightarrow 2a = 9$$
$$\Rightarrow \left(\cancel{a^2 - a^2}\right) - (2a - 4) = -5 \qquad \Rightarrow a = \frac{9}{2}$$

Question 26 (**+)



The figure above shows the curve C with equation

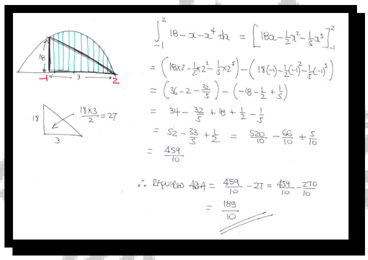
$$y = 18 - x - x^4 .$$

The curve crosses the x axis at $B(2,0)$ and the point $A(-1,18)$ lies on C.

The shaded region R is bounded by the curve and the straight line segment AB.

Find the area of the shaded region.

$\boxed{}$, $\boxed{\text{area} = 18.9}$

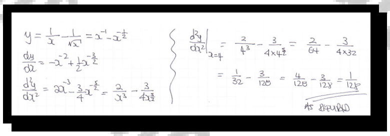


Question 27 (**+)

$$y = \frac{1}{x} - \frac{1}{\sqrt{x}}, \ x \in \mathbb{R}, \ x > 0.$$

Show that the value of $\frac{d^2y}{dx^2}$ where $x = 4$, is $\frac{1}{128}$.

$\boxed{\text{proof}}$



$y = \frac{1}{x} - \frac{1}{\sqrt{x}} = x^{-1} - x^{-\frac{1}{2}}$

$\frac{dy}{dx} = -x^{-2} + \frac{1}{2}x^{-\frac{3}{2}}$

$\frac{d^2y}{dx^2} = 2x^{-3} - \frac{3}{4}x^{-\frac{5}{2}} = \frac{2}{x^3} - \frac{3}{4x^{\frac{5}{2}}}$

$\left.\frac{d^2y}{dx^2}\right|_{x=4} = \frac{2}{4^3} - \frac{3}{4\times 4^{\frac{5}{2}}} = \frac{2}{64} - \frac{3}{4\times 32}$

$= \frac{1}{32} - \frac{3}{128} = \frac{4}{128} - \frac{3}{128} = \frac{1}{128}$

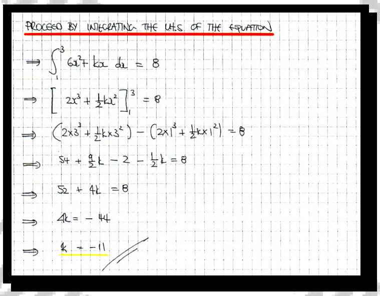
AS REQUIRED

Question 28 (**+)

Find the value of the constant k if

$$\int_1^3 6x^2 + kx \ dx = 8.$$

$\boxed{}$, $\boxed{k = -11}$



PROCEED BY INTEGRATING THE L.H.S OF THE EQUATION

$\Rightarrow \int_1^3 6x^2 + kx \ dx = 8$

$\Rightarrow \left[2x^3 + \frac{1}{2}kx^2\right]_1^3 = 8$

$\Rightarrow \left(2\times 3^3 + \frac{1}{2}k\times 3^2\right) - \left(2\times 1^3 + \frac{1}{2}k\times 1^2\right) = 8$

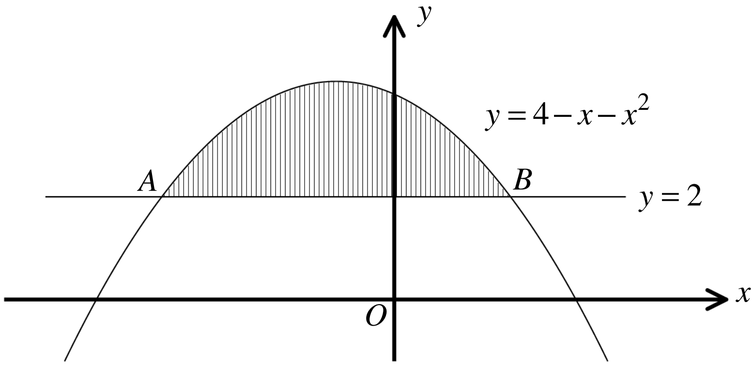
$\Rightarrow 54 + \frac{9}{2}k - 2 - \frac{1}{2}k = 8$

$\Rightarrow 52 + 4k = 8$

$\Rightarrow 4k = -44$

$\Rightarrow k = -11$

Question 29 (**+)



The figure above shows a quadratic curve and a straight line with respective equations

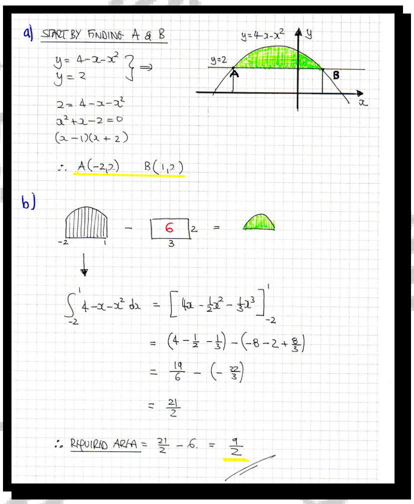
$$y = 4 - x - x^2 \quad \text{and} \quad y = 2 .$$

The points A and B are the points of intersection between the quadratic curve and the straight line.

a) Find the coordinates of A and B.

b) Determine the exact area of the finite region bounded by the quadratic curve and the straight line, shown shaded in the above figure.

$\boxed{}$, $\boxed{A(-2,2),\ B(1,2)}$, $\boxed{\text{area} = \frac{9}{2}}$

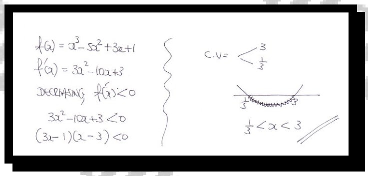


Question 30 (**+)

$$f(x) \equiv x^3 - 5x^2 + 3x + 1, \; x \in \mathbb{R}.$$

Find the range of values of x, for which $f(x)$ is decreasing.

$\boxed{\frac{1}{3} < x < 3}$

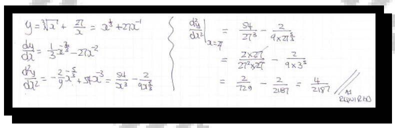


Question 31 (**+)

$$y = \sqrt[3]{x} + \frac{27}{x}, \; x \in \mathbb{R}, \; x > 0.$$

Show clearly that the value of $\frac{d^2y}{dx^2}$ where $x = 27$, is $\frac{4}{2187}$.

$\boxed{\text{proof}}$



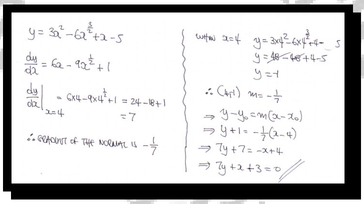
Question 32 (**+)

The curve C has equation

$$y = 3x^2 - 6x^{\frac{3}{2}} + x - 5, \ x \geq 0 .$$

Find an equation of the normal to C at the point where $x = 4$.

$\boxed{7y + x + 3 = 0}$

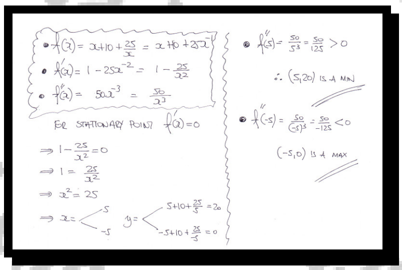


Question 33 (**+)

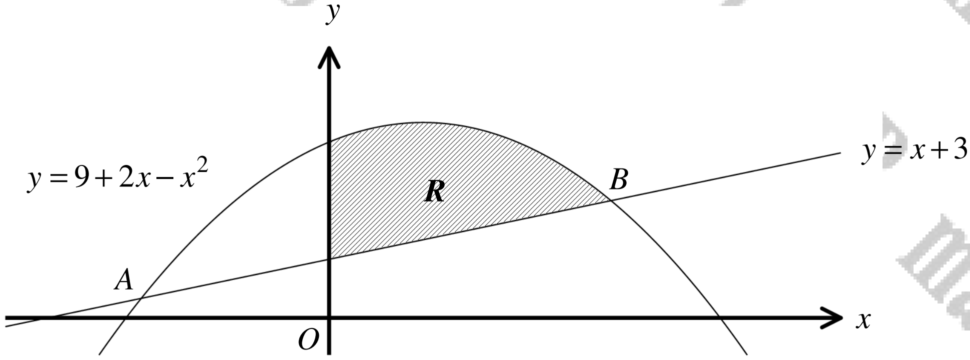
$$f(x) \equiv x + 10 + \frac{25}{x}, \ x \neq 0 .$$

Find the coordinates of the two stationary points of $f(x)$ and use $f''(x)$ to determine their nature.

$\boxed{\min(5,20)}$, $\boxed{\max(-5,0)}$



Question 34 (**+)



The figure above shows the graph of the curve C with equation

$$y = 9 + 2x - x^2,$$

and the straight line L with equation

$$y = x + 3.$$

The curve meets the straight line at the points A and B.

The finite region R, shown shaded in the figure, is bounded by the curve C, the straight line L and the coordinate axes.

Show that the area of R is 13.5 square units.

SYN, proof

- START BY FINDING THE CO-ORDINATES OF B

$\left.\begin{array}{l} y = x+3 \\ y = 9+2x-x^2 \end{array}\right\} \Rightarrow x+3 = 9+2x-x^2$

$\Rightarrow x^2 - x - 6 = 0$

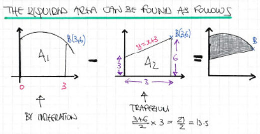
$\Rightarrow (x-3)(x+2) = 0$

$\Rightarrow x = \cancel{-2} \leftarrow A$, 3

$\Rightarrow y = 3+3 = 6$

$\therefore B(3,6)$

- THE REQUIRED AREA CAN BE FOUND AS FOLLOWS



- $A_1 = \int_0^3 9+2x-x^2 \, dx$

$A_1 = \left[9x + x^2 - \frac{1}{3}x^3\right]_0^3$

$A_1 = (27+9-9) - (0)$

$A_1 = 27$

- HENCE THE REQUIRED AREA IS

$27 - \frac{27}{2} = 27 - 13.5$

$= 13.5$

AS REQUIRED

Question 35 (**+)

A curve has equation

$$y = x^3 - 10x + 2,\ x \in \mathbb{R}.$$

The point $P(2,-10)$ lies on the curve.

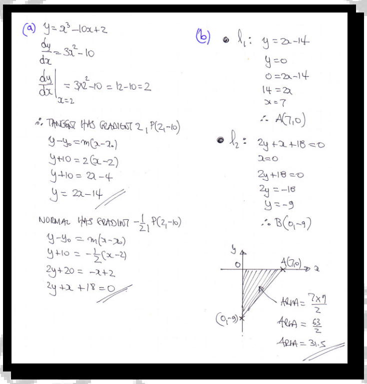
The straight lines l_1 and l_2 are the tangent and the normal to the curve at P, respectively.

a) Find an equation for l_1 and an equation for l_2.

l_1 crosses the x axis at A and l_2 crosses the y axis at B.

b) Find the area of the triangle OAB where O is the origin.

$\boxed{}$, $\boxed{y = 2x - 14}$, $\boxed{2y + x + 18 = 0}$, $\boxed{\text{area} = 31.5}$



(a) $y = x^3 - 10x + 2$

$\frac{dy}{dx} = 3x^2 - 10$

$\left.\frac{dy}{dx}\right|_{x=2} = 3\times2^2 - 10 = 12 - 10 = 2$

∴ TANGENT HAS GRADIENT 2, $P(2,-10)$

$y - y_0 = m(x - x_0)$

$y + 10 = 2(x - 2)$

$y + 10 = 2x - 4$

$y = 2x - 14$

NORMAL HAS GRADIENT $-\frac{1}{2}$, $P(2,-10)$

$y - y_0 = m(x - x_0)$

$y + 10 = -\frac{1}{2}(x - 2)$

$2y + 20 = -x + 2$

$2y + x + 18 = 0$

(b) • l_1: $y = 2x - 14$

$y = 0$

$0 = 2x - 14$

$14 = 2x$

$x = 7$

∴ $A(7,0)$

• l_2: $2y + x + 18 = 0$

$x = 0$

$2y + 18 = 0$

$2y = -18$

$y = -9$

∴ $B(0,-9)$

AREA $= \frac{7\times9}{2}$

AREA $= \frac{63}{2}$

AREA $= 31.5$

Question 36 **(**+)**

By showing clear workings, find the value of

$$\int_1^9 6\sqrt{x} - \frac{6}{\sqrt{x}} \, dx \, .$$

$\boxed{\quad}$, $\boxed{80}$

$$\int_1^9 6\sqrt{x} - \frac{6}{\sqrt{x}} \, dx = \int_1^9 6x^{\frac{1}{2}} - 6x^{-\frac{1}{2}} \, dx = \left[4x^{\frac{3}{2}} - 12x^{\frac{1}{2}}\right]_1^9$$

$$= (108-36) - (4-12) = 72 + 8 = 80$$

Question 37 **(**+)**

Find the value of the constant k, $k > 1$, given that

$$\int_1^k 4x - 3 \ \ dx = 28 \, .$$

$\boxed{k = \frac{9}{2}}$

$$\int_1^k 4x - 3 \, dx = 28$$

$$\Rightarrow \left[2x^2 - 3x\right]_1^k = 28$$

$$\Rightarrow (2k^2 - 3k) - (2 - 3) = 28$$

$$\Rightarrow 2k^2 - 3k + 1 = 28$$

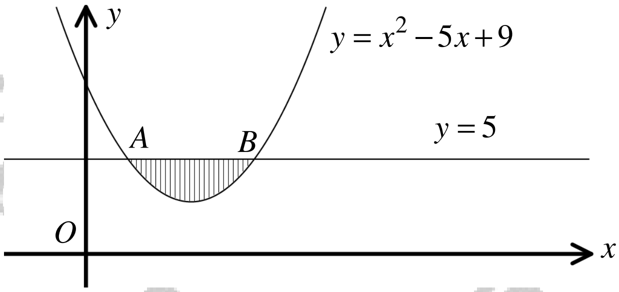
$$\Rightarrow 2k^2 - 3k - 27 = 0$$

$$\Rightarrow (2k-9)(k+3) = 0$$

$$\Rightarrow k = \begin{cases} \frac{9}{2} \\ \cancel{-3} \end{cases}$$

$$k = \frac{9}{2}$$

Question 38 (**+)



The figure above shows a quadratic curve and a straight line with respective equations

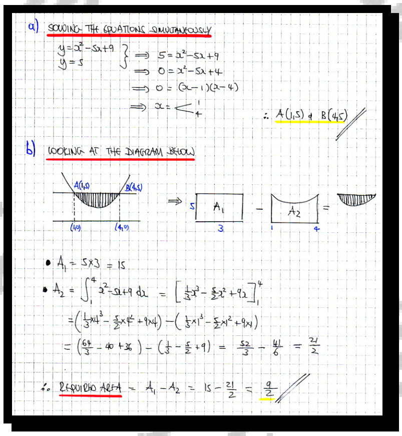
$$y = x^2 - 5x + 9 \quad \text{and} \quad y = 5.$$

The points A and B are the points of intersection between the straight line and the quadratic curve.

a) Find the coordinates of A and B.

b) Calculate the exact area of the finite region bounded by the quadratic curve and the straight line, shown shaded in the above figure.

$\boxed{A(1,5), B(4,5)}$, $\boxed{\text{area} = \frac{9}{2}}$



a) SOLVING THE EQUATIONS SIMULTANEOUSLY

$$\left.\begin{array}{l} y = x^2 - 5x + 9 \\ y = 5 \end{array}\right\} \Rightarrow 5 = x^2 - 5x + 9$$
$$\Rightarrow 0 = x^2 - 5x + 4$$
$$\Rightarrow 0 = (x-1)(x-4)$$
$$\Rightarrow x = \begin{cases} 1 \\ 4 \end{cases}$$

$\therefore A(1,5)$ & $B(4,5)$

b) LOOKING AT THE DIAGRAM BELOW

- $A_1 = 5 \times 3 = 15$
- $A_2 = \int_1^4 x^2 - 5x + 9 \, dx = \left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 9x\right]_1^4$

$$= \left(\tfrac{1}{3}\times 4^3 - \tfrac{5}{2}\times 4^2 + 9\times 4\right) - \left(\tfrac{1}{3}\times 1^3 - \tfrac{5}{2}\times 1^2 + 9\times 1\right)$$
$$= \left(\tfrac{64}{3} - 40 + 36\right) - \left(\tfrac{1}{3} - \tfrac{5}{2} + 9\right) = \tfrac{52}{3} - \tfrac{41}{6} = \tfrac{21}{2}$$

$\therefore$ REQUIRED AREA $= A_1 - A_2 = 15 - \frac{21}{2} = \frac{9}{2}$

Question 39 **(**+)**

The curve C with equation $y = f(x)$ has gradient function

$$\frac{dy}{dx} = 9x^2 + \frac{7}{x^2}, \quad x \neq 0.$$

The point $A(-1,-1)$ lies on C.

Find an equation for C.

$$\boxed{y = 3x^3 - \frac{7}{x} - 5}$$

$y = \int 9x^2 + \frac{7}{x^2}\,dx = \int 9x^2 + 7x^{-2}\,dx$

$y = 3x^3 - 7x^{-1} + C$

$y = 3x^3 - \frac{7}{x} + C$

- when $x=-1$ $y=-1$

$-1 = 3(-1)^3 - \frac{7}{(-1)} + C$

$-1 = -3 + 7 + C$

$C = -5$

Thus $y = 3x^3 - \frac{7}{x} - 5$

Question 40 **(**+)**

$$y = 2x + \frac{8}{x^2}, \quad x \neq 0.$$

Find the coordinates of the stationary point of y and determine its nature.

$\boxed{}$, $\boxed{\min(2,6)}$

- $y = 2x + \frac{8}{x^2} = 2x + 8x^{-2}$
- $\frac{dy}{dx} = 2 - 16x^{-3} = 2 - \frac{16}{x^3}$
- $\frac{d^2y}{dx^2} = 48x^{-4} = \frac{48}{x^4}$

FOR STATIONARY POINTS $\frac{dy}{dx} = 0$

$\Rightarrow 2 - \frac{16}{x^3} = 0$

$\Rightarrow 2 = \frac{16}{x^3}$

$\Rightarrow 2x^3 = 16$

$\Rightarrow x^3 = 8$

$\Rightarrow \boxed{x = 2}$

USING $y = 2x + \frac{8}{x^2}$

$y = 2 \times 2 + \frac{8}{2^2}$

$y = 4 + \frac{8}{4}$

$\boxed{y = 6}$

$\left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{48}{2^4} = 3 > 0$

$\therefore$ (2,6) IS A MIN

Question 41 (***)

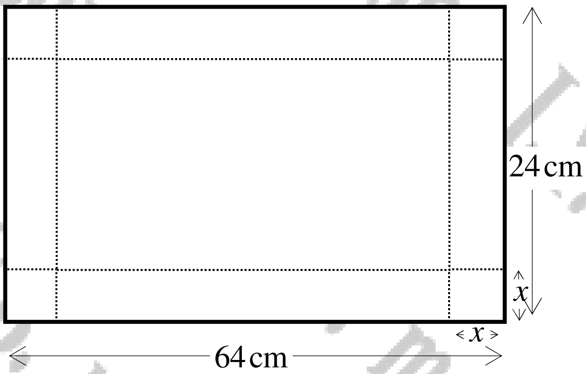


figure 1

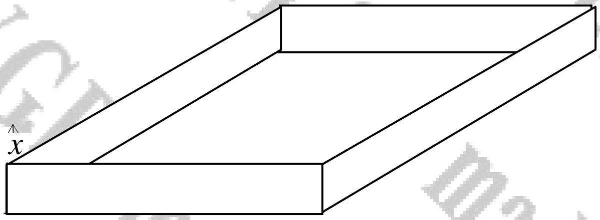


figure 2

An **open** box is to be made out of a rectangular piece of card measuring 64 cm by 24 cm. Figure 1 shows how a square of side length x cm is to be cut out of each corner so that the box can be made by folding, as shown in figure 2.

a) Show that the volume of the box, V cm^3, is given by

$$V = 4x^3 - 176x^2 + 1536x .$$

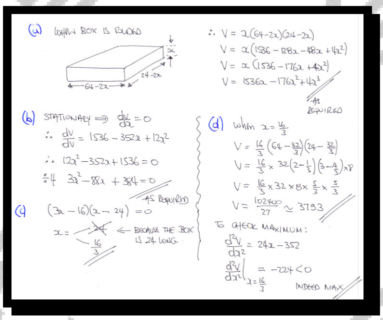
b) Show further that the stationary points of V occur when

$$3x^2 - 88x + 384 = 0 .$$

c) Find the value of x for which V is stationary.
(You may find the fact $24 \times 16 = 384$ useful.)

d) Find, to the nearest cm^3, the maximum value for V, justifying that it is indeed the maximum value.

$\boxed{x = \frac{16}{3}}$, $\boxed{V_{max} \approx 3793}$



(a) WHEN BOX IS FOLDED

64 − 2x, 24 − 2x, x

$\therefore V = x(64-2x)(24-2x)$

$V = x(1536 - 128x - 48x + 4x^2)$

$V = x(1536 - 176x + 4x^2)$

$V = 1536x - 176x^2 + 4x^3$

AS REQUIRED

(b) STATIONARY $\Rightarrow \frac{dV}{dx} = 0$

$\therefore \frac{dV}{dV} = 1536 - 352x + 12x^2$

$\therefore 12x^2 - 352x + 1536 = 0$

$\div 4 \quad 3x^2 - 88x + 384 = 0$

AS REQUIRED

(c) $(3x-16)(x-24) = 0$

$x = \cancel{24}$ ← BECAUSE THE BOX IS 24 LONG

$x = \frac{16}{3}$

(d) when $x = \frac{16}{3}$

$V = \frac{16}{3}\left(64 - \frac{32}{3}\right)\left(24 - \frac{32}{3}\right)$

$V = \frac{16}{3} \times 32\left(2 - \frac{1}{3}\right)\left(3 - \frac{4}{3}\right) \times 8$

$V = \frac{16}{3} \times 32 \times 8 \times \frac{5}{3} \times \frac{5}{3}$

$V = \frac{102400}{27} \simeq 3793$

TO CHECK MAXIMUM:

$\frac{d^2V}{dx^2} = 24x - 352$

$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{16}{3}} = -224 < 0$

INDEED MAX

Question 42 (***)

A curve C has the following equation

$$f(x) = 4x\sqrt{x} - \frac{25x^2}{16}, \quad x \geq 0.$$

a) Find a simplified expression for $f'(x)$.

b) Determine an equation of the tangent to the curve at the point where $x = 4$, giving the answer in the form $ax + by = c$, where a, b and c are integers.

$\boxed{}$, $\boxed{f'(x) = 6x^{\frac{1}{2}} - \frac{25}{8}x}$, $\boxed{x + 2y = 18}$

(a) $f(x) = 4x\sqrt{x} - \frac{25x^2}{16}$

$\Rightarrow f(x) = 4x^1x^{\frac{1}{2}} - \frac{25}{16}x^2$

$\Rightarrow f(x) = 4x^{\frac{3}{2}} - \frac{25}{16}x^2$

$\Rightarrow f'(x) = 6x^{\frac{1}{2}} - \frac{25}{8}x$

(b) $f'(4) = 6 \times 4^{\frac{1}{2}} - \frac{25}{\cancel{8}_2} \times \cancel{4}$

$= 6 \times 2 - \frac{25}{2}$

$= 12 - \frac{25}{2}$

$= \frac{24}{2} - \frac{25}{2}$

$= -\frac{1}{2}$

WHEN $x = 4$

$y = f(4) = 4 \times 4 \times \sqrt{4} - \frac{25}{\cancel{16}} \times 4^2$

$= 32 - 25$

$= 7$

$\therefore (4,7)$ & GRADIENT $-\frac{1}{2}$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 7 = -\frac{1}{2}(x - 4)$

$\Rightarrow 2y - 14 = -x + 4$

$\Rightarrow 2y + x = 18$

Question 43 (***)

The curve C has equation

$$y = x^3 - 6x^2 + 12x - 5.$$

Find the coordinates of the stationary point of C and use a clear method to determine its nature.

$\boxed{}$, $\boxed{\text{point of inflexion at } (2,3)}$

$y = x^3 - 6x^2 + 12x - 5$

- $\frac{dy}{dx} = 3x^2 - 12x + 12$

SOLVE FOR ZERO

$3x^2 - 12x + 12 = 0$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$\boxed{x = 2}$

WHEN $x = 2$

$y = 8 - \cancel{24} + \cancel{24} - 5$

$\boxed{y = 3}$

$\therefore (2,3)$

DETERMINE THE NATURE

$\frac{d^2y}{dx^2} = 6x - 12$

$\frac{d^2y}{dx^2} = 6 \times 2 - 12 = 0$

$\therefore$ POSSIBLY A POINT OF INFLEXION

$\frac{d^3y}{dx^3} = 6$

$\left.\frac{d^3y}{dx^3}\right|_{x=2} = 6 \neq 0$

$\therefore (2,3)$ IS A POINT OF INFLEXION

Question 44 (***)

$$f(x) = \frac{5\sqrt{x}\left(3x^2-2\right)}{x}, \quad x > 0.$$

Show clearly that

$$\int f(x)\,dx = P\sqrt{x} + Qx^{\frac{5}{2}} + C,$$

where P and Q are integers to be found, and C is an arbitrary constant.

$\boxed{P=-20}$, $\boxed{Q=6}$

$$\int f(x)\,dx = \int \frac{5\sqrt{x}(3x^2-2)}{x}\,dx = \int \frac{5x^{\frac{1}{2}}(3x^2-2)}{x}\,dx$$
$$= \int \frac{15x^{\frac{5}{2}} - 10x^{\frac{1}{2}}}{x}\,dx = \int \frac{15x^{\frac{5}{2}}}{x} - \frac{10x^{\frac{1}{2}}}{x}\,dx$$
$$= \int 15x^{\frac{3}{2}} - 10x^{-\frac{1}{2}}\,dx = \frac{15}{\frac{5}{2}}x^{\frac{5}{2}} - \frac{10}{\frac{1}{2}}x^{\frac{1}{2}} + C$$
$$= 6x^{\frac{5}{2}} - 20x^{\frac{1}{2}} + C$$
$$= 6x^{\frac{5}{2}} - 20\sqrt{x} + C$$

ie $P = -20$
$Q = 6$

Question 45 (***)

The point $P(3,0)$ lies on the curve C whose gradient function is given by

$$\frac{dy}{dx} = 3x^2 - 14x + 12.$$

a) Find an equation of the tangent to C at the point P.

The point Q lies on C, so that the tangent at Q is parallel to the tangent at P.

b) Find the x coordinate of Q.

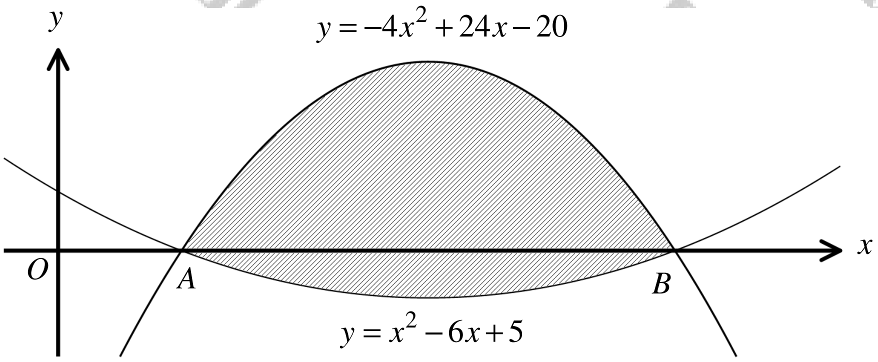
$\boxed{y = 9-3x}$, $\boxed{x=\frac{5}{3}}$

(a)
$$\left.\frac{dy}{dx}\right|_{x=3} = 3\times3^2 - 14\times3 + 12$$
$$\left.\frac{dy}{dx}\right|_{x=3} = 27 - 42 + 12 = -3$$
$$\therefore m = -3 \quad \& \quad (3,0)$$
$$\Rightarrow y - y_0 = m(x - x_0)$$
$$\Rightarrow y - 0 = -3(x-3)$$
$$\Rightarrow y = -3x + 9$$

(b) PARALLEL $\Longrightarrow$ SAME GRADIENT AT AT POINTS P & Q

THUS $3x^2 - 14x + 12 = -3$
$$\Rightarrow 3x^2 - 14x + 15 = 0$$
$$\Rightarrow (x-3)(3x-5) = 0$$
$$\Rightarrow x = 3 \leftarrow \text{POINT P (ALREADY KNOWN)}$$
$$ \tfrac{5}{3} \leftarrow Q$$

Question 46 (***)



The figure above shows the graph of the curves with equations

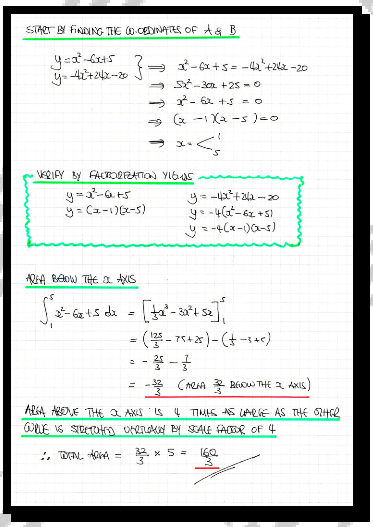
$$y = -4x^2 + 24x - 20 \quad \text{and} \quad y = x^2 - 6x + 5 .$$

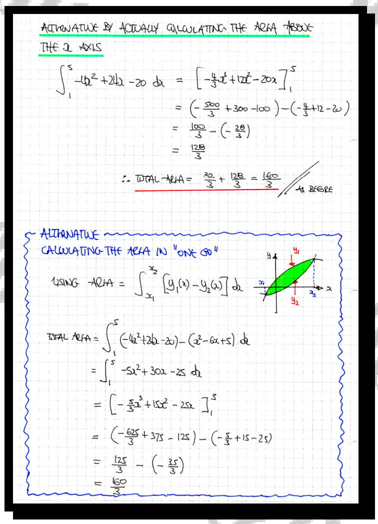
The two curves intersect each other at the points A and B.

The finite region R bounded by the two curves is shown shaded in the figure.

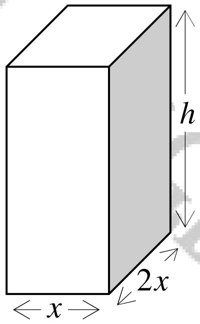
Find the exact area of R.

$\boxed{}$, $\boxed{\frac{160}{3}}$





Question 47 (***)



The figure above shows the design of a fruit juice carton with capacity of 1000 cm^3.

The design of the carton is that of a closed cuboid whose base measures x cm by $2x$ cm, and its height is h cm.

a) Show that the surface area of the carton, $A \text{ cm}^2$, is given by

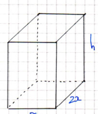
$$A = 4x^2 + \frac{3000}{x}.$$

b) Find the value of x for which A is stationary.

c) Calculate the minimum value for A, justifying fully the fact that it is indeed the minimum value of A.

$\boxed{}$, $\boxed{x = \sqrt[3]{375} \approx 7.21}$, $\boxed{A_{\text{min}} \approx 624}$

a)



CONSTRAINT $V = 1000 \text{ cm}^3$

$\Rightarrow V = x(2x)h$

$\Rightarrow 1000 = 2x^2h$

$\Rightarrow x^2h = 500$

$xh = \frac{500}{x}$

$6xh = \frac{3000}{x}$

SURFACE AREA ($A \text{ cm}^2$)

$\Rightarrow A = 2[2x^2 + xh + 2xh]$

$\Rightarrow A = 4x^2 + 6xh$

$\Rightarrow A = 4x^2 + \frac{3000}{x}$ // AS REQUIRED

b) $A = 4x^2 + 3000x^{-1}$

$\Rightarrow \frac{dA}{dx} = 8x - 3000x^{-2}$

FOR STATIONARY VALUES $\frac{dA}{dx} = 0$

$\Rightarrow 8x - \frac{3000}{x^2} = 0$

$\Rightarrow 8x = \frac{3000}{x^2}$

$\Rightarrow 8x^3 = 3000$

$\Rightarrow x^3 = 375$

$\Rightarrow x = \sqrt[3]{375} \approx 7.21 \text{ cm}$ //

c) $A = 4x^2 + \frac{3000}{x}$

$\Rightarrow A_{\text{MIN}} = 4(7.21\ldots)^2 + \frac{3000}{7.21\ldots}$

$\Rightarrow A_{\text{MIN}} \approx 624 \text{ cm}^2$ //

TO JUSTIFY IT IS A MIN, USE 2ND DERIVATIVE

$\Rightarrow \frac{dA}{dx} = 8x - 3000x^{-2}$

$\Rightarrow \frac{d^2A}{dx^2} = 8 + 6000x^{-3}$

$\Rightarrow \frac{d^2A}{dx^2} = 8 + \frac{6000}{x^3}$

$\Rightarrow \left.\frac{d^2A}{dx^2}\right|_{x=7.21\ldots} = 8 + \frac{6000}{(7.21\ldots)^3} = 24 > 0$

INDEED A MINIMUM //

Question 48 (***)

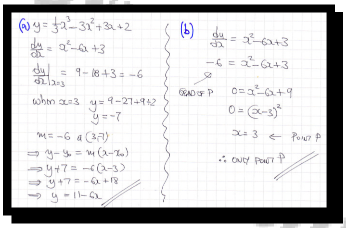
The point P, where $x=3$, lies on the curve C whose equation is

$$y=\tfrac{1}{3}x^3-3x^2+3x+2.$$

a) Find an equation of the tangent to C at the point P.

b) Show that there is no other point on C where the gradient is the same as the gradient at P.

$$\boxed{y=11-6x}$$


(a) $y=\frac{1}{3}x^3-3x^2+3x+2$
$\frac{dy}{dx}=x^2-6x+3$
$\left.\frac{dy}{dx}\right|_{x=3}=9-18+3=-6$
when $x=3$ $y=9-27+9+2$
$y=-7$
$m=-6$ & $(3,-7)$
$\Rightarrow y-y_0=m(x-x_0)$
$\Rightarrow y+7=-6(x-3)$
$\Rightarrow y+7=-6x+18$
$\Rightarrow y=11-6x$

(b) $\frac{dy}{dx}=x^2-6x+3$
$-6=x^2-6x+3$
GRAD OF P
$0=x^2-6x+9$
$0=(x-3)^2$
$x=3 \leftarrow$ POINT P
$\therefore$ ONLY POINT P

Question 49 (***)

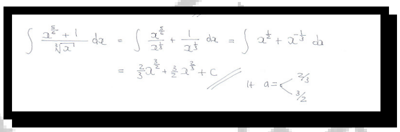
$$y=\frac{x^{\frac{5}{6}}+1}{\sqrt[3]{x}},\quad x>0.$$

Show clearly that

$$\int y\,dx = ax^{\frac{1}{a}}+\frac{1}{a}x^{a}+C,$$

where a is a rational constant to be found, and C is an arbitrary constant.

$$\boxed{a=\frac{2}{3}\ \text{or}\ \frac{3}{2}}$$


$\int \frac{x^{\frac{5}{6}}+1}{\sqrt[3]{x^1}}\,dx=\int \frac{x^{\frac{5}{6}}}{x^{\frac{1}{3}}}+\frac{1}{x^{\frac{1}{3}}}\,dx=\int x^{\frac{1}{2}}+x^{-\frac{1}{3}}\,dx$
$=\frac{2}{3}x^{\frac{3}{2}}+\frac{3}{2}x^{\frac{2}{3}}+C$
If $a=\frac{2}{3}$ or $\frac{3}{2}$

Question 50 (***)

The cubic equation C passes through the origin O and its gradient function is

$$\frac{dy}{dx} = 6x^2 - 6x - 20 .$$

a) Show clearly that the equation of C can be written as

$$y = x(2x + a)(x + b),$$

where a and b are constants.

b) Sketch the graph of C, indicating clearly the coordinates of the points where the graph meets the coordinate axes.

$\boxed{}$, $\boxed{a = 5}$, $\boxed{b = -4}$

(a) IF $\frac{dy}{dx} = 6x^2 - 6x - 20$

THEN $y = \int 6x^2 - 6x - 20 \, dx$

$y = 2x^3 - 3x^2 - 20x + C$

CURVE PASSES THROUGH $(0,0)$

$0 = 0 - 0 - 0 + C$

$\boxed{C = 0}$

$\therefore y = 2x^3 - 3x^2 - 20x$

$y = x(2x^2 - 3x - 20)$

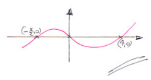
$y = x(2x+5)(x-4)$

$a = 5$
$b = -4$

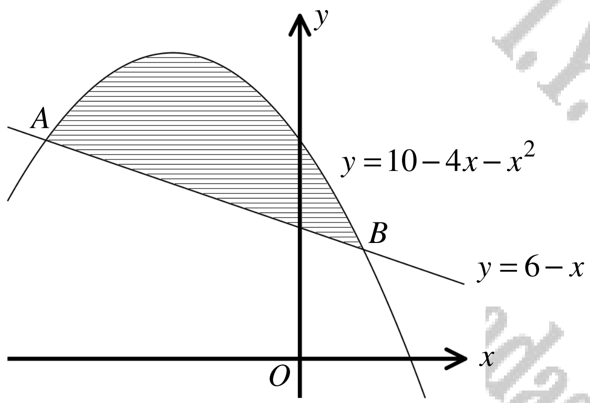
(b) • $+2x^3 \Rightarrow$

• $x=0, y=0 \Rightarrow \boxed{(0,0)}$

• $y=0, x = 0, 4, -\frac{5}{2}$ $\boxed{(0,0) \; (4,0) \; (-\frac{5}{2},0)}$



Question 51 (***)



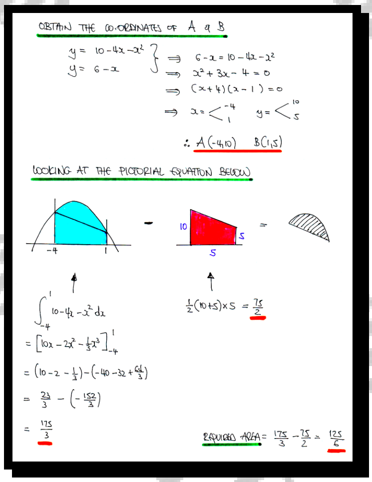
The figure above shows a quadratic curve and a straight line with respective equations

$$y = 10 - 4x - x^2 \qquad \text{and} \qquad y = 6 - x .$$

The points A and B, are the points of intersection between the quadratic curve and the straight line.

Calculate the exact area of the finite region bounded by the quadratic curve and the straight line, shown shaded in the above figure.

$\boxed{}$, $\boxed{\text{area} = \frac{125}{6}}$



OBTAIN THE CO-ORDINATES OF A & B

$y = 10 - 4x - x^2$, $y = 6 - x$ $\Rightarrow 6 - x = 10 - 4x - x^2$
$\Rightarrow x^2 + 3x - 4 = 0$
$\Rightarrow (x+4)(x-1) = 0$
$\Rightarrow x = -4, 1 \quad y = 10, 5$

$\therefore A(-4,10) \quad B(1,5)$

LOOKING AT THE PICTORIAL EQUATION BELOW

$\int_{-4}^{1} 10 - 4x - x^2 \, dx$
$= \left[10x - 2x^2 - \frac{1}{3}x^3\right]_{-4}^{1}$
$= \left(10 - 2 - \frac{1}{3}\right) - \left(-40 - 32 + \frac{64}{3}\right)$
$= \frac{23}{3} - \left(-\frac{152}{3}\right)$
$= \frac{175}{3}$

$\frac{1}{2}(10+5) \times 5 = \frac{75}{2}$

REQUIRED AREA $= \frac{175}{3} - \frac{75}{2} = \frac{125}{6}$

Question 52 (***)

The gradient function of the curve with equation

$$y = 2(x+a)^2,$$

where a is a non zero constant, is given by

$$\frac{dy}{dx} = 4x + 10.$$

Determine the value of a.

$\boxed{}$, $\boxed{a = \frac{5}{2}}$

EXPAND & DIFFERENTIATE

$\Rightarrow y = 2(x+a)^2$

$\Rightarrow y = 2(x^2 + 2ax + a^2)$

$\Rightarrow y = 2x^2 + 4ax + 2a^2$

$\Rightarrow \frac{dy}{dx} = 4x + 4a$

BY DIRECT COMPARISON

$\Rightarrow 4a = 10$

$\Rightarrow a = \frac{5}{2}$

Question 53 (***)

The point $P(4,9)$ lies on the curve with equation $y = f(x)$, whose gradient function is given by

$$f'(x) = 1 + \frac{2}{\sqrt{x}}, \quad x > 0.$$

Find an equation for $f(x)$.

$\boxed{f(x) = x + 4\sqrt{x} - 3}$

IF $f'(x) = 1 + \frac{2}{\sqrt{x}} = 1 + 2x^{-\frac{1}{2}}$

THEN

$\Rightarrow f(x) = \int 1 + 2x^{-\frac{1}{2}}\,dx$

$\Rightarrow f(x) = x + \frac{2}{\frac{1}{2}}x^{\frac{1}{2}} + C$

$\Rightarrow f(x) = x + 4x^{\frac{1}{2}} + C$

WHEN $x=4$ $y=9 \leftarrow f(x)$

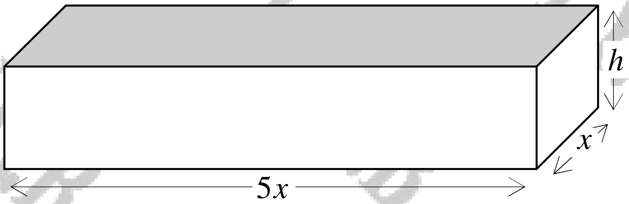
$\therefore 9 = 4 + 4 \times 4^{\frac{1}{2}} + C$

$9 = 4 + 8 + C$

$C = -3$

$\therefore f(x) = x + 4x^{\frac{1}{2}} - 3$

Question 54 (***)



The figure above shows a **solid** brick, in the shape of a cuboid, measuring $5x$ cm by x cm by h cm. The total surface area of the brick is 720 cm^2.

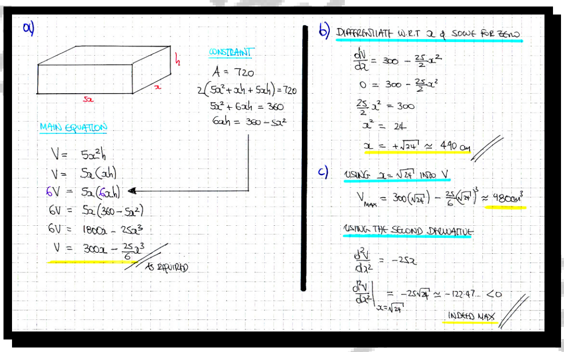
a) Show that the volume of the brick, $V \text{ cm}^3$, is given by

$$V = 300x - \frac{25}{6}x^3.$$

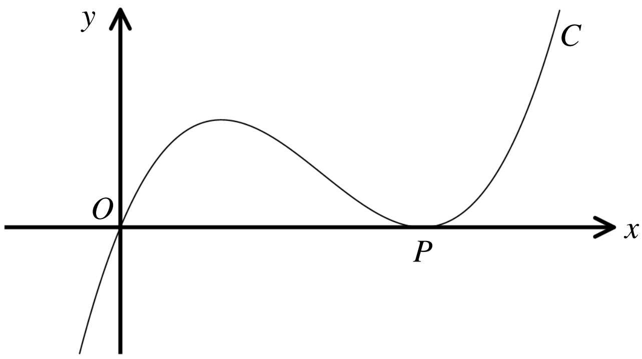
b) Find the value of x for which V is stationary.

c) Calculate the maximum value for V, fully justifying the fact that it is indeed the maximum value.

$\boxed{}$, $\boxed{x = 2\sqrt{6} \approx 4.90}$, $\boxed{V_{\text{max}} = 400\sqrt{6} \approx 980}$



Question 55 (***)



The figure above shows the cubic curve C which meets the coordinates axes at the origin O and at the point P.

The gradient function of C is given by

$$f'(x) = 3x^2 - 8x + 4.$$

a) Find an equation for C.

b) Determine the coordinates of P.

$\boxed{}$, $\boxed{f(x) = x^3 - 4x^2 + 4x}$, $\boxed{P(2,0)}$

(a) IF $f'(x) = 3x^2 - 8x + 4$

THEN

$f(x) = \int 3x^2 - 8x + 4 \, dx$

$f(x) = x^3 - 4x^2 + 4x + C$

BUT CUBIC GOES THROUGH $(0,0)$

$0 = 0 - 0 + 0 + C$

$\therefore C = 0$

$\therefore f(x) = x^3 - 4x^2 + 4x$

(b) $f(x) = x^3 - 4x^2 + 4x$

$f(x) = x(x^2 - 4x + 4)$

$f(x) = x(x-2)^2$

$\therefore$ when $y = 0$

$x = \begin{cases} 0 \\ 2 \end{cases}$

$\therefore P(2,0)$

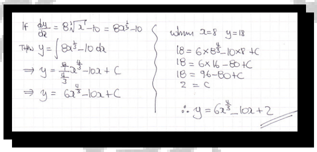
Question 56 **(***)**

The point $P(8,18)$ lies on the curve C, whose gradient function is given by

$$\frac{dy}{dx} = 8\sqrt[3]{x} - 10, \ x \geq 0 .$$

Find an equation for C.

$\boxed{}$, $\boxed{y = 6x^{\frac{4}{3}} - 10x + 2}$



If $\frac{dy}{dx} = 8\sqrt[3]{x^1} - 10 = 8x^{\frac{1}{3}} - 10$

Then $y = \int 8x^{\frac{1}{3}} - 10 \, dx$

$\Rightarrow y = \frac{\frac{8}{1}}{\frac{4}{3}} x^{\frac{4}{3}} - 10x + C$

$\Rightarrow y = 6x^{\frac{4}{3}} - 10x + C$

when $x = 8$ $y = 18$

$18 = 6 \times 8^{\frac{4}{3}} - 10 \times 8 + C$

$18 = 6 \times 16 - 80 + C$

$18 = 96 - 80 + C$

$2 = C$

$\therefore y = 6x^{\frac{4}{3}} - 10x + 2$

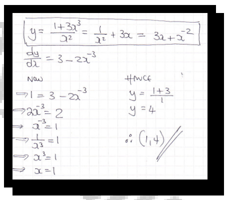
Question 57 **(***)**

A curve C has equation given by

$$y = \frac{1 + 3x^3}{x^2}, \ x \neq 0$$

Find the coordinates of the point on C where the gradient is 1.

$\boxed{(1,4)}$



$\boxed{y = \frac{1+3x^3}{x^2} = \frac{1}{x^2} + 3x = 3x + x^{-2}}$

$\frac{dy}{dx} = 3 - 2x^{-3}$

Now

$\Rightarrow 1 = 3 - 2x^{-3}$

$\Rightarrow 2x^{-3} = 2$

$\Rightarrow x^{-3} = 1$

$\Rightarrow \frac{1}{x^3} = 1$

$\Rightarrow x^3 = 1$

$\Rightarrow x = 1$

Hence

$y = \frac{1+3}{1}$

$y = 4$

$\therefore (1,4)$

Question 58 (***)

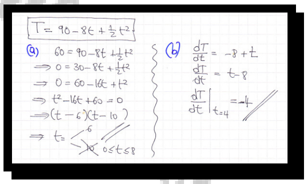
The temperature, T in °C, of a hot drink t minutes after it was made is given by

$$T = 90 - 8t + \frac{1}{2}t^2 \ , \ 0 \le t \le 8 .$$

a) Calculate after how many minutes the drink has a temperature of 60 °C.

b) Find the rate of change of temperature of the drink 4 minutes after it was made.

$\boxed{t = 6}$, $\boxed{-4 \text{ °C/min}}$



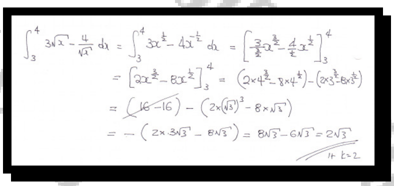
Question 59 (***)

Show clearly that

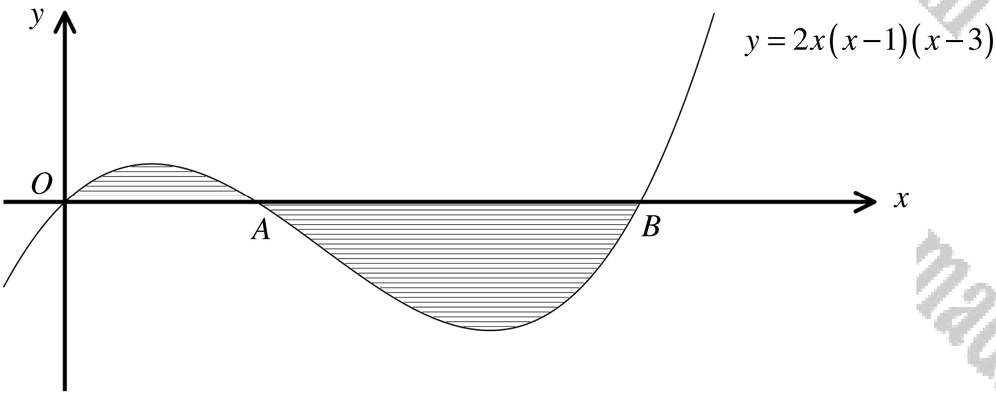
$$\int_3^4 3\sqrt{x} - \frac{4}{\sqrt{x}} \ dx = k\sqrt{3} ,$$

where k is an integer to be found.

$\boxed{k = 2}$



Question 60 (***)



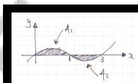
The figure above shows part of the curve with equation

$$y = 2x(x-1)(x-3), \; x \in \mathbb{R}.$$

The curve meets the x axis at the origin and at the points A and B.

Determine the exact area of the finite region bounded by the curve and the x axis, shown shaded in the figure above.

$\boxed{}$, $\boxed{\text{area} = \frac{37}{6}}$



$y = 2x(x-1)(x-3)$

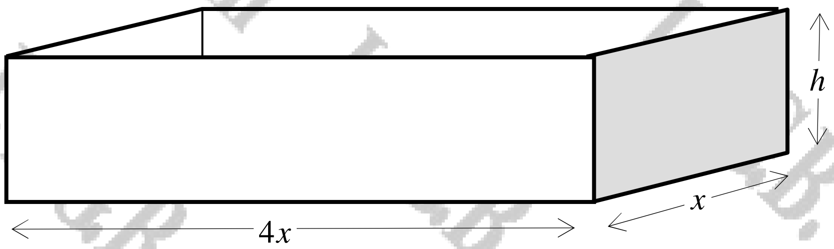
$y = 2x(x^2-4x+3)$

$y = 2x^3-8x^2+6x$

- $A_1 = \int_0^1 2x^3-8x^2+6x \, dx = \left[\frac{1}{2}x^4 - \frac{8}{3}x^3 + 3x^2\right]_0^1 = \left(\frac{1}{2} - \frac{8}{3} + 3\right) - (0) = \frac{5}{6}$
- $A_2 = \int_1^3 2x^3-8x^2+6x \, dx = \left[\frac{1}{2}x^4 - \frac{8}{3}x^3 + 3x^2\right]_1^3 = \left(\frac{81}{2} - 72 + 27\right) - \left(\frac{5}{6}\right) = -\frac{32}{6}$

$\therefore$ REQUIRED AREA $= \frac{5}{6} + \frac{32}{6} = \frac{37}{6}$

Question 61 (***)



The figure above shows a box in the shape of a cuboid with a rectangular base x cm by $4x$ cm and **no top**. The height of the box is h cm.

It is given that the surface area of the box is 1728 cm^2.

a) Show clearly that

$$h = \frac{864 - 2x^2}{5x}.$$

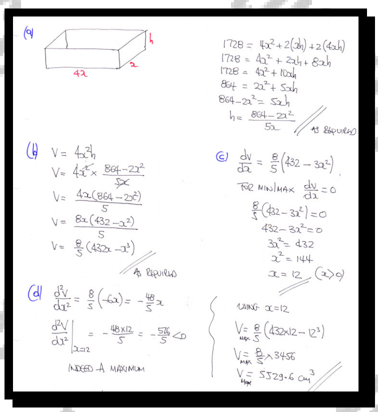
b) Use part (a) to show that the volume of the box, $V \text{ cm}^3$, is given by

$$V = \frac{8}{5}\left(432x - x^3\right).$$

c) Find the value of x for which V is stationary.

d) Find the maximum value for V, justifying the fact that it is the maximum.

$\boxed{x = 12}$, $\boxed{V_{\max} = 5529.6}$


(a)

$1728 = 4x^2 + 2(xh) + 2(4xh)$

$1728 = 4x^2 + 2xh + 8xh$

$1728 = 4x^2 + 10xh$

$864 = 2x^2 + 5xh$

$864 - 2x^2 = 5xh$

$h = \frac{864 - 2x^2}{5x}$ AS REQUIRED

(b) $V = 4x^2 h$

$V = 4x^2 \times \frac{864 - 2x^2}{5x}$

$V = \frac{4x(864 - 2x^2)}{5}$

$V = \frac{8x(432 - x^2)}{5}$

$V = \frac{8}{5}(432x - x^3)$ AS REQUIRED

(c) $\frac{dV}{dx} = \frac{8}{5}(432 - 3x^2)$

FOR MIN/MAX $\frac{dV}{dx} = 0$

$\frac{8}{5}(432 - 3x^2) = 0$

$432 - 3x^2 = 0$

$3x^2 = 432$

$x^2 = 144$

$x = 12$ $(x > 0)$

(d) $\frac{d^2V}{dx^2} = \frac{8}{5}(-6x) = -\frac{48}{5}x$

$\left.\frac{d^2V}{dx^2}\right|_{x=12} = -\frac{48 \times 12}{5} = -\frac{576}{5} < 0$

INDEED A MAXIMUM

USING $x = 12$

$V_{MAX} = \frac{8}{5}(432 \times 12 - 12^3)$

$V_{MAX} = \frac{8}{5} \times 3456$

$V_{MAX} = 5529.6 \text{ cm}^3$

Question 62 **(***)**

The curve C and the line L have equations

$$C:\ y = 16\sqrt{x} + \frac{32}{x} - 35 \quad \text{and} \quad L:\ 2y + x = 14\,.$$

Show that L is a normal to C at the point where $x = 4$.

$\boxed{\text{proof}}$

- $y = 16\sqrt{x} + \frac{32}{x} - 35$

  $y = 16x^{\frac{1}{2}} + 32x^{-1} - 35$
- $\frac{dy}{dx} = 8x^{-\frac{1}{2}} - 32x^{-2} = \frac{8}{\sqrt{x}} - \frac{32}{x^2}$
- when $x=4$ $y = 16\sqrt{4} + \frac{32}{4} - 35$

  $y = 32 + 8 - 35$

  $y = 5$
- $\left.\frac{dy}{dx}\right|_{x=4} = \frac{8}{\sqrt{4}} - \frac{32}{4^2} = 4 - 2 = 2$

Tangent gradient $= 2$
Normal gradient $= -\frac{1}{2}$
Point of normality $(4,5)$

$\Rightarrow y - y_0 = m(x - x_0)$
$\Rightarrow y - 5 = -\frac{1}{2}(x-4)$
$\Rightarrow 2y - 10 = -x + 4$
$\Rightarrow 2y + x = 14$

AS REQUIRED

Question 63 **(***)**

The curve C has equation

$$y = ax^2 - 4\sqrt{x} + \frac{8}{x}\,,\ x > 0\,,$$

where a is a non zero constant.

Given that $\frac{dy}{dx} = 0$ at the point on C where $x = 4$, find the value of a.

$\boxed{}$, $\boxed{a = \frac{3}{16}}$

$y = ax^2 - 4\sqrt{x} + \frac{8}{x}$
$y = ax^2 - 4x^{\frac{1}{2}} + 8x^{-1}$
$\frac{dy}{dx} = 2ax - 2x^{-\frac{1}{2}} - 8x^{-2}$
$\frac{dy}{dx} = 2ax - \frac{2}{\sqrt{x}} - \frac{8}{x^2}$

Now $x = 4$, $dy/dx = 0$
$0 = 2a \times 4 - \frac{2}{\sqrt{4}} - \frac{8}{4^2}$
$0 = 8a - 1 - \frac{1}{2}$
$0 = 8a - \frac{3}{2}$
$\frac{3}{2} = 8a$
$a = \frac{3}{16}$

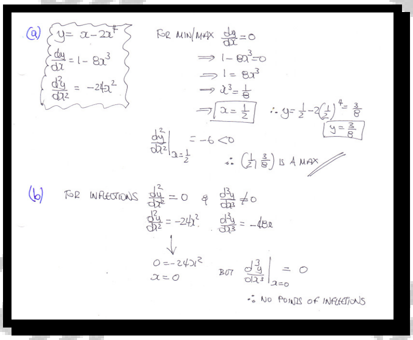
Question 64 (***)

$$y = x - 2x^4, \; x \in \mathbb{R}.$$

a) Find the coordinates of the stationary point of y and determine its nature.

b) Show clearly that y has no points of inflection.

$\boxed{}$, $\boxed{\max\left(\frac{1}{2}, \frac{3}{8}\right)}$



(a) $y = x - 2x^4$

$\frac{dy}{dx} = 1 - 8x^3$

$\frac{d^2y}{dx^2} = -24x^2$

FOR MIN/MAX $\frac{dy}{dx} = 0$

$\Rightarrow 1 - 8x^3 = 0$

$\Rightarrow 1 = 8x^3$

$\Rightarrow x^3 = \frac{1}{8}$

$\Rightarrow \boxed{x = \frac{1}{2}}$ $\therefore y = \frac{1}{2} - 2\left(\frac{1}{2}\right)^4 = \frac{3}{8}$ $\boxed{y = \frac{3}{8}}$

$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = -6 < 0$

$\therefore \left(\frac{1}{2}, \frac{3}{8}\right)$ IS A MAX

(b) FOR INFLECTIONS $\frac{d^2y}{dx^2} = 0$ & $\frac{d^3y}{dx^3} \neq 0$

$\frac{d^2y}{dx^2} = -24x^2$ $\frac{d^3y}{dx^3} = -48x$

$0 = -24x^2$

$x = 0$

BUT $\left.\frac{d^3y}{dx^3}\right|_{x=0} = 0$

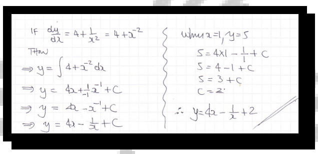
$\therefore$ NO POINTS OF INFLECTIONS

Question 65 (***)

$$\frac{dy}{dx} = 4 + \frac{1}{x^2}, \; x \neq 0.$$

Given that $y = 5$ when $x = 1$, express y in terms of x.

$\boxed{}$, $\boxed{y = 4x - \frac{1}{x} + 2}$



IF $\frac{dy}{dx} = 4 + \frac{1}{x^2} = 4 + x^{-2}$

THEN

$\Rightarrow y = \int 4 + x^{-2} \, dx$

$\Rightarrow y = 4x + \frac{1}{-1}x^{-1} + C$

$\Rightarrow y = 4x - x^{-1} + C$

$\Rightarrow y = 4x - \frac{1}{x} + C$

When $x = 1$, $y = 5$

$5 = 4 \times 1 - \frac{1}{1} + C$

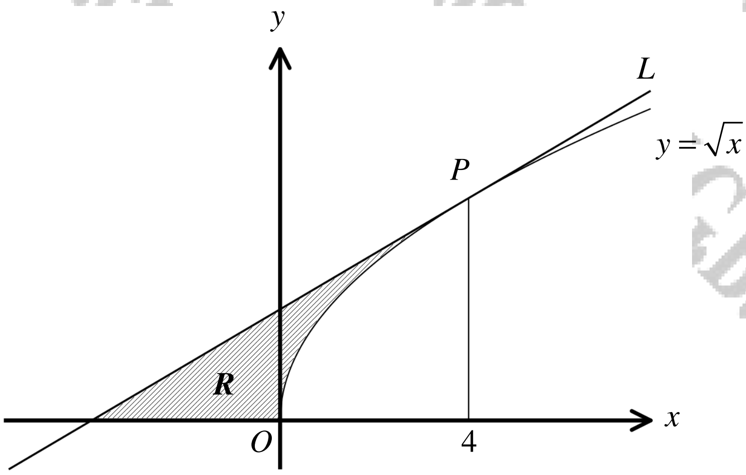
$5 = 4 - 1 + C$

$5 = 3 + C$

$C = 2$

$\therefore y = 4x - \frac{1}{x} + 2$

Question 66 (***)



The figure above shows the graph of the curve C with equation

$$y=\sqrt{x}\,,\ x \geq 0\,.$$

The point P lies on C where $x=4$.

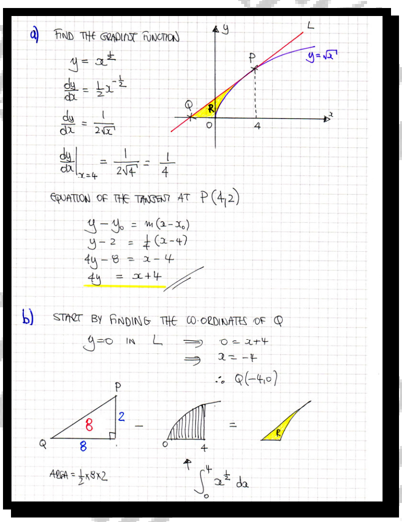
The straight line L is the tangent to C at P.

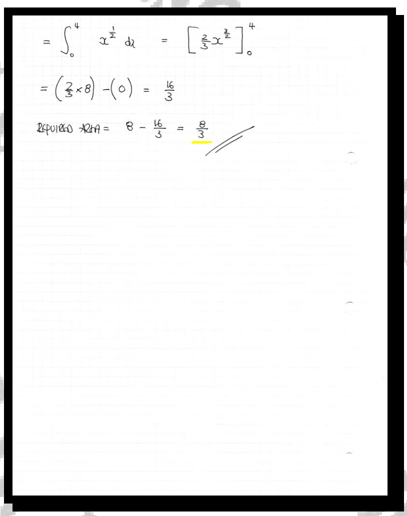
a) Find an equation of L.

The finite region R, shown shaded in the figure, is bounded by C, L and the x axis.

b) Find the exact area of R.

$\boxed{}$, $\boxed{y=\frac{1}{4}x+1}$, $\boxed{\frac{8}{3}}$





Question 67 **(***)**

The curve C has equation

$$f(x) = \frac{(2x-3)(x+2)}{\sqrt{x}}, \quad x > 0.$$

a) Express $f(x)$ in the form $Ax^{\frac{3}{2}} + Bx^{\frac{1}{2}} + Cx^{-\frac{1}{2}}$, where A, B and C are constants to be found.

b) Show that the tangent to C at the point where $x = 1$ is parallel to the line with equation

$$2y = 13x + 2.$$

$\boxed{A=2}$, $\boxed{B=1}$, $\boxed{C=-6}$

(a) $f(x) = \frac{(2x-3)(x+2)}{\sqrt{x}} = \frac{2x^2+x-6}{x^{\frac{1}{2}}} = \frac{2x^2}{x^{\frac{1}{2}}} + \frac{x}{x^{\frac{1}{2}}} - \frac{6}{x^{\frac{1}{2}}}$

$= 2x^{\frac{3}{2}} + x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$

$A = 2$
$B = 1$
$C = -6$

(b) $f'(x) = 3x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} + 3x^{-\frac{3}{2}}$

$f'(1) = 3 + \frac{1}{2} + 3$

$f'(1) = \frac{13}{2}$

(IE GRADIENT OF TANGENT IS $\frac{13}{2}$)

ALSO $2y = 13x + 2$

$y = \frac{13}{2}x + 1$

SAME GRADIENT AS TANGENT

$\therefore$ INDEED PARALLEL

Question 68 (***)

A curve with equation $y = f(x)$ passes through the point $(2,3)$.

The gradient function of the curve is given by

$$f'(x) = (x-3)(3x-1).$$

a) Find an equation of the curve, giving the answer as a polynomial in its simplest form.

b) Show clearly that

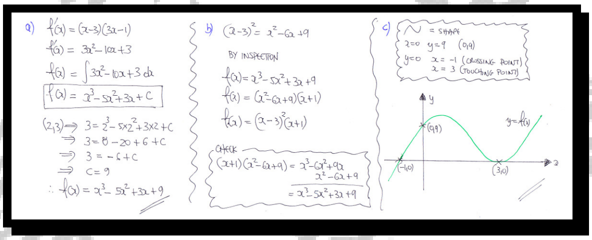
$$f(x) \equiv (x+k)(x-3)^2,$$

where k is a constant to be found.

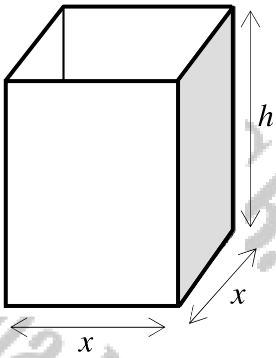
c) Sketch the graph of $f(x)$.

The sketch must show the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

$\boxed{}$, $\boxed{f(x) \equiv x^3 - 5x^2 + 3x + 9}$, $\boxed{k=1}$



Question 69 **(***)**



The figure above shows the design of a large water tank in the shape of a cuboid with a square base and **no top**.

The square base is of length x metres and its height is h metres.

It is given that the volume of the tank is 500 m^3.

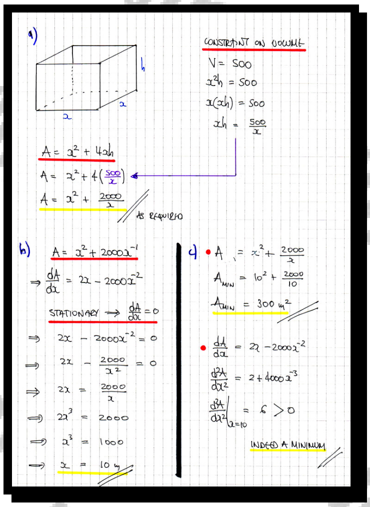
a) Show that the surface area of the tank, $A \text{ m}^2$, is given by

$$A = x^2 + \frac{2000}{x}.$$

b) Find the value of x for which A is stationary.

c) Find the minimum value for A, fully justifying the fact that it is the minimum.

$\boxed{\quad}$, $\boxed{x = 10}$, $\boxed{A_{\text{min}} = 300}$



a)

CONSTRAINT ON VOLUME

$V = 500$

$x^2 h = 500$

$x(xh) = 500$

$xh = \frac{500}{x}$

$A = x^2 + 4xh$

$A = x^2 + 4\left(\frac{500}{x}\right)$

$A = x^2 + \frac{2000}{x}$ // AS REQUIRED

b)

$A = x^2 + 2000x^{-1}$

$\Rightarrow \frac{dA}{dx} = 2x - 2000x^{-2}$

STATIONARY $\rightarrow \frac{dA}{dx} = 0$

$\Rightarrow 2x - 2000x^{-2} = 0$

$\Rightarrow 2x - \frac{2000}{x^2} = 0$

$\Rightarrow 2x = \frac{2000}{x}$

$\Rightarrow 2x^3 = 2000$

$\Rightarrow x^3 = 1000$

$\Rightarrow x = 10 \text{ m}$ //

c)

• $A = x^2 + \frac{2000}{x}$

$A_{\text{MIN}} = 10^2 + \frac{2000}{10}$

$A_{\text{MIN}} = 300 \text{ m}^2$ //

• $\frac{dA}{dx} = 2x - 2000x^{-2}$

$\frac{d^2A}{dx^2} = 2 + 4000x^{-3}$

$\left.\frac{d^2A}{dx^2}\right|_{x=10} = 6 > 0$

INDEED A MINIMUM //

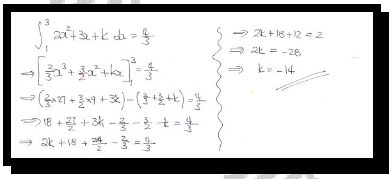
Question 70 (***)

$$f(x) = 2x^2 + 3x + k \text{, where } k \text{ is a constant.}$$

Find the value of k, given that

$$\int_1^3 f(x)\ dx = \frac{4}{3}.$$

$\boxed{k = -14}$



$\int_1^3 2x^2+3x+k\ dx = \frac{4}{3}$

$\Rightarrow \left[\frac{2}{3}x^3+\frac{3}{2}x^2+kx\right]_1^3 = \frac{4}{3}$

$\Rightarrow \left(\frac{2}{3}\times 27+\frac{3}{2}\times 9+3k\right)-\left(\frac{2}{3}+\frac{3}{2}+k\right) = \frac{4}{3}$

$\Rightarrow 18+\frac{27}{2}+3k-\frac{2}{3}-\frac{3}{2}-k = \frac{4}{3}$

$\Rightarrow 2k+18+\frac{24}{2}-\frac{2}{3} = \frac{4}{3}$

$\Rightarrow 2k+18+12 = 2$

$\Rightarrow 2k = -28$

$\Rightarrow k = -14$

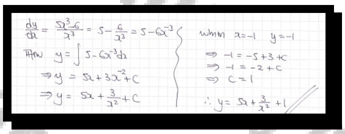
Question 71 (***)

The point $P(-1,-1)$ lies on the curve C, whose gradient function is given by

$$\frac{dy}{dx} = \frac{5x^3-6}{x^3},\ x \neq 0.$$

Find an equation for C.

$\boxed{y = 5x + \frac{3}{x^2} + 1}$



$\frac{dy}{dx} = \frac{5x^3-6}{x^3} = 5-\frac{6}{x^3} = 5-6x^{-3}$

Then $y = \int 5-6x^{-3}\,dx$

$\Rightarrow y = 5x+3x^{-2}+C$

$\Rightarrow y = 5x+\frac{3}{x^2}+C$

when $x=-1$ $y=-1$

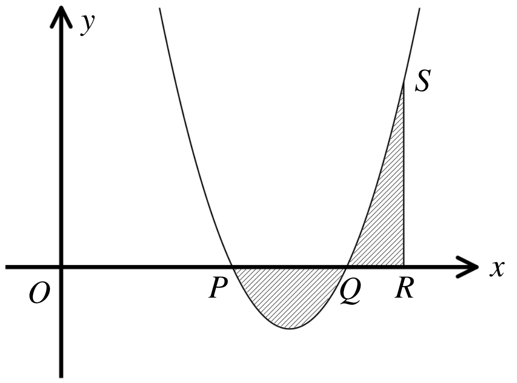
$\Rightarrow -1 = -5+3+C$

$\Rightarrow -1 = -2+C$

$\Rightarrow C = 1$

$\therefore y = 5x+\frac{3}{x^2}+1$

Question 72 (***)



The figure above shows the graph of the quadratic curve C with equation

$$y = 3x^2 - 8x + 5.$$

The curve crosses the x axis at the points P and Q.

a) Find the coordinates of P and Q.

b) Show clearly that

$$\int_1^2 3x^2 - 8x + 5 \; dx = 0.$$

The point R lies on the x axis where $x = 2$ and the point S lies on C so that RS is parallel to the y axis.

c) Find the area of the shaded region.

$\boxed{P(1,0),\ Q\left(\frac{5}{3},0\right)}$ $\boxed{\frac{8}{27}}$

(a) $3x^2 - 8x + 5 = (3x-5)(x-1)$ $\therefore x = \begin{cases} 1 \\ \frac{5}{3} \end{cases}$ $\therefore P(1,0)$, $Q\left(\frac{5}{3},0\right)$

(b) $\int_1^2 3x^2 - 8x + 5 \, dx = \left[x^3 - 4x^2 + 5x\right]_1^2 = (8 - 16 + 10) - (1 - 4 + 5)$
$= 2 - 2 = 0$

(c) AREA UNDER THE AXIS
$\int_1^{\frac{5}{3}} 3x^2 - 8x + 5 \, dx = \left[x^3 - 4x^2 + 5x\right]_1^{5/3} = \left(\frac{125}{27} - \frac{100}{9} + \frac{25}{3}\right) - (1 - 4 + 5)$
$= \frac{50}{27} - 2 = -\frac{4}{27}$

$\therefore$ SHADED AREA $= 2 \times \frac{4}{27} = \frac{8}{27}$

Question 73 (***)

A cubic curve has equation

$$f(x) = 2x^3 - 7x^2 + 6x + 1.$$

The point $P(2,1)$ lies on the curve.

a) Find an equation of the tangent to the curve at P.

The point Q lies on the curve so that the tangent to the curve at Q is parallel to the tangent to the curve at P.

b) Determine the x coordinate of Q.

$\boxed{y = 2x - 3}$, $\boxed{x_Q = \frac{1}{3}}$

(a) $f(x) = 2x^3 - 7x^2 + 6x + 1$

$f'(x) = 6x^2 - 14x + 6$ $\qquad f'(2) = 24 - 28 + 6 = 2$

$P(2,1)$, $m = 2$

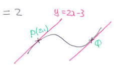
$y - y_0 = m(x - x_0)$

$y - 1 = 2(x - 2)$

$y - 1 = 2x - 4$

$y = 2x - 3$

(b) PARALLEL TANGENTS $\Rightarrow$ SAME GRADIENT $= 2$



$2 = 6x^2 - 14x + 6$

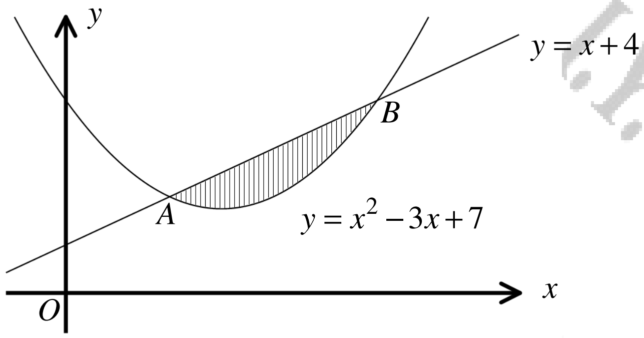
$0 = 6x^2 - 14x + 4$

$0 = 3x^2 - 7x + 2$

$0 = (x - 2)(3x - 1)$

$x = \begin{cases} 2 & \leftarrow \text{POINT } P \text{ (ALREADY KNOWN)} \\ \frac{1}{3} & \leftarrow \text{POINT } Q \end{cases}$

Question 74 (***)



The figure above shows a curve C and a straight line L with respective equations

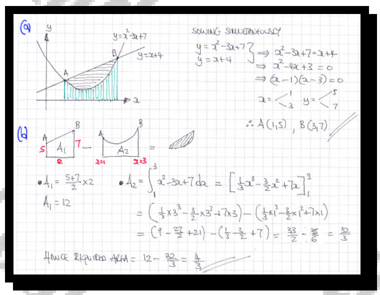
$$y = x^2 - 3x + 7 \quad \text{and} \quad y = x + 4 .$$

The curve and the straight line meet at the points A and B.

a) Find the coordinates of A and B.

b) Find the exact area of the finite region bounded by C and L, shown shaded in the figure above.

$\boxed{A(1,5), B(3,7)}$, $\boxed{\text{area} = \frac{4}{3}}$



Question 75 (***)

The curve C has equation

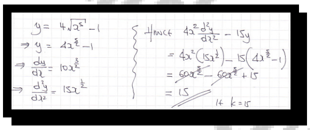
$$y = 4\sqrt{x^5} - 1, \ \ x \geq 0$$

Show clearly that

$$4x^2 \frac{d^2 y}{dx^2} - 15y = k,$$

where k is an integer to be found.

$\boxed{}$, $\boxed{k = 15}$



$y = 4\sqrt{x^5} - 1$

$\Rightarrow y = 4x^{\frac{5}{2}} - 1$

$\Rightarrow \frac{dy}{dx} = 10x^{\frac{3}{2}}$

$\Rightarrow \frac{d^2y}{dx^2} = 15x^{\frac{1}{2}}$

Hence $4x^2\frac{d^2y}{dx^2} - 15y$

$= 4x^2(15x^{\frac{1}{2}}) - 15(4x^{\frac{5}{2}} - 1)$

$= \cancel{60x^{\frac{5}{2}}} - \cancel{60x^{\frac{5}{2}}} + 15$

$= 15$

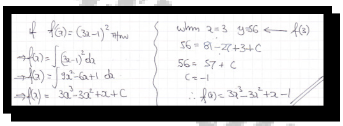
$k = 15$

Question 76 (***)

$$f'(x) = (3x - 1)^2.$$

Given that $f(3) = 56$, find an expression for $f(x)$.

$\boxed{}$, $\boxed{f(x) = 3x^3 - 3x^2 + x - 1}$



If $f'(x) = (3x-1)^2$ then

$\Rightarrow f(x) = \int (3x-1)^2 \, dx$

$\Rightarrow f(x) = \int 9x^2 - 6x + 1 \, dx$

$\Rightarrow f(x) = 3x^3 - 3x^2 + x + C$

when $x = 3$ $y = 56 \leftarrow f(3)$

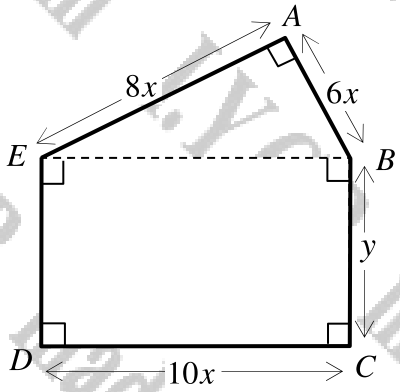
$56 = 81 - 27 + 3 + C$

$56 = 57 + C$

$C = -1$

$\therefore f(x) = 3x^3 - 3x^2 + x - 1$

Question 77 (***)



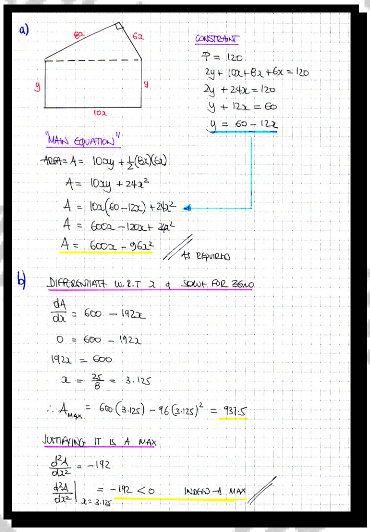
The figure above shows a pentagon $ABCDE$ whose measurements, in cm, are given in terms of x and y.

a) If the perimeter of the pentagon is 120 cm, show clearly that its area, A cm^2, is given by

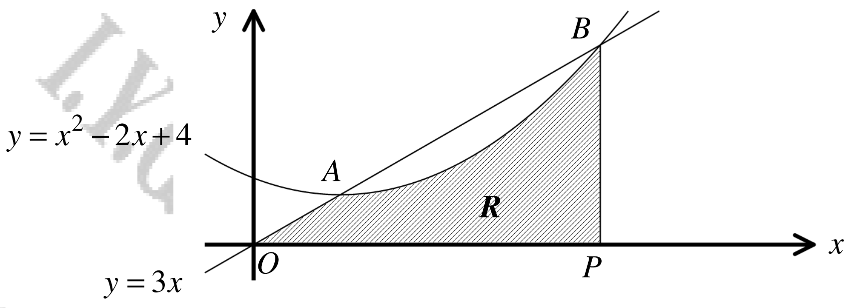
$$A = 600x - 96x^2.$$

b) Use a method based on differentiation to calculate the maximum value for A, fully justifying the fact that it is indeed the maximum value.

$\boxed{}$, $\boxed{A_{max} = 937.5}$



Question 78 (***)



The figure above shows the graph of the curve C with equation

$$y = x^2 - 2x + 4,\ x \in \mathbb{R}$$

intersected by the straight line L with equation

$$y = 3x,\ x \in \mathbb{R}.$$

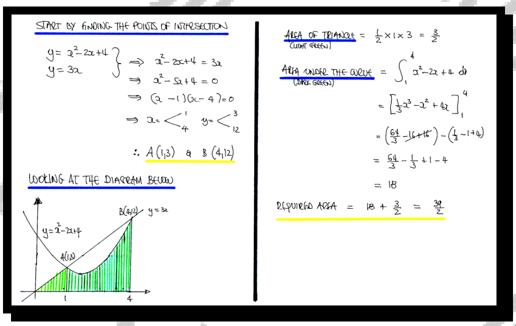
The curve meets the straight line at the points A and B.

The point P is located on the x axis so that the straight line segment BP is parallel to the y axis.

The finite region R is bounded by C, L, BP and the x axis.

Show that the area of R, shown shaded in the figure, is $\frac{39}{2}$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 79 (***)

The curve C has equation

$$y = x^4 - 2x^3 + 1, \ x \in \mathbb{R}.$$

Find the coordinates of the stationary points of C and determine their nature.

$\boxed{}$, $\boxed{\text{min at } \left(\frac{3}{2}, -\frac{11}{16}\right), \quad \text{point of inflexion } (0,1)}$

$y = x^4 - 2x^3 + 1$

$\frac{dy}{dx} = 4x^3 - 6x^2$

$\frac{d^2y}{dx^2} = 12x^2 - 12x$

FOR STATIONARY POINTS $\frac{dy}{dx} = 0$

$4x^3 - 6x^2 = 0$

$2x^2(2x-3) = 0$

$x = \begin{cases} 0 \\ \frac{3}{2} \end{cases} \quad y = \begin{cases} 1 \\ \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 + 1 = -\frac{11}{16} \end{cases}$

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 0$ POSSIBLE POINT OF INFLEXION

USE $\frac{d^3y}{dx^3} = 24x - 12$

$\left.\frac{d^3y}{dx^3}\right|_{x=0} = -12 \neq 0$

$\therefore (0,1)$ IS A POINT OF INFLEXION

$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{3}{2}} = 12\left(\frac{3}{2}\right)^2 - 12 \times \frac{3}{2} = 9 > 0 \quad \therefore \left(\frac{3}{2}, -\frac{11}{16}\right)$ IS A MIN

Question 80 (***) non calculator

The curve C has equation

$$y = x^2 - \frac{16}{3x^2}, \; x \neq 0 .$$

Show that C has two points of inflection and determine their coordinates.

$$\boxed{\left(2, \frac{8}{3}\right), \; \left(-2, \frac{8}{3}\right)}$$

$$y = x^2 - \frac{16}{3x^2} = x^2 - \frac{16}{3}x^{-2}$$

$$\frac{dy}{dx} = 2x + \frac{32}{3}x^{-3}$$

$$\frac{d^2y}{dx^2} = 2 - 32x^{-4}$$

$$\frac{d^3y}{dx^3} = 128x^{-5}$$

- $\frac{d^2y}{dx^2} = 0 \Rightarrow 2 - 32x^{-4} = 0$

$\Rightarrow 2 = \frac{32}{x^4}$

$\Rightarrow x^4 = 16$

$\Rightarrow x = \pm\sqrt[4]{16}$

$\Rightarrow x = 2, -2$

- $\frac{d^3y}{dx^3} = \frac{128}{x^5}$

$\left.\frac{d^3y}{dx^3}\right|_{x=\pm 2} = \pm 4 \neq 0$

- $y = x^2 - \frac{16}{3x^2}$

when $x = \pm 2$ $\quad y = 4 - \frac{16}{3 \times 4}$

$y = 4 - \frac{4}{3}$

$y = \frac{8}{3}$

$\therefore$ POINTS OF INFLECTION AT $\left(2, \frac{8}{3}\right)$ & $\left(-2, \frac{8}{3}\right)$ //

Question 81 (***)

$$y=\left(\frac{2x+1}{3x^{2}}\right)^{2},\ x>0.$$

a) Express y in the form $Ax^{-2}+Bx^{-3}+Cx^{-4}$, where A, B and C are fractions to be found.

b) Hence determine simplified expressions for …

i. … $\dfrac{dy}{dx}$.

ii. … $\displaystyle\int y\,dx$

$\boxed{y=\frac{4}{9}x^{-2}+\frac{4}{9}x^{-3}+\frac{1}{9}x^{-4}}$, $\boxed{\frac{dy}{dx}=-\frac{8}{9}x^{-3}-\frac{4}{3}x^{-4}-\frac{4}{9}x^{-5}}$, $\boxed{\int y\,dx=-\frac{4}{9}x^{-1}-\frac{2}{9}x^{-2}-\frac{1}{27}x^{-3}+C}$

a) $y=\left(\frac{2x+1}{3x^2}\right)^2=\frac{(2x+1)^2}{9x^4}$

$y=\frac{4x^2+4x+1}{9x^4}$

$y=\frac{4x^2}{9x^4}+\frac{4x}{9x^4}+\frac{1}{9x^4}$

$y=\frac{4}{9}x^{-2}+\frac{4}{9}x^{-3}+\frac{1}{9}x^{-4}$

(b)(I) $\frac{dy}{dx}=-\frac{8}{9}x^{-3}-\frac{12}{9}x^{-4}-\frac{4}{9}x^{-5}$

$\frac{dy}{dx}=-\frac{8}{9}x^{-3}-\frac{4}{3}x^{-4}-\frac{4}{9}x^{-5}$

(II) $\int y\,dx=\int \frac{4}{9}x^{-2}+\frac{4}{9}x^{-3}+\frac{1}{9}x^{-4}\,dx$

$=\frac{\frac{4}{9}}{-1}x^{-1}+\frac{\frac{4}{9}}{-\frac{2}{1}}x^{-2}+\frac{\frac{1}{9}}{-\frac{3}{1}}x^{-3}+C$

$=-\frac{4}{9}x^{-1}-\frac{4}{18}x^{-2}-\frac{1}{27}x^{-3}+C$

$=-\frac{4}{9}x^{-1}-\frac{2}{9}x^{-2}-\frac{1}{27}x^{-3}+C$

Question 82 (***)

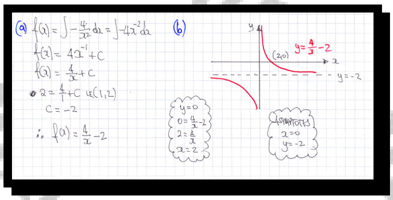
The curve C with equation $y = f(x)$ satisfies

$$f'(x) = -\frac{4}{x^2}, \quad x \neq 0.$$

a) Given that $f(1) = 2$, find an expression for $f(x)$.

b) Sketch the graph of $f(x)$, indicating clearly the asymptotes of the curve and the coordinates of any points where the curve crosses the coordinate axes.

$\boxed{}$, $\boxed{f(x) = \frac{4}{x} - 2}$, $\boxed{(2,0)}$



Question 83 (***)

The total cost C, in £, for a certain car journey, is modelled by

$$C = \frac{200}{V} + \frac{2V}{25}, \quad V > 30,$$

where V is the average speed in miles per hour.

a) Find the value of V for which C is stationary.

b) Justify that this value of V minimizes C.

c) Hence determine the minimum total cost of the journey.

$\boxed{}$, $\boxed{V = 50}$, $\boxed{£8}$

(a) $C = \frac{200}{V} + \frac{2V}{25} = 200V^{-1} + \frac{2}{25}V$

$\frac{dC}{dV} = -200V^{-2} + \frac{2}{25}$

SOLVE FOR ZERO

$\Rightarrow -200V^{-2} + \frac{2}{25} = 0$

$\Rightarrow -\frac{200}{V^2} + \frac{2}{25} = 0$

$\Rightarrow \frac{2}{25} = \frac{200}{V^2}$

$\Rightarrow 2V^2 = 5000$

$\Rightarrow V^2 = 2500$

$\Rightarrow V = 50 \quad (V > 30)$

(b) $\frac{d^2C}{dV^2} = 400V^{-3}$

$\frac{d^2C}{dV^2} = \frac{400}{V^3}$

$\left.\frac{d^2C}{dV^2}\right|_{V=50} = \frac{400}{50^3} = \frac{2}{625} > 0$

INDEED MINIMUM

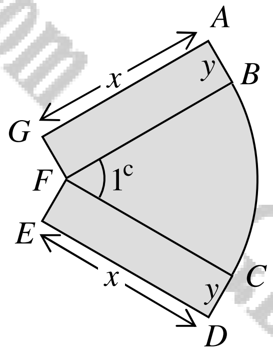
(c) WHEN $V = 50$

$C = \frac{200}{50} + \frac{2 \times 50}{25}$

$C = 4 + 4$

$C = £8$

Question 84 (***)



The figure above shows a clothes design consisting of two identical rectangles attached to each of the straight sides of a circular sector of radius x cm.

The rectangles measure x cm by y cm and the circular sector subtends an angle of one radian at the centre.

The perimeter of the design is 40 cm.

a) Show that the area of the design, A cm^2, is given by

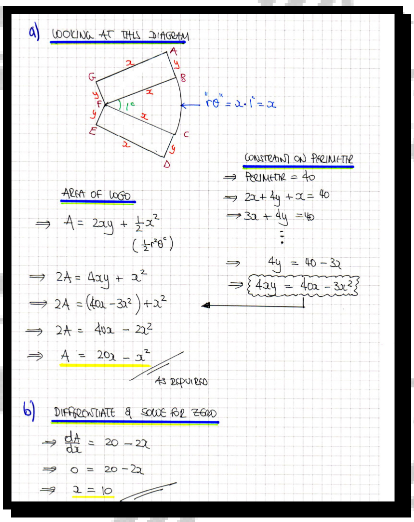
$$A = 20x - x^2 .$$

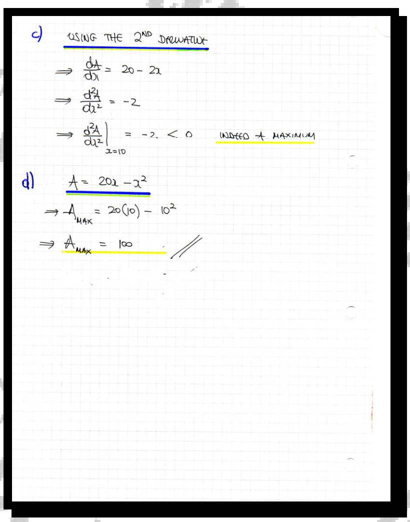
b) Determine **by differentiation** the value of x for which A is stationary.

c) Show that the value of x found in part **(b)** gives the maximum value for A.

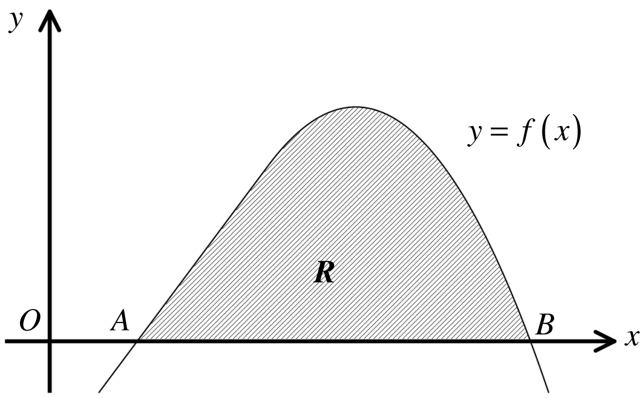
d) Find the maximum area of the design.

$\boxed{}$, $\boxed{x = 10}$, $\boxed{A_{\max} = 100}$





Question 85 **(***+)**



The figure above shows the graph of the curve with equation

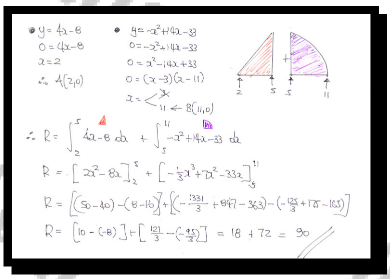
$$f(x) = \begin{cases} 4x - 8 & x < 5 \\ -x^2 + 14x - 33 & x \geq 5 \end{cases}.$$

The curve meets the x axis at the points A and B.

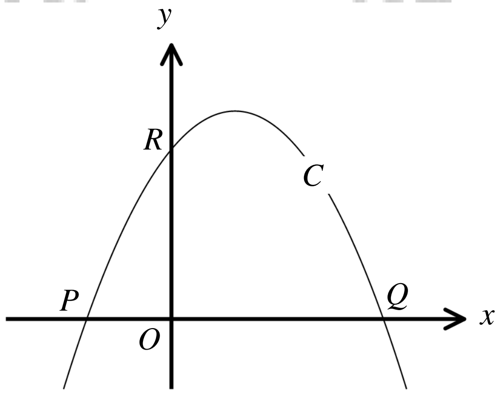
The finite region R, shown shaded in the figure above, is bounded by the curve and the x axis.

Find the area of R.

$\boxed{}$, $\boxed{90}$



Question 86 (***+)



The figure above shows the curve C which meets the coordinates axes at the points P, Q and R.

Given the gradient function of C is given by

$$f'(x) = 3 - 4x,$$

and that $f(1) = 2f(2)$, determine the coordinates of P, Q and R.

$\boxed{}$, $\boxed{P(-1,0)}$, $\boxed{P\left(\frac{5}{2},0\right)}$, $\boxed{R(0,5)}$

$f'(x) = 3 - 4x$

$f(x) = \int 3 - 4x \, dx = 3x - 2x^2 + k$

$\therefore \boxed{f(x) = -2x^2 + 3x + k}$

$f(1) = 2f(2)$

$-2 + 3 + k = 2(-8 + 6 + k)$

$1 + k = -4 + 2k$

$5 = k$

$\therefore \boxed{f(x) = -2x^2 + 3x + 5}$

when $x = 0$ $f(0) = 5$ $\therefore R(0,5)$

when $y = 0$ $0 = -2x^2 + 3x + 5$

$2x^2 - 3x - 5 = 0$

$(2x - 5)(x + 1) = 0$

$\therefore x = -1$ $P(-1,0)$

$\quad x = \frac{5}{2}$ $Q\left(\frac{5}{2},0\right)$

Question 87 **(***+)**

The curve C has equation

$$y=(x-1)\left(x^2+4x+5\right),\ x\in\mathbb{R}.$$

a) Show that C meets the x axis at only one point.

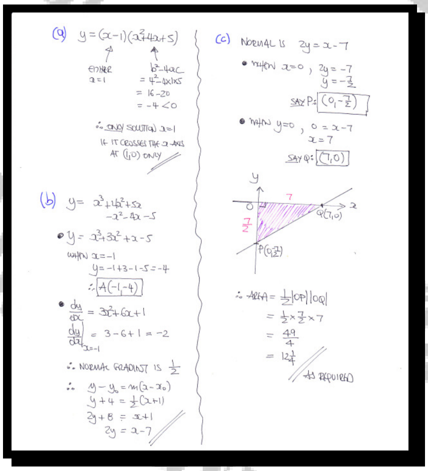
The point A, where $x=-1$, lies on C.

b) Find an equation of the normal to C at A.

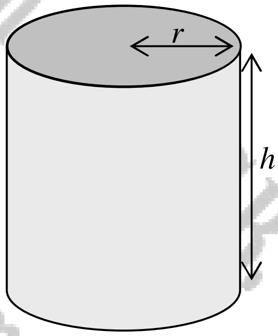
The normal to C at A meets the coordinate axes at the points P and Q.

c) Show further that the area of the triangle OPQ, where O is the origin, is $12\frac{1}{4}$ square units.

$\boxed{}$, $\boxed{2y=x-7}$



Question 88 (***+)



The figure above shows a **closed** cylindrical can of radius r cm and height h cm.

a) Given that the surface area of the can is 192π cm^2, show that the volume of the can, V cm^3, is given by

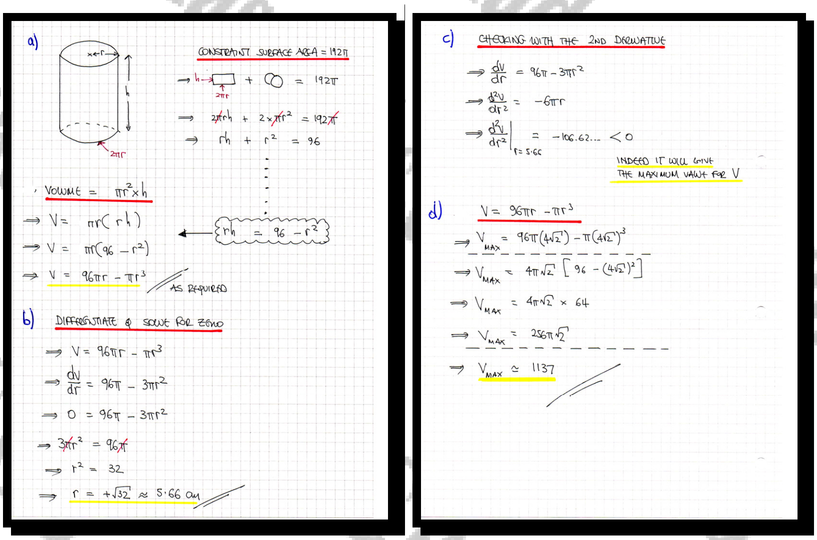
$$V = 96\pi r - \pi r^3 .$$

b) Find the value of r for which V is stationary.

c) Justify that the value of r found in part **(b)** gives the maximum value for V.

d) Calculate the maximum value of V.

$\boxed{\text{[illegible]}}$, $\boxed{r = 4\sqrt{2} \approx 5.66}$, $\boxed{V_{\max} = 256\pi\sqrt{2} \approx 1137}$



Question 89 (***+)

The curve C has equation

$$y = 2x^3 - 9x^2 + 12x - 10 .$$

a) Find the coordinates of the two points on the curve where the gradient is zero.

The point P lies on C and its x coordinate is -1.

b) Determine the gradient of C at the point P.

The point Q lies on C so the gradient at Q is the same as the gradient at P.

c) Find the coordinates of Q.

$\boxed{\text{C1G}}$, $\boxed{(1,-5),(2,-6)}$, $\boxed{36}$, $\boxed{Q(4,22)}$

$y = 2x^3 - 9x^2 + 12x - 10$

(a) $\frac{dy}{dx} = 6x^2 - 18x + 12$

$\frac{dy}{dx} = 0$

$0 = 6x^2 - 18x + 12$

$0 = x^2 - 3x + 2$

$0 = (x-1)(x-2)$

$x = \begin{cases} 1 \\ 2 \end{cases}$ $y = \begin{cases} 2 - 9 + 12 - 10 = 14 - 19 = -5 \\ 16 - 36 + 24 - 10 = 40 - 46 = -6 \end{cases}$ $\therefore \begin{matrix} (1,-5) \\ (2,-6) \end{matrix}$

(b) $\frac{dy}{dx} = 6x^2 - 18x + 12$

$\left.\frac{dy}{dx}\right|_{x=-1} = 6(-1)^2 - 18(-1) + 12 = 6 + 18 + 12 = 36$

(c) $\frac{dy}{dx} = 36$

$6x^2 - 18x + 12 = 36$

$6x^2 - 18x - 24 = 0$

$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

$x = \begin{cases} 4 \\ -1 \end{cases}$ $y = 2\times4^3 - 9\times4^2 + 12\times4 - 10$

ALREADY KNOWN

$y = 128 - 144 + 48 - 10$

$y = 176 - 154 = 22$

$\therefore (4,22)$

Question 90 (***+)

The gradient function at every point on a curve C is given by

$$\frac{dy}{dx} = (kx-3)\sqrt{x}\,,$$

where k is a non zero constant.

The point $P(4,40)$ lies on C and the gradient at P is 34.

Determine an equation of C.

$\boxed{}$, $\boxed{y = 2x^{\frac{5}{2}} - 2x^{\frac{3}{2}} - 8}$

$\frac{dy}{dx} = (kx-3)\sqrt{x}$

GRADIENT AT $(4,40)$ IS 34 IE $\left.\frac{dy}{dx}\right|_{x=4} = 34$

$34 = (k\times4-3)\times\sqrt{4}$
$34 = (4k-3)\times2$
$17 = 4k-3$
$20 = 4k$
$\boxed{k=5}$

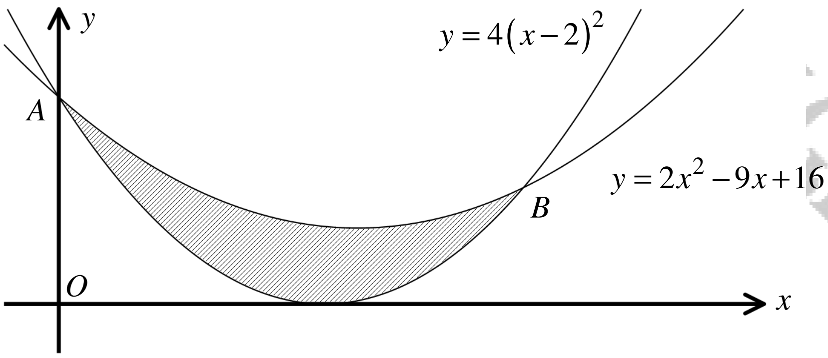
NOW $\frac{dy}{dx} = (5x-3)\sqrt{x} = (5x-3)x^{\frac{1}{2}} = 5x^{\frac{3}{2}} - 3x^{\frac{1}{2}}$

THUS
$y = \int 5x^{\frac{3}{2}} - 3x^{\frac{1}{2}}\,dx = \frac{5}{\frac{5}{2}}x^{\frac{5}{2}} - \frac{3}{\frac{3}{2}}x^{\frac{3}{2}} + C = 2x^{\frac{5}{2}} - 2x^{\frac{3}{2}} + C$

WHEN $x=4$ $y=40$
$40 = 2\times4^{\frac{5}{2}} - 2\times4^{\frac{3}{2}} + C$
$40 = 64 - 16 + C$
$40 = 48 + C$
$C = -8$

$\therefore y = 2x^{\frac{5}{2}} - 2x^{\frac{3}{2}} - 8$

Question 91 (***+)



The figure above shows the graph of the curves with equations

$$y = 4(x-2)^2 \quad \text{and} \quad y = 2x^2 - 9x + 16 .$$

The curves meet each other at the points A and B.

a) Determine the coordinates of A and B.

b) Find the exact area of the finite region bounded by the two curves, shown shaded in the above figure.

$\boxed{}$, $\boxed{A(0,16) \;\; B\left(\frac{7}{2},9\right)}$, $\boxed{\frac{343}{24}}$

a) SOLVING SIMULTANEOUSLY

$$\left.\begin{aligned} y &= 4(x-2)^2 \\ y &= 2x^2-9x+16 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow 4(x-2)^2 = 2x^2-9x+16$

$\Rightarrow 4(x^2-4x+4) = 2x^2-9x+16$

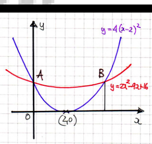
$\Rightarrow 4x^2-16x+\cancel{16} = 2x^2-9x+\cancel{16}$

$\Rightarrow 2x^2-7x=0$

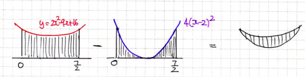
$\Rightarrow x(2x-7)=0$

$x = \begin{cases} 0 \\ \frac{7}{2} \end{cases} \quad y = \begin{cases} 16 \\ 9 \end{cases}$

$\therefore A(0,16)$ & $B\left(\frac{7}{2},9\right)$



b)



LOOKING AT THE ABOVE DIAGRAM

REQUIRED AREA $= \int_0^{\frac{7}{2}} 2x^2-9x+16 \, dx - \int_0^{\frac{7}{2}} 4(x-2)^2 \, dx$

$= \int_0^{\frac{7}{2}} 2x^2-9x+16 \, dx - \int_0^{\frac{7}{2}} 4x^2-16x+16 \, dx$

COMBINING INTEGRALS

$\dots = \int_0^{\frac{7}{2}} (2x^2-9x+16)-(4x^2-16x+16) \, dx$

$= \int_0^{\frac{7}{2}} -2x^2+7x \, dx$

$= \left[-\frac{2}{3}x^3+\frac{7}{2}x^2\right]_0^{\frac{7}{2}}$

$= \left[-\frac{2}{3}\times\left(\frac{7}{2}\right)^3+\frac{7}{2}\left(\frac{7}{2}\right)^2\right]-[0]$

$= -\frac{343}{12}+\frac{343}{8}$

$= \frac{343}{24}$

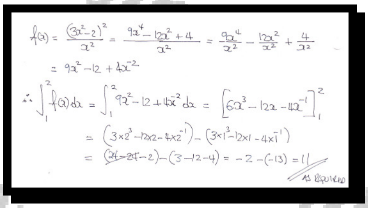
Question 92 (***+)

$$f(x)=\frac{\left(3x^2-2\right)^2}{x^2},\ x\neq 0.$$

Show clearly that

$$\int_1^2 f(x)\ dx=11.$$

proof



$$f(x)=\frac{(3x^2-2)^2}{x^2}=\frac{9x^4-12x^2+4}{x^2}=\frac{9x^4}{x^2}-\frac{12x^2}{x^2}+\frac{4}{x^2}$$
$$=9x^2-12+4x^{-2}$$
$$\therefore \int_1^2 f(x)\,dx=\int_1^2 9x^2-12+4x^{-2}\,dx=\left[6x^3-12x-4x^{-1}\right]_1^2$$
$$=\left(3\times2^3-12\times2-4\times2^{-1}\right)-\left(3\times1^3-12\times1-4\times1^{-1}\right)$$
$$=(\cancel{24}-\cancel{24}-2)-(3-12-4)=-2-(-13)=11$$
AS REQUIRED

Question 93 (***+)

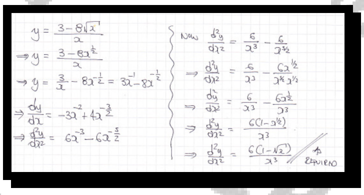
The curve C has equation

$$y=\frac{3-8\sqrt{x}}{x},\ x>0$$

Show clearly that

$$\frac{d^2y}{dx^2}=\frac{6\left(1-\sqrt{x}\right)}{x^3}.$$

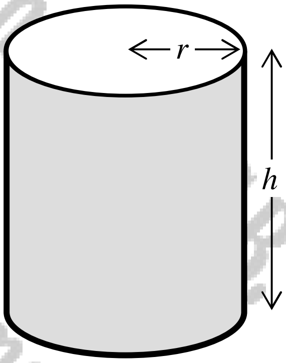
proof



$$y=\frac{3-8\sqrt{x}}{x}$$
$$\Rightarrow y=\frac{3-8x^{\frac{1}{2}}}{x}$$
$$\Rightarrow y=\frac{3}{x}-8x^{-\frac{1}{2}}=3x^{-1}-8x^{-\frac{1}{2}}$$
$$\Rightarrow \frac{dy}{dx}=-3x^{-2}+4x^{-\frac{3}{2}}$$
$$\Rightarrow \frac{d^2y}{dx^2}=6x^{-3}-6x^{-\frac{5}{2}}$$

Now $\frac{d^2y}{dx^2}=\frac{6}{x^3}-\frac{6}{x^{5/2}}$
$$\Rightarrow \frac{d^2y}{dx^2}=\frac{6}{x^3}-\frac{6x^{1/2}}{x^{5/2}x^{1/2}}$$
$$\Rightarrow \frac{d^2y}{dx^2}=\frac{6}{x^3}-\frac{6x^{\frac{1}{2}}}{x^3}$$
$$\Rightarrow \frac{d^2y}{dx^2}=\frac{6(1-x^{1/2})}{x^3}$$
$$\Rightarrow \frac{d^2y}{dx^2}=\frac{6(1-\sqrt{x})}{x^3}$$
AS REQUIRED

Question 94 (***+)



A pencil holder is in the shape of a right circular cylinder, which is **open** at one of its circular ends.

The cylinder has radius r cm and height h cm and the total surface area of the cylinder, including its base, is 360 cm^2.

a) Show that the volume, $V \text{ cm}^3$, of the cylinder is given by

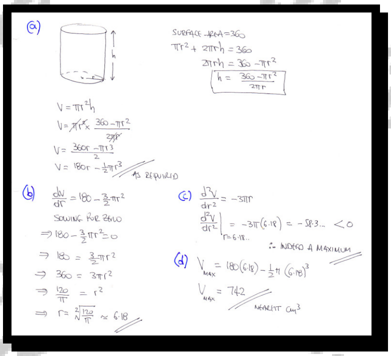
$$V = 180r - \frac{1}{2}\pi r^3.$$

b) Determine by differentiation the value of r for which V has a stationary value.

c) Show that the value of r found in part (b) gives the maximum value for V.

d) Calculate, to the nearest cm^3, the maximum volume of the pencil holder.

$\boxed{r = \sqrt{\frac{120}{\pi}} \approx 6.18}$, $\boxed{V_{\text{max}} \approx 742}$



(a)

SURFACE AREA = 360

$\pi r^2 + 2\pi r h = 360$

$2\pi r h = 360 - \pi r^2$

$\boxed{h = \frac{360 - \pi r^2}{2\pi r}}$

$V = \pi r^2 h$

$V = \cancel{\pi} r^{\cancel{2}} \times \frac{360 - \pi r^2}{2\cancel{\pi}\cancel{r}}$

$V = \frac{360r - \pi r^3}{2}$

$V = 180r - \frac{1}{2}\pi r^3$ AS REQUIRED

(b) $\frac{dV}{dr} = 180 - \frac{3}{2}\pi r^2$

SOLVING FOR ZERO

$\Rightarrow 180 - \frac{3}{2}\pi r^2 = 0$

$\Rightarrow 180 = \frac{3}{2}\pi r^2$

$\Rightarrow 360 = 3\pi r^2$

$\Rightarrow \frac{120}{\pi} = r^2$

$\Rightarrow r = \sqrt[2]{\frac{120}{\pi}} \simeq 6.18$

(c) $\frac{d^2V}{dr^2} = -3\pi r$

$\left.\frac{d^2V}{dr^2}\right|_{r=6.18..} = -3\pi(6.18) = -58.3... < 0$

$\therefore$ INDEED A MAXIMUM

(d) $V_{\text{MAX}} = 180(6.18) - \frac{1}{2}\pi(6.18)^3$

$V_{\text{MAX}} = 742$ NEAREST cm^3

Question 95 (***+)

A curve has equation

$$y = x - 8\sqrt{x}, \quad x \in \mathbb{R}, \quad x \geq 0.$$

The curve meets the x axis at the origin and at the point P.

a) Determine the coordinates of P.

The point Q, where $x = 4$, lies on the curve.

b) Find an equation of the normal to curve at Q.

c) Show that the normal to the curve at Q does not meet the curve again.

$\boxed{}$, $\boxed{P(64,0)}$, $\boxed{y = x - 16}$

a) SETTING $y=0$

$y = x - 8\sqrt{x}$

$0 = x - 8\sqrt{x}$

$8\sqrt{x} = x$

$64x = x^2$

$x^2 - 64x = 0$

$x(x-64) = 0$

$x = \begin{cases} 0 & \leftarrow O \\ 64 & \leftarrow P \end{cases}$

$\therefore P(64,0)$

b) OBTAIN THE y CO-ORDINATE OF Q

$y = 4 - 8 \times \sqrt{4}$

$y = 4 - 8 \times 2$

$y = -12$

$\therefore Q(4,-12)$

FIND THE GRADIENT AT Q

$y = x - 8x^{\frac{1}{2}}$

$\frac{dy}{dx} = 1 - 4x^{-\frac{1}{2}}$

$\left.\frac{dy}{dx}\right|_{x=4} = 1 - 4 \times 4^{-\frac{1}{2}} = 1 - 4 \times \frac{1}{2}$

$\frac{dy}{dx} = -1$

NORMAL GRADIENT IS $+1$

$y - y_0 = m(x - x_0)$

$y - (-12) = +1(x-4)$

$y + 12 = x - 4$

$y = x - 16$

c) SUPPOSE THAT IT DOES MEET THE CURVE

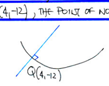
$\left.\begin{array}{l} y = x - 16 \\ y = x - 8\sqrt{x} \end{array}\right\} \Rightarrow \left.\begin{array}{l} 16 = x - y \\ 8\sqrt{x} = x - y \end{array}\right\} \Rightarrow$

$\Rightarrow 16 = 8\sqrt{x}$

$\Rightarrow 2 = \sqrt{x}$

$\Rightarrow x = 4$

$\therefore$ ONLY POINT OF INTERSECTION IS AT $x=4$, IE AT THE POINT $Q(4,-12)$, THE POINT OF NORMALITY



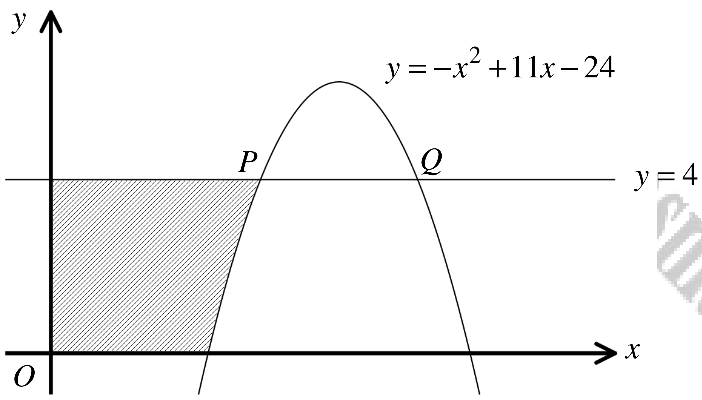
$\therefore$ THE NORMAL DOES NOT MEET THE CURVE AGAIN

Question 96 (***+)

The diagram below shows a parabola and a straight line with respective equations

$$y = -x^2 + 11x - 24 \quad \text{and} \quad y = 4 .$$

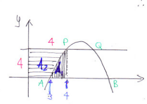
The points P and Q are the intersections between the parabola and the straight line.



Find the exact area of the shaded region, bounded by the curve, the coordinate axes and the straight line with equation $y = 4$.

$\boxed{}$, $\boxed{\text{area} = \frac{83}{6}}$

$$\left.\begin{array}{l} y = -x^2 + 11x - 24 \\ y = 4 \end{array}\right\} \Rightarrow -x^2 + 11x - 24 = 4$$
$$0 = x^2 - 11x + 28$$
$$0 = (x-4)(x-7)$$
$$\therefore x = \begin{cases} 4 \\ 7 \end{cases} \quad \therefore \begin{array}{l} P(4,4) \\ Q(7,4) \end{array}$$



$$y = -x^2 + 11x - 24$$
$$0 = -x^2 + 11x - 24$$
$$x^2 - 11x + 24 = 0$$
$$(x-3)(x-8) = 0$$
$$x = \begin{cases} 3 \leftarrow A \\ 8 \leftarrow B \end{cases}$$

$$A_1 = \int_3^4 -x^2 + 11x - 24 \, dx = \left[-\frac{1}{3}x^3 + \frac{11}{2}x^2 - 24x \right]_3^4$$
$$= \left(-\frac{64}{3} + 88 - 96 \right) - \left(-9 + \frac{99}{2} - 72 \right)$$
$$= -\frac{88}{3} - \left(-\frac{63}{2} \right) = \frac{13}{6}$$

$\therefore$ REQUIRED AREA $A_2 = 16 - \frac{13}{6} = \frac{83}{6}$

Question 97 **(***+)**

Given that n is a positive integer greater than 2, show clearly that

$$\int_4^{n^2} 1-\frac{2}{\sqrt{x}}\ dx = (n-2)^2,\ x>0.$$

$\boxed{\text{proof}}$

$$\int_4^{n^2} 1-\frac{2}{\sqrt{x}}\,dx = \int_4^{n^2} 1-2x^{-\frac{1}{2}}\,dx$$

$$= \left[x-\frac{2}{\frac{1}{2}}x^{\frac{1}{2}}\right]_4^{n^2} = \left[x-4x^{\frac{1}{2}}\right]_4^{n^2}$$

$$= \left[n^2-4(n^2)^{\frac{1}{2}}\right]-\left[4-4\times 4^{\frac{1}{2}}\right]$$

$$(n^2-4n)-(4-8)$$

$$n^2-4n+4$$

$$(n-2)^2$$

Question 98 **(***+)**

The point $P\left(4,\frac{1}{3}\right)$ lies on the curve C whose gradient function is given by

$$\frac{dy}{dx}=\frac{x^{\frac{5}{2}}+24}{x^2},\ x\neq 0.$$

a) Determine an equation of the tangent to C at P.

b) Find an equation of C.

$\boxed{6y=21x-82}$, $\boxed{y=\frac{2}{3}x^{\frac{3}{2}}-\frac{24}{x}+1}$

a) $\boxed{\frac{dy}{dx}=\frac{x^{\frac{5}{2}}+24}{x^2}}$

- $\left.\frac{dy}{dx}\right|_{x=4} = \frac{4^{\frac{5}{2}}+24}{4^2} = \frac{32+24}{16} = \frac{56}{16} = \frac{28}{8} = \frac{14}{4} = \frac{7}{2}$

- EQUATION OF TANGENT $y-y_0=m(x-x_0)$

$$y-\tfrac{1}{3}=\tfrac{7}{2}(x-4)$$
$$y-\tfrac{1}{3}=\tfrac{7}{2}x-14$$
$$6y-2=21x-84$$
$$6y=21x-82$$

b) $\frac{dy}{dx}=\frac{x^{\frac{5}{2}}}{x^2}+\frac{24}{x^2}=x^{\frac{1}{2}}+24x^{-2}$

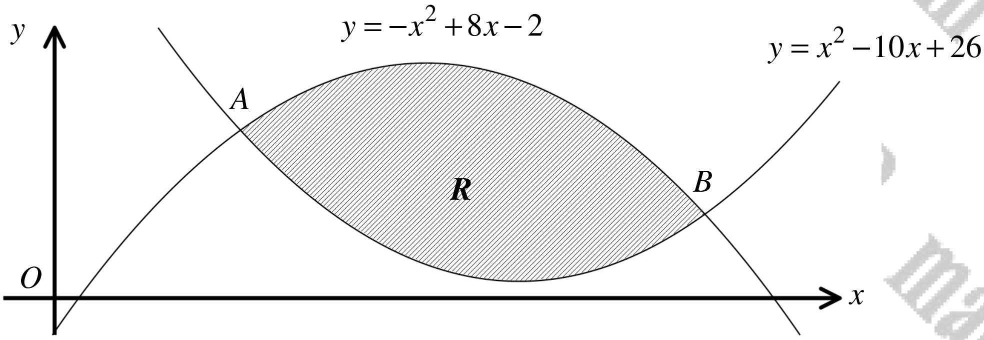
$$y=\int x^{\frac{1}{2}}+24x^{-2}\,dx = \tfrac{2}{3}x^{\frac{3}{2}}-24x^{-1}+C$$

$$\boxed{y=\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{24}{x}+C}$$

BUT $\left(4,\frac{1}{3}\right)$

$$\tfrac{1}{3}=\tfrac{2}{3}\times 4^{\frac{3}{2}}-\tfrac{24}{4}+C$$
$$\tfrac{1}{3}=\tfrac{16}{3}-6+C$$
$$\tfrac{1}{3}-\tfrac{16}{3}+6=C$$
$$C=1$$
$$\therefore y=\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{24}{x}+1$$

Question 99 (***+)



The figure above shows the graphs of the curves with equations

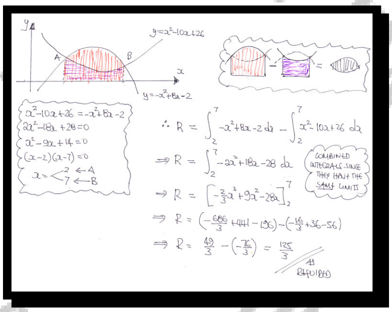
$$y = -x^2 + 8x - 2 \quad \text{and} \quad y = x^2 - 10x + 26.$$

The two curves intersect each other at the points A and B.

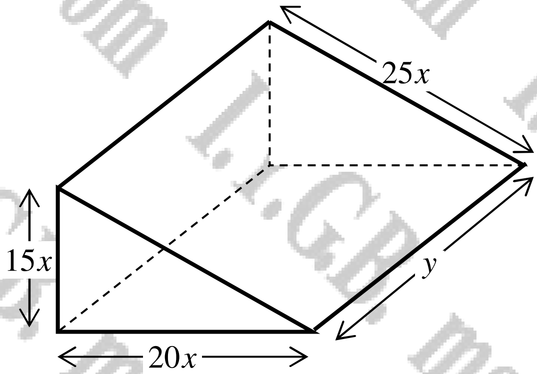
The finite region R bounded by the two curves is shown shaded in the figure above.

Show that the area of R is exactly $\frac{125}{3}$.

, proof



Question 100 (***+)



The figure above shows a solid triangular prism with a **total** surface area of 3600 cm^2.

The triangular faces of the prism are right angled with a base of $20x$ cm and a height of $15x$ cm. The length of the prism is y cm.

a) Show that the volume of the prism, V cm^3, is given by

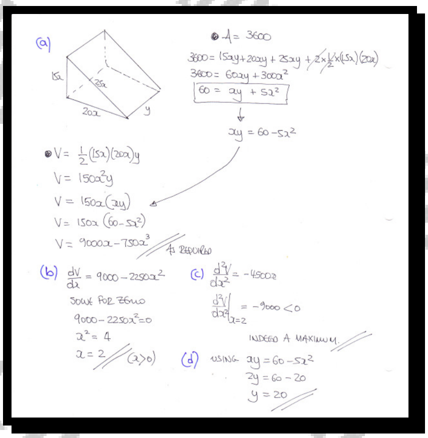
$$V = 9000x - 750x^3 .$$

b) Find the value of x for which V is stationary.

c) Show that the value of x found in part (b) gives the maximum value for V.

d) Determine the value of y when V is maximum.

[illegible], $\boxed{x=2}$, $\boxed{y=20}$



Question 101 (***+)

The curve C has equation

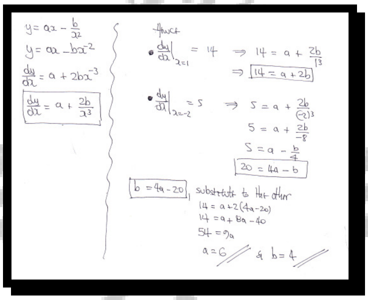
$$y = ax - \frac{b}{x^2}, \ x \neq 0,$$

where a and b are non zero constants.

The gradient of C at the points where $x = 1$ and $x = -2$ is 14 and 5, respectively.

Find the value of a and the value of b.

$\boxed{a = 6, \ b = 4}$



$y = ax - \frac{b}{x^2}$

$y = ax - bx^{-2}$

$\frac{dy}{dx} = a + 2bx^{-3}$

$\boxed{\frac{dy}{dx} = a + \frac{2b}{x^3}}$

Thus

- $\left.\frac{dy}{dx}\right|_{x=1} = 14 \Rightarrow 14 = a + \frac{2b}{1^3}$

  $\Rightarrow \boxed{14 = a + 2b}$

- $\left.\frac{dy}{dx}\right|_{x=-2} = 5 \Rightarrow 5 = a + \frac{2b}{(-2)^3}$

  $5 = a + \frac{2b}{-8}$

  $5 = a - \frac{b}{4}$

  $\boxed{20 = 4a - b}$

$\boxed{b = 4a - 20}$, substitute to the other

$14 = a + 2(4a - 20)$

$14 = a + 8a - 40$

$54 = 9a$

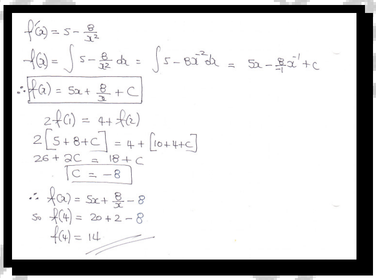
$a = 6$ & $b = 4$

Question 102 (***+)

$$f'(x) = 5 - \frac{8}{x^2}, \ x \neq 0.$$

Find the value of $f(4)$, given that $2f(1) = 4 + f(2)$.

$\boxed{}$, $\boxed{f(4) = 14}$



$f'(x) = 5 - \frac{8}{x^2}$

$f(x) = \int 5 - \frac{8}{x^2} \, dx = \int 5 - 8x^{-2} \, dx = 5x - \frac{8}{-1}x^{-1} + C$

$\therefore \boxed{f(x) = 5x + \frac{8}{x} + C}$

$2f(1) = 4 + f(2)$

$2[5 + 8 + C] = 4 + [10 + 4 + C]$

$26 + 2C = 18 + C$

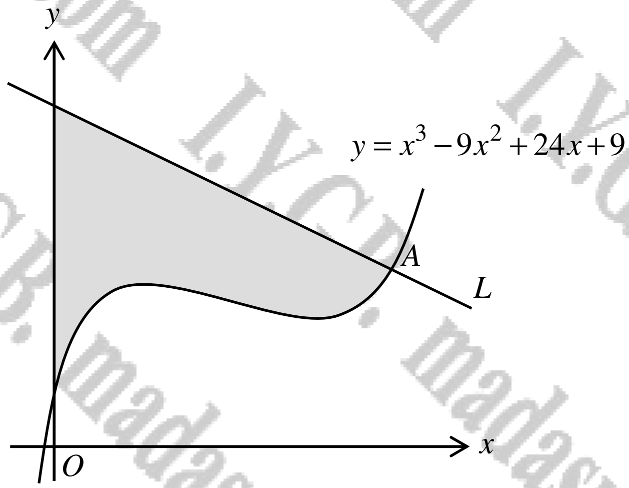
$\boxed{C = -8}$

$\therefore f(x) = 5x + \frac{8}{x} - 8$

So $f(4) = 20 + 2 - 8$

$f(4) = 14$

Question 103 **(***+)**



The figure above shows the graph of the curve C with equation

$$y = x^3 - 9x^2 + 24x + 9 .$$

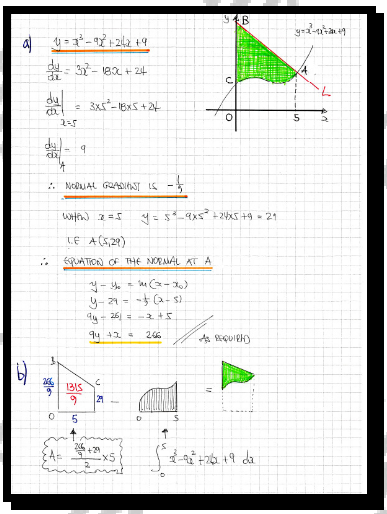
The straight line L is the normal to C at the point A, whose x coordinate is 5.

a) Show that an equation of L is

$$x + 9y = 266 .$$

b) Show further that the area of the finite region bounded by C, L and the y axis is approximately 20 square units.

$\boxed{}$, $\boxed{\text{proof}}$



a) $y = x^3 - 9x^2 + 24x + 9$

$\frac{dy}{dx} = 3x^2 - 18x + 24$

$\left.\frac{dy}{dx}\right|_{x=5} = 3\times5^2 - 18\times5 + 24$

$\left.\frac{dy}{dx}\right|_{A} = 9$

$\therefore$ NORMAL GRADIENT IS $-\frac{1}{9}$

WHEN $x = 5$ $\quad y = 5^3 - 9\times5^2 + 24\times5 + 9 = 29$

I.E $A(5,29)$

$\therefore$ EQUATION OF THE NORMAL AT A

$y - y_0 = m(x - x_0)$

$y - 29 = -\frac{1}{9}(x - 5)$

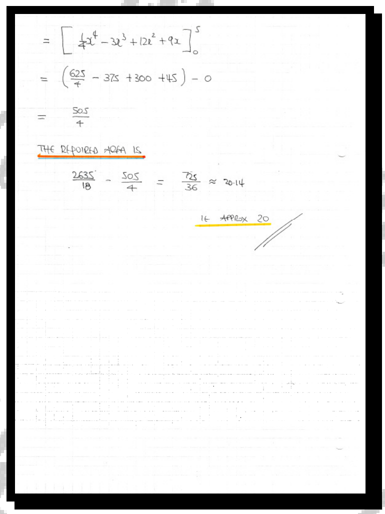
$9y - 261 = -x + 5$

$9y + x = 266$ // AS REQUIRED

b)

$A = \frac{\frac{266}{9} + 29}{2} \times 5$

$\int_0^5 x^3 - 9x^2 + 24x + 9 \, dx$



$= \left[\frac{1}{4}x^4 - 3x^3 + 12x^2 + 9x\right]_0^5$

$= \left(\frac{625}{4} - 375 + 300 + 45\right) - 0$

$= \frac{505}{4}$

THE REQUIRED AREA IS

$\frac{2635}{18} - \frac{505}{4} = \frac{725}{36} \approx 20.14$

IE APPROX 20 //

Question 104 (***+)

The curve C has equation

$$y = \left(1+\sqrt{x}\right)^2 \ , \ x \geq 0 .$$

a) Find an expression for $\dfrac{dy}{dx}$.

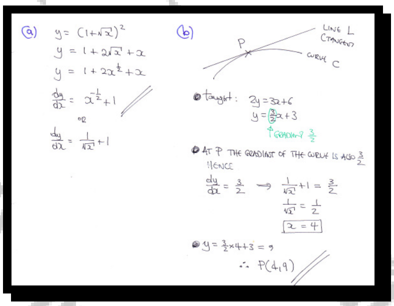
The straight line L with equation

$$2y = 3x + 6$$

is a tangent to C at the point P.

b) Use a calculus method to determine the coordinates of P.

$\boxed{}$, $\boxed{\dfrac{dy}{dx} = 1 + \dfrac{1}{\sqrt{x}}}$, $\boxed{P(4,9)}$



(a) $y = (1+\sqrt{x})^2$

$y = 1 + 2\sqrt{x} + x$

$y = 1 + 2x^{\frac{1}{2}} + x$

$\dfrac{dy}{dx} = x^{-\frac{1}{2}} + 1$

OR

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}} + 1$

(b)

- tangent: $2y = 3x + 6$
  $y = \frac{3}{2}x + 3$
  GRADIENT $\frac{3}{2}$
- AT P THE GRADIENT OF THE CURVE IS ALSO $\frac{3}{2}$
  HENCE
  $\dfrac{dy}{dx} = \dfrac{3}{2} \longrightarrow \dfrac{1}{\sqrt{x}} + 1 = \dfrac{3}{2}$
  $\dfrac{1}{\sqrt{x}} = \dfrac{1}{2}$
  $\boxed{x = 4}$
- $y = \frac{3}{2} \times 4 + 3 = 9$
  $\therefore P(4,9)$

Question 105 (***+)

$$y = \frac{x^2 + 4}{4x}, \quad x \neq 0.$$

Find the range of values of x for which y is increasing.

$\boxed{}$, $\boxed{x < -2 \cup x > 2}$

$y = \frac{x^2+4}{4x} = \frac{x^2}{4x} + \frac{4}{4x} = \frac{1}{4}x + \frac{1}{x} = \frac{1}{4}x + x^{-1}$

$\frac{dy}{dx} = \frac{1}{4} - x^{-2}$

INCREASING $\Rightarrow \frac{dy}{dx} > 0$

$\Rightarrow \frac{1}{4} - \frac{1}{x^2} > 0$

$\Rightarrow -\frac{1}{x^2} > -\frac{1}{4}$

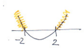
$\Rightarrow \frac{1}{x^2} < \frac{1}{4}$

$\Rightarrow 4 < x^2$

$\Rightarrow x^2 > 4$

$\Rightarrow x^2 - 4 > 0$

$\Rightarrow (x-2)(x+2) > 0$



$x < -2$ $\underline{\underline{\text{OR}}}$ $x > 2$

Question 106 (***+)

The curve C has equation

$$y = x^3 - 9x^2 + 24x - 19,\ x \in \mathbb{R}.$$

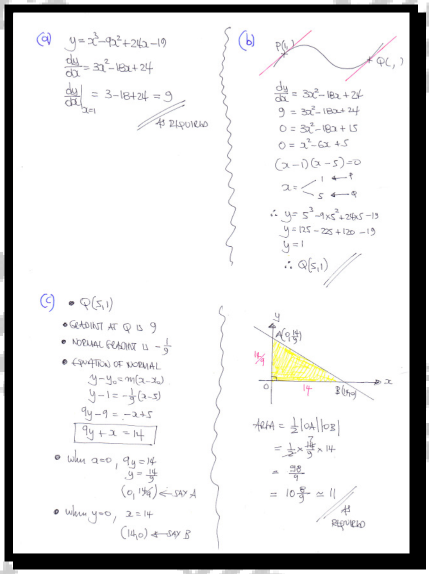
a) Show that the tangent to C at the point P, where $x = 1$, has gradient 9.

b) Find the coordinates of another point Q on C at which the tangent also has gradient 9.

The normal to C at Q meets the coordinate axes at the points A and B.

c) Show further that the **approximate** area of the triangle OAB, where O is the origin, is 11 square units.

$\boxed{Q(5,1)}$



(a) $y = x^3 - 9x^2 + 24x - 19$

$\frac{dy}{dx} = 3x^2 - 18x + 24$

$\left.\frac{dy}{dx}\right|_{x=1} = 3 - 18 + 24 = 9$

AS REQUIRED

(b)

$\frac{dy}{dx} = 3x^2 - 18x + 24$

$9 = 3x^2 - 18x + 24$

$0 = 3x^2 - 18x + 15$

$0 = x^2 - 6x + 5$

$(x-1)(x-5) = 0$

$x = \begin{cases} 1 \leftarrow P \\ 5 \leftarrow Q \end{cases}$

$\therefore y = 5^3 - 9\times5^2 + 24\times5 - 19$

$y = 125 - 225 + 120 - 19$

$y = 1$

$\therefore Q(5,1)$

(c) • $Q(5,1)$

• GRADIENT AT Q IS 9

• NORMAL GRADIENT IS $-\frac{1}{9}$

• EQUATION OF NORMAL

$y - y_0 = m(x - x_0)$

$y - 1 = -\frac{1}{9}(x-5)$

$9y - 9 = -x + 5$

$\boxed{9y + x = 14}$

• when $x = 0$, $9y = 14$

$y = \frac{14}{9}$

$(0, 14/9) \leftarrow$ SAY A

• when $y = 0$, $x = 14$

$(14, 0) \leftarrow$ SAY B

AREA $= \frac{1}{2}|OA||OB|$

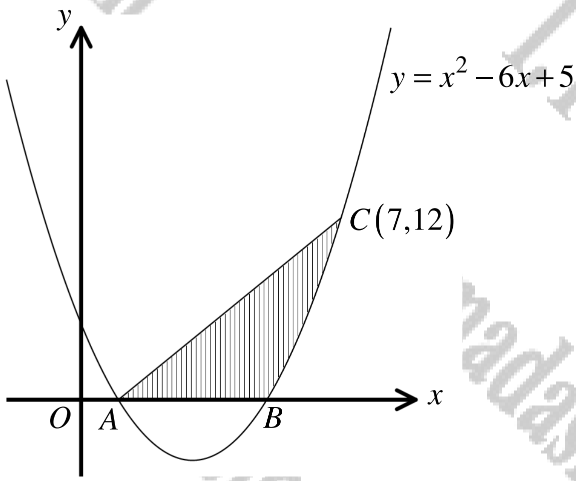
$= \frac{1}{2} \times \frac{14}{9} \times 14$

$= \frac{98}{9}$

$= 10\frac{8}{9} \simeq 11$

AS REQUIRED

Question 107 (***+)



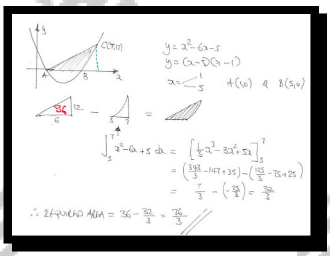
The diagram above shows the curve with equation

$$y = x^2 - 6x + 5.$$

The point $C(7,12)$ lies on the curve while A and B are the points of intersection of the curve and the x axis.

Find the exact area of the shaded region, bounded by the curve, the straight line segment AC and the x axis.

$\boxed{}$, $\boxed{\text{area} = \frac{76}{3}}$



Question 108 (*+)**

$$y = \frac{x^{\frac{1}{2}}(3x^2+1)}{x^2}, \ x > 0.$$

Show clearly that

$$\int_1^4 y \ dx = 15.$$

$\boxed{\text{proof}}$

$y = \frac{x^{\frac{1}{2}}(3x^2+1)}{x^2} = \frac{3x^{\frac{5}{2}}+x^{\frac{1}{2}}}{x^2} = 3x^{\frac{1}{2}}+x^{-\frac{3}{2}}$

$\therefore \int_1^4 y \, dx = \int_1^4 3x^{\frac{1}{2}}+x^{-\frac{3}{2}} \, dx = \left[2x^{\frac{3}{2}}-2x^{-\frac{1}{2}}\right]_1^4$

$= \left(2\times4^{\frac{3}{2}}-2\times4^{-\frac{1}{2}}\right)-\left(2\times1^{\frac{3}{2}}-2\times1^{-\frac{1}{2}}\right)$

$= (16-1)-(\cancel{2}-\cancel{2}) = 15$

AS REQUIRED

Question 109 (*+)**

The gradient of every point on the curve C, with equation $y = f(x)$ satisfies

$$f'(x) = 3x^2 - 4x + k,$$

where k is a non zero constant.

The points $P(0,-3)$ and $Q(2,7)$ both lie on C.

Find an equation for C.

$\boxed{y = x^3 - 2x^2 + 5x - 3}$

$f'(x) = 3x^2-4x+k$

THEN

$f(x) = \int 3x^2-4x+k \, dx$

$f(x) = x^3-2x^2+kx+C$

$(0,-3) \Rightarrow -3 = 0 + C$

$\Rightarrow C = -3$

$\therefore f(x) = x^3-2x^2+kx-3$

$(2,7) \Rightarrow 7 = \cancel{8-8}+2k-3$

$10 = 2k$

$k = 5$

$\therefore f(x) = x^3-2x^2+5x-3$

Question 110 (***+)

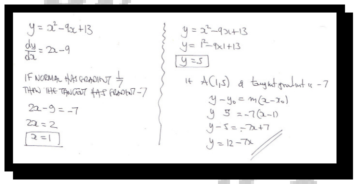
The point A lies on the curve with equation

$$y = x^2 - 9x + 13.$$

The gradient of the normal to the curve at the point A is $\frac{1}{7}$.

Find an equation of the tangent to the curve at A.

$$\boxed{y = 12 - 7x}$$



$y = x^2 - 9x + 13$

$\frac{dy}{dx} = 2x - 9$

IF NORMAL HAS GRADIENT $\frac{1}{7}$
THEN THE TANGENT HAS GRADIENT -7

$2x - 9 = -7$

$2x = 2$

$\boxed{x = 1}$

$y = x^2 - 9x + 13$

$y = 1^2 - 9 \times 1 + 13$

$\boxed{y = 5}$

If $A(1,5)$ & tangent gradient is -7

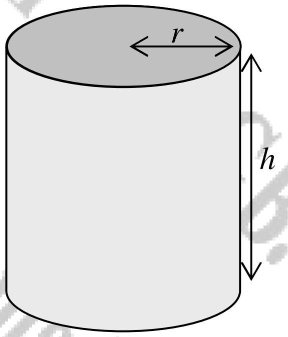
$y - y_0 = m(x - x_0)$

$y - 5 = -7(x - 1)$

$y - 5 = -7x + 7$

$y = 12 - 7x$

Question 111 (***+)



The figure above shows a **closed** cylindrical can, of radius r cm and height h cm.

a) If the volume of the can is 330 cm^3, show that surface area of the can, $A \text{ cm}^2$, is given by

$$A = 2\pi r^2 + \frac{660}{r}.$$

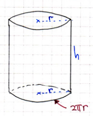
b) Find the value of r for which A is stationary.

c) Justify that the value of r found in part **(b)** gives the minimum value for A.

d) Hence calculate the minimum value of A.

$\boxed{}$, $\boxed{r \approx 3.745}$, $\boxed{A_{\text{min}} \approx 264}$

a)



CONSTRAINT ON THE VOLUME

$V = 330$

$\pi r^2 h = 330$

$(\pi r h) r = 330$

$\pi r h = \frac{330}{r}$

$2\pi r h = \frac{660}{r}$

$A = \pi r^2 \times 2 + (2\pi r \times h)$

$A = 2\pi r^2 + 2\pi r h$

$A = 2\pi r^2 + \frac{660}{r}$

// AS REQUIRED

b) DIFFERENTIATE & SOLVE FOR ZERO

$A = 2\pi r^2 + 660 r^{-1}$

$\frac{dA}{dr} = 4\pi r - 660 r^{-2}$

$\frac{dA}{dr} = 4\pi r - \frac{660}{r^2}$

FOR MIN/MAX $\frac{dA}{dr} = 0$

$0 = 4\pi r - \frac{660}{r^2}$

$\frac{660}{r^2} = 4\pi r$

$660 = 4\pi r^3$

$r^3 = \frac{165}{\pi}$

$r = 3.745 \text{ cm}$ //

c) USING THE SECOND DERIVATIVE

$\frac{dA}{dr} = 4\pi r - 660 r^{-2}$

$\frac{d^2A}{dr^2} = 4\pi + 1320 r^{-3}$

$\left.\frac{d^2A}{dr^2}\right|_{r=3.745\ldots} = 12\pi \approx 37.7 > 0$

INDEED $r = 3.745$ MINIMIZES A

d) FINALLY USING

$A = 2\pi r^2 + \frac{660}{r}$

$A_{\text{MIN}} = 2\pi (3.745\ldots)^2 + \frac{660}{3.745}$

$A_{\text{MIN}} \approx 264 \text{ cm}^2$ //

Question 112 (***+)

The point $A(2,1)$ lies on the curve with equation

$$y = \frac{(x-1)(x+2)}{2x}, \quad x \in \mathbb{R}, \ x \neq 0.$$

a) Find the gradient of the curve at A.

b) Show that the tangent to the curve at A has equation

$$3x - 4y - 2 = 0.$$

The tangent to the curve at the point B is parallel to the tangent to the curve at A.

c) Determine the coordinates of B.

$\boxed{\text{gradient at } A = \frac{3}{4}}$, $\boxed{B(-2,0)}$

(a) $y = \frac{(x-1)(x+2)}{2x} = \frac{x^2+x-2}{2x} = \frac{x^2}{2x} + \frac{x}{2x} - \frac{2}{2x} = \frac{1}{2}x + \frac{1}{2} - \frac{1}{x}$

$\therefore y = \frac{1}{2}x + \frac{1}{2} - x^{-1}$

$\frac{dy}{dx} = \frac{1}{2} + x^{-2} = \frac{1}{2} + \frac{1}{x^2}$

$\left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{2} + \frac{1}{2^2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ //

(b) $A(2,1)$ GRADIENT $= \frac{3}{4} \Rightarrow$

$y - y_0 = m(x - x_0)$

$y - 1 = \frac{3}{4}(x-2)$

$4y - 4 = 3x - 6$

$0 = 3x - 4y - 2$ // AS REQUIRED

(c) PARALLEL TANGENTS $\Rightarrow$ SAME GRADIENT OF $\frac{3}{4}$

$\frac{dy}{dx} = \frac{1}{2} + \frac{1}{x^2}$

$\Rightarrow \frac{3}{4} = \frac{1}{2} + \frac{1}{x^2}$

$\Rightarrow \frac{1}{4} = \frac{1}{x^2}$

$\Rightarrow 4 = x^2$

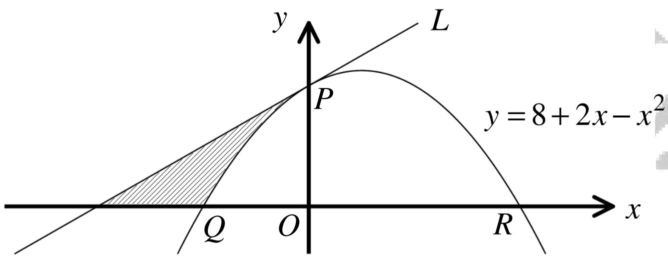
$\Rightarrow x = 2 \leftarrow A$, $\quad x = -2 \leftarrow B$

USING $y = \frac{(x-1)(x+2)}{2x}$ WITH $x = -2$

$y = \frac{(-2-1)(-2+2)}{2(-2)} = 0$

$\therefore B(-2,0)$ //

Question 113 (***+)



The figure above shows the graph of a curve with equation

$$y = 8 + 2x - x^2 .$$

The curve meets the y axis at the point P, and the x axis at the points Q and R.

a) Determine the coordinates of P, Q and R.

The straight line L is the tangent to the curve at P.

b) Find an equation of L.

c) Show that the area of the finite region bounded by the curve, the tangent L and the x axis is $\frac{20}{3}$.

$\boxed{P(0,8)}$, $\boxed{Q(-2,0)}$, $\boxed{R(4,0)}$, $\boxed{y = 2x + 8}$

a) $y = 8 + 2x - x^2$

• WHEN $x = 0$
$y = 8$
$\therefore P(0,8)$

• WHEN $y = 0$
$0 = 8 + 2x - x^2$
$x^2 - 2x - 8 = 0$
$(x+2)(x-4) = 0$
$x = -2 \leftarrow Q(-2,0)$
$x = 4 \leftarrow R(4,0)$

b) DIFFERENTIATING

$\frac{dy}{dx} = 2 - 2x$

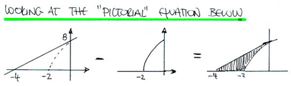
$\left.\frac{dy}{dx}\right|_{x=0} = 2 - 2 \times 0 = 2 \leftarrow$ tangent gradient

$\therefore$ EQUATION OF TANGENT : $y = 2x + 8$ P(0,8)

c) THE x INTERCEPT OF THE TANGENT IS GIVEN BY $y = 0$

$\Rightarrow 0 = 2x + 8$
$\Rightarrow -8 = 2x$
$\Rightarrow x = -4$

LOOKING AT THE "PICTORIAL" EQUATION BELOW



AREA OF TRIANGLE

$\frac{1}{2} \times 4 \times 8 = 16$

AREA "UNDER THE CURVE" BETWEEN -2 & 0

$$\int_{-2}^{0} 8 + 2x - x^2 \, dx = \left[8x + x^2 - \tfrac{1}{3}x^3\right]_{-2}^{0}$$
$$= (0 + 0 - 0) - (-16 + 4 + \tfrac{8}{3})$$
$$= 16 - 4 - \tfrac{8}{3}$$
$$= \tfrac{28}{3}$$

THE REQUIRED AREA $= 16 - \frac{28}{3} = \frac{20}{3}$

AS REQUIRED

Question 114 (***+)

On a given lorry journey the cost C, in pence per mile, is modelled by

$$C = \frac{192}{V} + \frac{V^2}{144}, \quad V > 6,$$

where V is the lorry's average speed in metres per second.

a) Find the speed, in metres per second, for which the cost, in pence per mile, is stationary.

b) Justify that this value of the speed minimizes the cost.

c) Hence determine the minimum cost of a 600 mile journey.

$\boxed{}$, $\boxed{V = 24}$, $\boxed{£72}$

a) $C = \frac{192}{V} + \frac{V^2}{144} = 192V^{-1} + \frac{1}{144}V^2$

$\frac{dC}{dV} = -192V^{-2} + \frac{1}{72}V = \frac{V}{72} - \frac{192}{V^2}$

SOLVE FOR ZERO

$\Rightarrow \frac{V}{72} - \frac{192}{V^2} = 0$

$\Rightarrow \frac{V}{72} = \frac{192}{V^2}$

$\Rightarrow V^3 = 13824$

$\Rightarrow V = 24$

b) $\frac{d^2C}{dV^2} = 384V^{-3} + \frac{1}{72} = \frac{384}{V^3} + \frac{1}{72}$

$\left.\frac{d^2C}{dV^2}\right|_{V=24} = \frac{384}{24^3} + \frac{1}{72} = \frac{1}{24} > 0$

INDEED A MINIMUM

c) WHEN $V = 24$

$C = \frac{192}{24} + \frac{24^2}{144} = 12 \leftarrow$ PENCE PER MILE

$\therefore$ TOTAL COST IS $600 \times 12 = 7200$ PENCE $= £72$

Question 115 (***+)

A certain chemical industrial process is carried out at low temperatures.

The wastage cost £C during this chemical process and the average temperature T °C are related by the equation

$$C = \frac{36}{T} + \frac{2T^2}{3}, \ T > 0.$$

Find the average temperature during which the wastage cost is increasing.

$\boxed{}, \ \boxed{T > 3}$

$C = 36T^{-1} + \frac{2}{3}T^2$

$\Rightarrow \frac{dC}{dT} = -36T^{-2} + \frac{4}{3}T$

INCREASING $\Rightarrow \frac{dC}{dT} > 0$

$\Rightarrow -\frac{36}{T^2} + \frac{4}{3}T > 0$

$\Rightarrow \frac{4}{3}T > \frac{36}{T^2}$

$\Rightarrow T > \frac{27}{T^2}$ $(T^2 > 0)$

$\Rightarrow T^3 > 27$

$\Rightarrow T > 3$ //

Question 116 (***+)

The curve C has equation $y = f(x)$ given by

$$f(x) = 2(x-2)^3, \quad x \in \mathbb{R}.$$

a) Sketch the graph of $f(x)$.

b) Find an expression for $f'(x)$.

The point $P(3,2)$ lies on C and the straight line l_1 is the tangent to C at P.

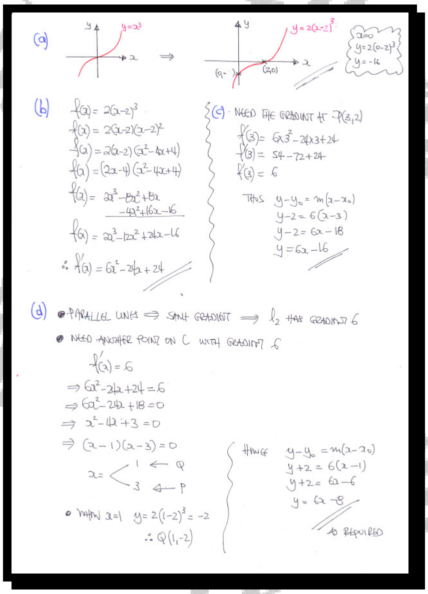
c) Find an equation of l_1.

The straight line l_2 is another tangent at a different point Q on C.

d) Given that l_1 is parallel to l_2 show that an equation of l_2 is

$$y = 6x - 8.$$

$\boxed{}$, $\boxed{f'(x) = 6x^2 - 24x + 24}$, $\boxed{y = 6x - 16}$

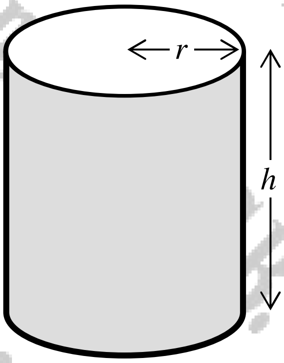

(a) $y=x^3$ $\Rightarrow$ $y=2(x-2)^3$, $(2,0)$, $(0,-\ )$; $x=0$, $y=2(0-2)^3$, $y=-16$

(b) $f(x) = 2(x-2)^3$
$f(x) = 2(x-2)(x-2)^2$
$f(x) = 2(x-2)(x^2-4x+4)$
$f(x) = (2x-4)(x^2-4x+4)$
$f(x) = 2x^3 - 8x^2 + 8x - 4x^2 + 16x - 16$
$f(x) = 2x^3 - 12x^2 + 24x - 16$
$\therefore f'(x) = 6x^2 - 24x + 24$

(c) NEED THE GRADIENT AT $P(3,2)$
$f'(3) = 6 \times 3^2 - 24 \times 3 + 24$
$f'(3) = 54 - 72 + 24$
$f'(3) = 6$
THUS $y - y_0 = m(x - x_0)$
$y - 2 = 6(x-3)$
$y - 2 = 6x - 18$
$y = 6x - 16$

(d) • PARALLEL LINES $\Rightarrow$ SAME GRADIENT $\Rightarrow$ l_2 HAS GRADIENT 6
• NEED ANOTHER POINT ON C WITH GRADIENT 6
$f'(x) = 6$
$\Rightarrow 6x^2 - 24x + 24 = 6$
$\Rightarrow 6x^2 - 24x + 18 = 0$
$\Rightarrow x^2 - 4x + 3 = 0$
$\Rightarrow (x-1)(x-3) = 0$
$x = 1 \leftarrow Q$, $x = 3 \leftarrow P$
• WHEN $x=1$ $y = 2(1-2)^3 = -2$
$\therefore Q(1,-2)$
HENCE $y - y_0 = m(x - x_0)$
$y + 2 = 6(x-1)$
$y + 2 = 6x - 6$
$y = 6x - 8$
AS REQUIRED

Question 117 (***+)



A hollow container, made of thin sheet metal, is in the shape of a right circular cylinder, which is **open** at one of its circular ends.

The container has radius r cm, height h cm and a capacity of 1500 cm^3.

a) Show that the surface area, A cm^2, of the container is given by

$$A = \pi r^2 + \frac{3000}{r}.$$

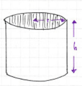
b) Determine the value of r for which A has a stationary value.

c) Show that the value of r found in part **(b)** gives the minimum value for A.

d) Calculate, to the nearest cm^2, the minimum surface area of the container.

$\boxed{}$, $\boxed{r \approx 7.816}$, $\boxed{A_{\text{min}} \approx 576}$

a)



$\boxed{\text{CAPACITY} = 1500}$

$\Rightarrow \text{VOLUME} = 1500$

$\Rightarrow \pi r^2 h = 1500$

$\Rightarrow \pi r h = \frac{1500}{r}$

$\boxed{A = \pi r^2 + 2\pi r h}$

BASE CURVED SURFACE

$\Rightarrow A = \pi r^2 + 2\pi r h$

$\Rightarrow A = \pi r^2 + 2(\pi r h)$

$\Rightarrow A = \pi r^2 + 2\left(\frac{1500}{r}\right)$

$\Rightarrow A = \pi r^2 + \frac{3000}{r}$ // AS REQUIRED

b) DIFFERENTIATE THE "AREA" EXPRESSION WITH RESPECT TO r

$\Rightarrow A = \pi r^2 + 3000 r^{-1}$

$\Rightarrow \frac{dA}{dr} = 2\pi r - 3000 r^{-2}$

FOR STATIONARY VALUES $\frac{dA}{dr} = 0$

$\Rightarrow 2\pi r - \frac{3000}{r^2} = 0$

$\Rightarrow 2\pi r = \frac{3000}{r^2}$

$\Rightarrow 2\pi r^3 = 3000$

$\Rightarrow \pi r^3 = 1500$

$\Rightarrow r = \sqrt[3]{\frac{1500}{\pi}}$

$\Rightarrow r \approx 7.82$ cm //

c) USING THE SECOND DERIVATIVE TEST

$\frac{dA}{dr} = 2\pi r - 3000 r^{-2}$

$\frac{d^2A}{dr^2} = 2\pi + 6000 r^{-3}$

$\left.\frac{d^2A}{dr^2}\right|_{r=7.82} = 2\pi + 6000(7.82\ldots)^{-3} = 6\pi > 0$

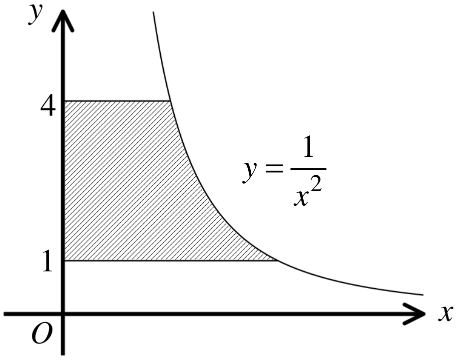
INDEED THE MIN VALUE OF A //

d) FINALLY USING $A = \pi r^2 + \frac{3000}{r}$ WITH $r = 7.82$

$A_{\text{MIN}} = \pi(7.82\ldots)^2 + \frac{3000}{(7.82\ldots)}$

$A_{\text{MIN}} \approx 576 \text{ cm}^2$ //

Question 118 (***+)

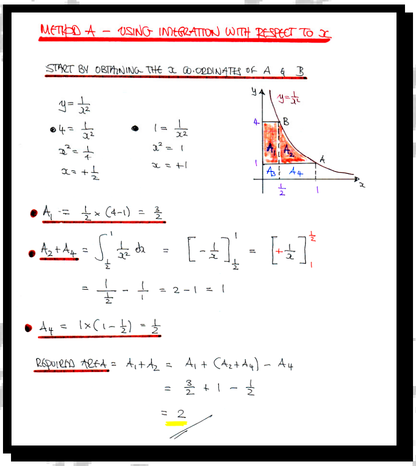


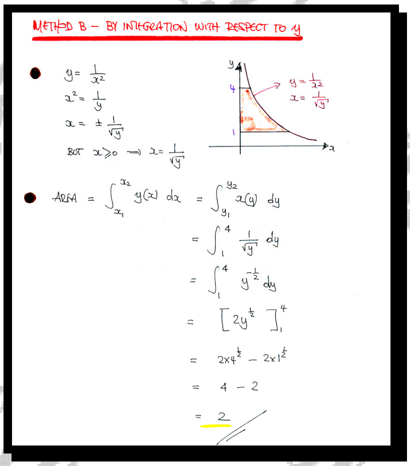
The figure above shows part of the graph of the curve with equation

$$y = \frac{1}{x^2}, \ x \in \mathbb{R}, \ x > 0 .$$

Find the area bounded by the curve, the y axis, and the straight lines with equations $y = 1$ and $y = 4$.

$\boxed{}$, $\boxed{2}$





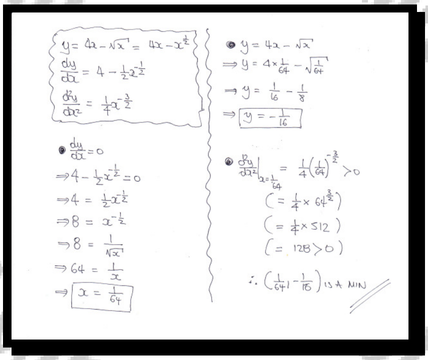
Question 119 (***+) non calculator

The curve C has equation

$$y = 4x - \sqrt{x}, \quad x \geq 0.$$

Find the coordinates of the stationary point of C, and determine its nature.

$$\boxed{\min\left(\frac{1}{64}, -\frac{1}{16}\right)}$$



$y = 4x - \sqrt{x} = 4x - x^{\frac{1}{2}}$
$\frac{dy}{dx} = 4 - \frac{1}{2}x^{-\frac{1}{2}}$
$\frac{d^2y}{dx^2} = \frac{1}{4}x^{-\frac{3}{2}}$

- $\frac{dy}{dx} = 0$
$\Rightarrow 4 - \frac{1}{2}x^{-\frac{1}{2}} = 0$
$\Rightarrow 4 = \frac{1}{2}x^{-\frac{1}{2}}$
$\Rightarrow 8 = x^{-\frac{1}{2}}$
$\Rightarrow 8 = \frac{1}{\sqrt{x}}$
$\Rightarrow 64 = \frac{1}{x}$
$\Rightarrow \boxed{x = \frac{1}{64}}$

- $y = 4x - \sqrt{x}$
$\Rightarrow y = 4 \times \frac{1}{64} - \sqrt{\frac{1}{64}}$
$\Rightarrow y = \frac{1}{16} - \frac{1}{8}$
$\Rightarrow \boxed{y = -\frac{1}{16}}$

- $\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{64}} = \frac{1}{4}\left(\frac{1}{64}\right)^{-\frac{3}{2}} > 0$
$\left(= \frac{1}{4} \times 64^{\frac{3}{2}}\right)$
$\left(= \frac{1}{4} \times 512\right)$
$(= 128 > 0)$

$\therefore \left(\frac{1}{64}, -\frac{1}{16}\right)$ IS A MIN

Question 120 (***+)

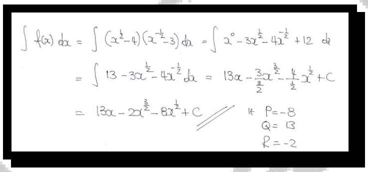
$$f(x) = \left(x^{\frac{1}{2}} - 4\right)\left(x^{-\frac{1}{2}} - 3\right), \quad x > 0.$$

Show clearly that

$$\int f(x)\, dx = P\sqrt{x} + Qx + Rx^{\frac{3}{2}} + C,$$

where P, Q and R are integers to be found, and C is an arbitrary constant.

$$\boxed{P = -8}, \ \boxed{Q = 13}, \ \boxed{R = -2}$$



$\int f(x)\,dx = \int \left(x^{\frac{1}{2}} - 4\right)\left(x^{-\frac{1}{2}} - 3\right) dx = \int x^0 - 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} + 12 \, dx$
$= \int 13 - 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} \, dx = 13x - \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{4}{\frac{1}{2}}x^{\frac{1}{2}} + C$
$= 13x - 2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + C$

$\therefore P = -8$, $Q = 13$, $R = -2$

Question 121 (***+)

$$f(x) \equiv x^3 - 2x^2 - x - 6, \ x \in \mathbb{R}.$$

a) Use the factor theorem to show that $(x-3)$ is a factor of $f(x)$.

b) Hence express $f(x)$ as the product of a linear and a quadratic factor.

The curve C has equation

$$y = 3x^4 - 8x^3 - 6x^2 - 72x + 240.$$

c) Show that C has a single stationary point, and determine its coordinates and its nature.

$\boxed{}$, $\boxed{f(x) \equiv (x-3)(x^2+x+2)}$, $\boxed{\min(3,-3)}$

a) $f(x) = x^3 - 2x^2 - x - 6$

$f(3) = 3^3 - 2\times 3^2 - 3 - 6 = 27 - 18 - 9 = 0 \quad \therefore (x-3)$ IS A FACTOR OF $f(x)$

b) $x - 3 \,\overline{)\, x^3 - 2x^2 - x - 6}$, quotient $x^2 + x + 2$

$-x^3 + 3x^2$

$x^2 - x - 6$

$-x^2 + 3x$

$2x - 6$

$-2x + 6$

$\therefore f(x) = (x-3)(x^2+x+2)$

c) $y = 3x^4 - 8x^3 - 6x^2 - 72x + 240$

$\frac{dy}{dx} = 12x^3 - 24x^2 - 12x - 72$

SOLVE FOR ZERO

$12x^3 - 24x^2 - 12x - 72 = 0$

$x^3 - 2x^2 - x - 6 = 0$

$(x-3)(x^2+x+2) = 0$

$x = 3$ OR $x^2 + x + 2 = 0$

$b^2 - 4ac = 1^2 - 4\times1\times2$

$= -7$

NO MORE SOLUTIONS

$y = 3\times3^4 - 8\times3^3 - 6\times3^2 - 72\times3 + 240$

$y = 243 - 216 - 54 - 216 + 240$

$\boxed{y = -3}$

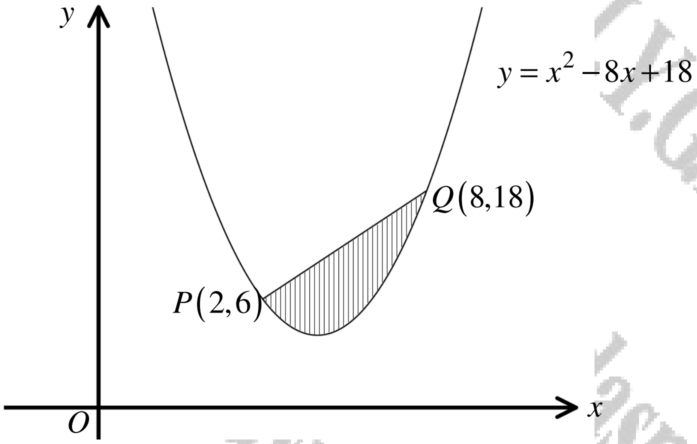
$\frac{d^2y}{dx^2} = 36x^2 - 48x - 12$

$\left.\frac{d^2y}{dx^2}\right|_{x=3} = 36\times3^2 - 48\times3 - 12$

$= 324 - 144 - 12 > 0$

$\therefore (3,-3)$ IS A MIN

Question 122 (***+)



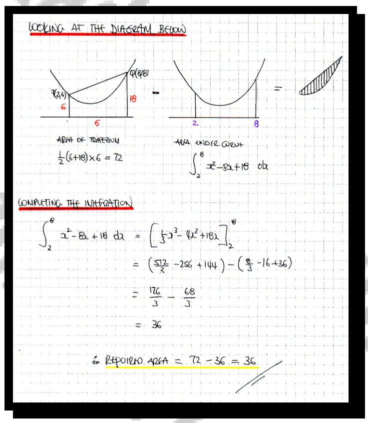
The figure above shows the parabola with equation

$$y = x^2 - 8x + 18, \ x \in \mathbb{R}.$$

The points $P(3,3)$ and $Q(6,6)$ both lie on the parabola.

Find the exact of the shaded region, bounded by the curve and the straight line segment between P and Q.

$\boxed{}$, $\boxed{\text{area} = 36}$



LOOKING AT THE DIAGRAM BELOW

AREA OF TRAPEZIUM

$\frac{1}{2}(6+18) \times 6 = 72$

AREA UNDER CURVE

$\int_2^8 x^2 - 8x + 18 \, dx$

COMPLETING THE INTEGRATION

$$\int_2^8 x^2 - 8x + 18 \, dx = \left[\frac{1}{3}x^3 - 4x^2 + 18x\right]_2^8$$
$$= \left(\frac{512}{3} - 256 + 144\right) - \left(\frac{8}{3} - 16 + 36\right)$$
$$= \frac{176}{3} - \frac{68}{3}$$
$$= 36$$

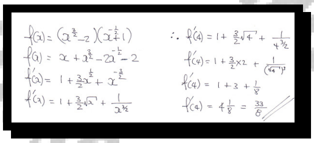
$\therefore$ REQUIRED AREA $= 72 - 36 = 36$

Question 123 (*+)**

$$f(x) = \left(x^{\frac{3}{2}} - 2\right)\left(x^{-\frac{1}{2}} + 1\right), \quad x > 0$$

Find the exact value of $f'(4)$.

$$\boxed{f'(4) = \frac{33}{8}}$$



Question 124 (*+)**

The curve C has equation

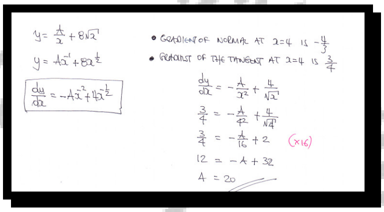
$$y = \frac{A}{x} + 8\sqrt{x}, \quad x > 0,$$

where A is a non zero constant.

The normal to the curve at C, at the point where $x = 4$, has gradient $-\frac{4}{3}$.

Find the value of A.

$$\boxed{A = 20}$$



Question 125 **(***+)**

Find the exact value of

$$\int_1^2 \left(3+2\sqrt{x}\right)^2 \ dx,$$

giving the answer in the form $a+b\sqrt{2}$, where a and b are integers.

$\boxed{7+16\sqrt{2}}$

$$\int_1^2 (3+2\sqrt{x})^2 dx = \int_1^2 9+12\sqrt{x}+4x \, dx = \int_1^2 9+12x^{\frac{1}{2}}+4x \ dx$$
$$= \left[9x+8x^{\frac{3}{2}}+2x^2\right]_1^2$$
$$= \left(9\times2+8\times2^{\frac{3}{2}}+2\times2^2\right)-\left(9\times1+8\times1^{\frac{3}{2}}+2\times1^2\right)$$
$$= \left(18+8\times2\sqrt{2}+8\right)-\left(9+8+2\right)$$
$$= 26+16\sqrt{2}-19$$
$$= 7+16\sqrt{2}$$

Question 126 **(***+)**

A cubic curve C passes through the points $P(-1,-9)$ and $Q(2,6)$ and its gradient function is given by

$$\frac{dy}{dx} = 3x^2+kx+7,$$

where k is a non zero constant.

Find an equation for C.

$\boxed{}$, $\boxed{y = x^3-5x^2+7x+4}$

IF $\frac{dy}{dx} = 3x^2+kx+7$

THEN

$y = \int 3x^2+kx+7 \, dx$

$\boxed{y = x^3+\frac{1}{2}kx^2+7x+C}$

$P(-1,-9) \Rightarrow -9 = -1+\frac{1}{2}k-7+C$
$Q(2,6) \Rightarrow 6 = 8+2k+14+C$
$\Rightarrow$ $-1 = \frac{1}{2}k+C$, $-16 = 2k+C$ $\Rightarrow$

$C = -1-\frac{1}{2}k$, $C = -16-2k$ $\Rightarrow$ $-1-\frac{1}{2}k = -16-2k$

$1+\frac{1}{2}k = 16+2k$

$2+k = 32+4k$

$-30 = 3k$

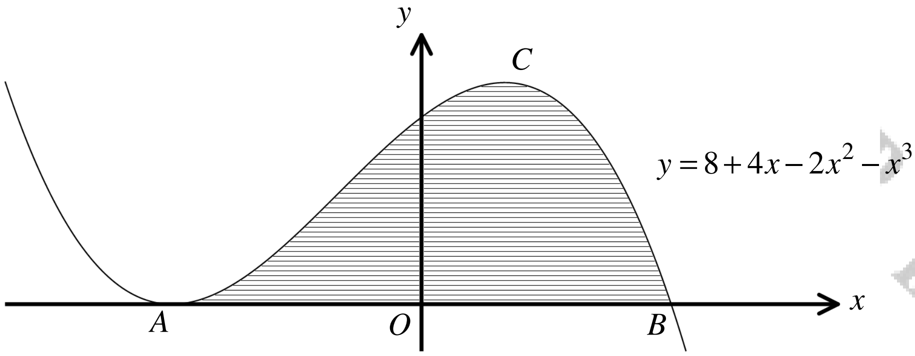
$\boxed{k=-10}$

$C = -16-2k = -16-2(-10) = -16+20 = 4$

$\boxed{C=4}$

$\therefore y = x^3-5x^2+7x+4$

Question 127 (***+)



The figure above shows part of the curve with equation

$$y = 8 + 4x - 2x^2 - x^3 .$$

The curve meets the x axis at A and B.

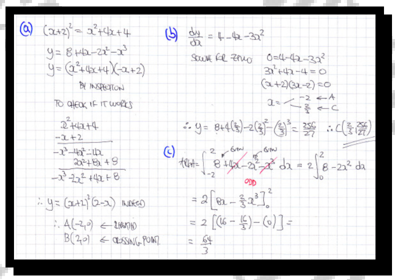
a) Verify that the coordinates of A are $(-2,0)$ and hence use algebra to show that the coordinates of B are $(2,0)$.

The point C is a stationary point of the curve.

b) Use calculus to determine the exact coordinates of C.

c) Find the exact area of the finite region bounded by the curve and the x axis.

$\boxed{}$, $\boxed{C\left(\frac{2}{3}, \frac{256}{27}\right)}$, $\boxed{\text{area} = \frac{64}{3}}$



Question 128 (***+)

The curve C has equation

$$y=(x-2)\left(x^2-2x-3\right),\ x\in\mathbb{R}.$$

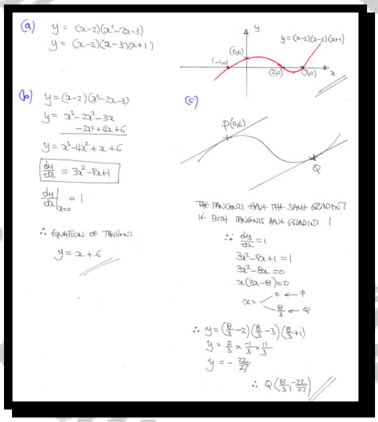
a) Sketch the graph of C, indicating the coordinates of any points where the curve meets the coordinate axes.

b) Find an equation of the tangent to C at the point P, where P is the point where C crosses the y axis.

The point Q lies on C so that the tangent to the curve at Q is parallel to the tangent to the curve at P.

c) Determine the exact coordinates of Q.

$\boxed{(-1,0),(2,0),(3,0),(0,6)}$, $\boxed{y=x+6}$, $\boxed{Q\left(\frac{8}{3},-\frac{22}{27}\right)}$



Question 129 **(***+)** **non calculator**

$$f(x) = 4\sqrt{x} - \sqrt[3]{x}, \ x \geq 0 .$$

Show clearly that

$$f'(64) = \frac{11}{48} .$$

proof

$$f(x) = 4\sqrt{x} - \sqrt[3]{x}$$
$$f(x) = 4x^{\frac{1}{2}} - x^{\frac{1}{3}}$$
$$f'(x) = 2x^{-\frac{1}{2}} - \frac{1}{3}x^{-\frac{2}{3}}$$
$$f'(x) = \frac{2}{\sqrt{x}} - \frac{1}{3x^{2/3}}$$
$$f'(x) = \frac{2}{\sqrt{x}} - \frac{1}{3(\sqrt[3]{x})^2}$$

Hence
$$f'(64) = \frac{2}{\sqrt{64}} - \frac{1}{3(\sqrt[3]{64})^2} = \frac{2}{8} - \frac{1}{3 \times 4^2} = \frac{1}{4} - \frac{1}{48} = \frac{12}{48} - \frac{1}{48} = \frac{11}{48}$$

Question 130 **(***+)**

The curve C has equation

$$y = 2\sqrt{x} - \sqrt{x^3}, \ x \geq 0 .$$

Show that the equation of the tangent to C at the point where $x = 2$, can be written as

$$y = \sqrt{2}(2 - x)$$

proof

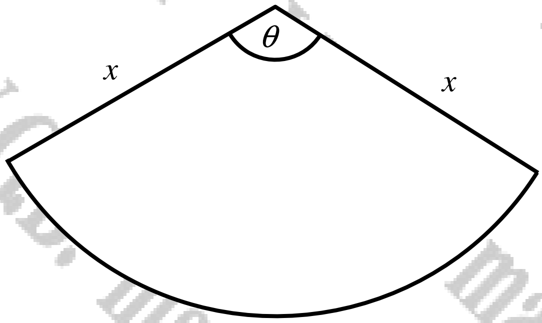
$$y = 2\sqrt{x} - \sqrt{x^3}$$
$$y = 2x^{\frac{1}{2}} - x^{\frac{3}{2}}$$
$$\frac{dy}{dx} = x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{x}} - \frac{3}{2}\sqrt{x}$$
$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{\sqrt{2}} - \frac{3}{2}\sqrt{2} = \frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} = \frac{-2\sqrt{2}}{2} = -\sqrt{2}$$

when $x = 2$
$$y = 2\sqrt{2} - \sqrt{2^3}$$
$$y = 2\sqrt{2} - \sqrt{8}$$
$$y = 2\sqrt{2} - 2\sqrt{2}$$
$$y = 0$$

$\therefore (2,0) \quad m = -\sqrt{2}$
$$y - y_0 = m(x - x_0)$$
$$y - 0 = -\sqrt{2}(x - 2)$$
$$y = -\sqrt{2}(x - 2)$$
$$y = \sqrt{2}(2 - x)$$

AS REQUIRED

Question 131 (***+)



A circular sector of radius x cm subtends an angle of θ radians at the centre.

The area of the sector is 36 cm^2 and its perimeter is P cm.

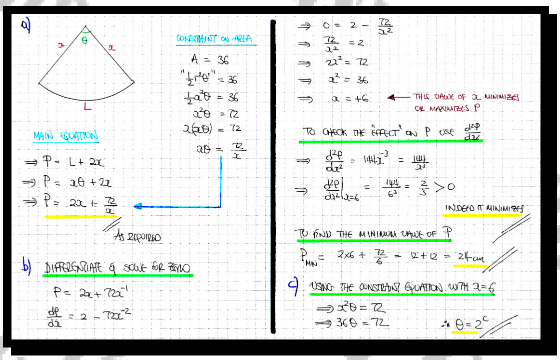
a) Show clearly that

$$P = 2x + \frac{72}{x}.$$

b) Find the minimum value of P, fully justifying the fact that it is a minimum.

c) Deduce the value of θ when P is minimum.

$\boxed{}$, $\boxed{P_{\text{min}} = 24}$, $\boxed{\theta = 2^{c}}$



Question 132 **(***+)**

Find the exact value of

$$\int_1^3 3\sqrt{x} - \frac{4}{\sqrt{x}}\, dx\,,$$

giving the answer in the form $p + q\sqrt{3}$, where p and q are integers.

$$\boxed{6 - 2\sqrt{3}}$$

$$\int_1^3 3\sqrt{x} - \frac{4}{\sqrt{x}}\, dx = \int_1^3 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}\, dx = \left[2x^{\frac{3}{2}} - 8x^{\frac{1}{2}}\right]_1^3$$
$$= \left(2 \times 3^{\frac{3}{2}} - 8 \times 3^{\frac{1}{2}}\right) - \left(2 \times 1^{\frac{3}{2}} - 8 \times 1^{\frac{1}{2}}\right)$$
$$= \left(2 \times 3\sqrt{3} - 8\sqrt{3}\right) - (2 - 8)$$
$$= 6\sqrt{3} - 8\sqrt{3} - 2 + 8$$
$$= 6 - 2\sqrt{3}$$

Question 133 (***+)

A curve C has equation

$$y = 4x^3 + 7x^2 + x + 11, \quad x \in \mathbb{R}.$$

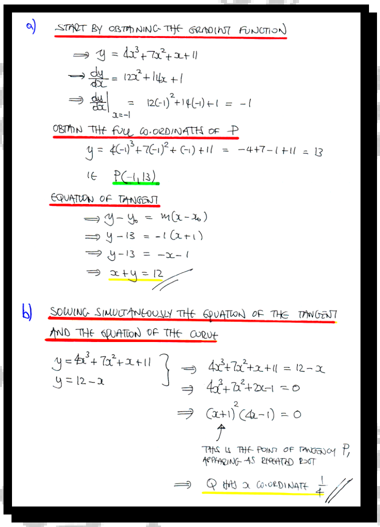
The point P lies on C, where $x = -1$.

a) Find an equation of the tangent to C at P.

The tangent to C at P meets C again at the point Q.

b) Determine the x coordinate of Q.

$\boxed{}$, $\boxed{y = 12 - x}$, $\boxed{x_Q = \frac{1}{4}}$



a) START BY OBTAINING THE GRADIENT FUNCTION

$$\Rightarrow y = 4x^3 + 7x^2 + x + 11$$
$$\Rightarrow \frac{dy}{dx} = 12x^2 + 14x + 1$$
$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=-1} = 12(-1)^2 + 14(-1) + 1 = -1$$

OBTAIN THE FULL CO-ORDINATES OF P

$$y = 4(-1)^3 + 7(-1)^2 + (-1) + 11 = -4 + 7 - 1 + 11 = 13$$

IE $P(-1, 13)$

EQUATION OF TANGENT

$$\Rightarrow y - y_0 = m(x - x_0)$$
$$\Rightarrow y - 13 = -1(x + 1)$$
$$\Rightarrow y - 13 = -x - 1$$
$$\Rightarrow x + y = 12$$

b) SOLVING SIMULTANEOUSLY THE EQUATION OF THE TANGENT AND THE EQUATION OF THE CURVE

$$\left.\begin{array}{l} y = 4x^3 + 7x^2 + x + 11 \\ y = 12 - x \end{array}\right\} \Rightarrow 4x^3 + 7x^2 + x + 11 = 12 - x$$
$$\Rightarrow 4x^3 + 7x^2 + 2x - 1 = 0$$
$$\Rightarrow (x+1)^2(4x - 1) = 0$$

THIS IS THE POINT OF TANGENCY P, APPEARING AS REPEATED ROOT

$\Rightarrow$ Q HAS x CO-ORDINATE $\frac{1}{4}$

Question 134 (***+)

A triangle PQR has $|PQ| = x+2$ cm, $|QR| = (2-x)^2$ cm and $\measuredangle PQR = 30°$.

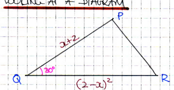
a) Show that the area of the triangle, A cm^2, is given by

$$A = \frac{1}{4}\left(x^3 - 2x^2 - 4x + 8\right).$$

b) Determine the value of x for which A is stationary and hence find, with justification, the greatest value of A.

$\boxed{}$, $\boxed{x = -\frac{2}{3}}$, $\boxed{A_{\text{max}} = \frac{64}{27} \approx 2.37}$

a) LOOKING AT A DIAGRAM



$\Rightarrow$ AREA $= \frac{1}{2}|PQ||QR|\sin 30°$

$\Rightarrow A = \frac{1}{2}(x+2)(2-x)^2 \times \frac{1}{2}$

$\Rightarrow A = \frac{1}{4}(x+2)(x^2-4x+4)$

$\Rightarrow A = \frac{1}{4}(x^3 - 4x^2 + 4x + 2x^2 - 8x + 8)$

$\Rightarrow A = \frac{1}{4}(x^3 - 2x^2 - 4x + 8)$ // AS REQUIRED

b) DIFFERENTIATING & SOLVING FOR ZERO

$\Rightarrow \frac{dA}{dx} = \frac{1}{4}(3x^2 - 4x - 4)$

$\Rightarrow 0 = \frac{1}{4}(3x^2 - 4x - 4)$

$\Rightarrow 0 = \frac{1}{4}(3x+2)(x-2)$

$\Rightarrow x = \cancel{2}$ or $-\frac{2}{3}$ //

CHECKING THE NATURE

$\frac{d^2A}{dx^2} = \frac{1}{4}(6x - 4)$

$\left.\frac{d^2A}{dx^2}\right|_{x=-\frac{2}{3}} = \frac{1}{4}\left(6 \times \frac{-2}{3} - 4\right) = -2 < 0$ IE A MAX //

FINALLY WE HAVE

$\Rightarrow A = \frac{1}{4}(x^3 - 2x^2 - 4x + 8)$

$\Rightarrow A_{\text{MAX}} = \frac{1}{4}\left(-\frac{8}{27} - \frac{8}{9} + \frac{8}{3} + 8\right)$

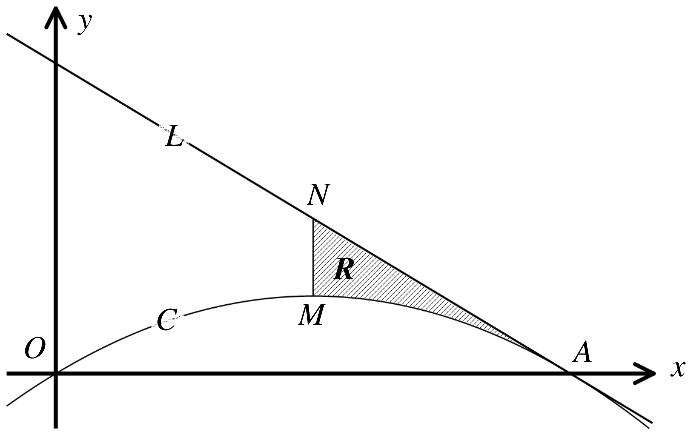
$\Rightarrow A_{\text{MAX}} = 2\left(1 + \frac{1}{3} - \frac{1}{9} - \frac{1}{27}\right)$

$\Rightarrow A_{\text{MAX}} = 2\left[\frac{27 + 9 - 3 - 1}{27}\right]$

$\Rightarrow A_{\text{MAX}} = 2 \times \frac{32}{27}$

$\Rightarrow A_{\text{MAX}} = \frac{64}{27}$ //

Question 135 (***+)



The figure above shows the graph of the curve C with equation

$$y = 6x - x^2, \ x \in \mathbb{R}.$$

The curve meets the x axis at the origin O and at the point A. The straight line L is the tangent to C at A.

a) Find an equation of L.

The point M is the maximum point of C. The point N lies on L so that MN is parallel to the y axis. The finite region R, shown shaded in the figure above, is bounded by C, L and the straight line segment MN.

b) Determine the area of R.

$\boxed{y = 36 - 6x}$, $\boxed{9}$

a) START BY FINDING THE CO-ORDINATES OF A

$y = 6x - x^2$
$y = x(6-x)$
$\therefore A(6,0)$

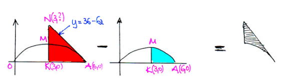
FIND THE GRADIENT AT A

$y = 6x - x^2$
$\frac{dy}{dx} = 6 - 2x$
$\left.\frac{dy}{dx}\right|_{x=6} = 6 - 2\times6 = -6$ ← TANGENT GRADIENT

EQUATION OF TANGENT

$y - y_0 = m(x - x_0)$
$y - 0 = -6(x-6)$
$y = -6x + 36$
$y = 36 - 6x$

b)



$y = 36 - 6\times3$
$y = 18$
$\therefore N(3,18)$

AREA $= \frac{1}{2}|KN||KA|$
$= \frac{1}{2}\times18\times3 = 27$

$\int_3^6 6x - x^2 \, dx$
$= \left[3x^2 - \frac{1}{3}x^3\right]_3^6$
$= (108-72) - (27-9)$
$= 18$

$= 9$

Question 136 (***+)

The curve C has equation

$$y = \frac{(x+4)^2}{\sqrt{x}}, \quad x > 0.$$

a) Show that the gradient function of C is

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 4x^{-\frac{1}{2}} - 8x^{-\frac{3}{2}}.$$

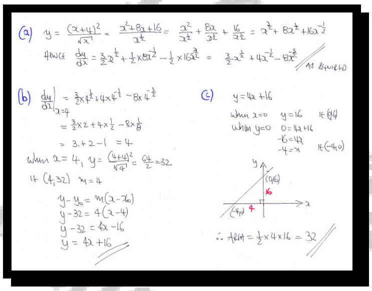
The point P lies on C where $x = 4$.

The straight line L is the tangent to C at the point P.

b) Find an equation of L.

c) Find the area of the triangle OAB, where A and B are the points where L crosses the coordinate axes, and O is the origin.

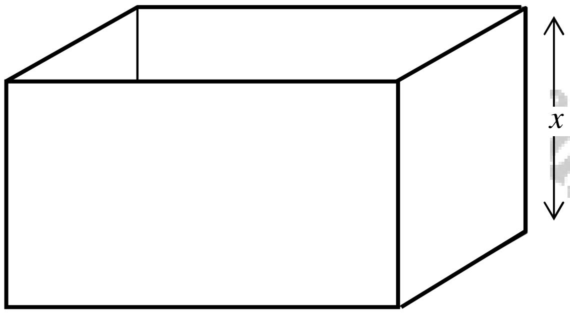
$\boxed{y = 4x + 16}$, $\boxed{\text{area} = 32}$



Question 137 (***+)

The figure below shows a large tank in the shape of a cuboid with a **rectangular** base and **no top**.

Two of the vertical opposite faces of the cuboid are square and the height of the cuboid is x metres.

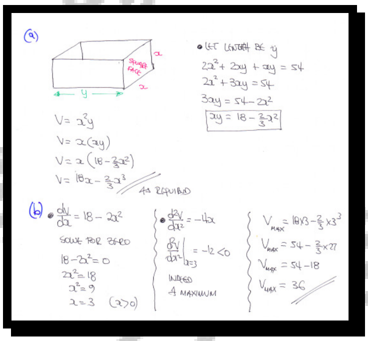


a) Given that the surface area of the tank is 54 m^2, show that the capacity of the tank, $V \text{ m}^3$, is given by

$$V = 18x - \frac{2}{3}x^3.$$

b) Find the maximum value for V, fully justifying the fact that it is indeed the maximum value.

$\boxed{V_{\max} = 36}$


(a)

SQUARE FACE

• LET LENGTH BE y

$2x^2 + 2xy + xy = 54$

$2x^2 + 3xy = 54$

$3xy = 54 - 2x^2$

$\boxed{xy = 18 - \frac{2}{3}x^2}$

$V = x^2 y$

$V = x(xy)$

$V = x\left(18 - \frac{2}{3}x^2\right)$

$V = 18x - \frac{2}{3}x^3$ AS REQUIRED

(b) • $\frac{dV}{dx} = 18 - 2x^2$

SOLVE FOR ZERO

$18 - 2x^2 = 0$

$2x^2 = 18$

$x^2 = 9$

$x = 3 \quad (x > 0)$

• $\frac{d^2V}{dx^2} = -4x$

$\left.\frac{d^2V}{dx^2}\right|_{x=3} = -12 < 0$

INDEED A MAXIMUM

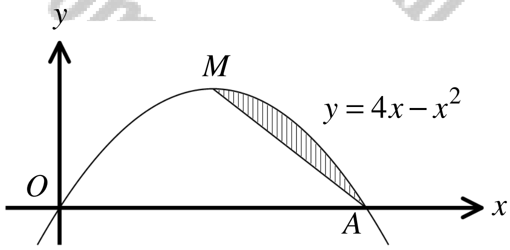
$V_{MAX} = 18 \times 3 - \frac{2}{3} \times 3^3$

$V_{MAX} = 54 - \frac{2}{3} \times 27$

$V_{MAX} = 54 - 18$

$V_{MAX} = 36$

Question 138 (***+)



The figure above shows the curve with equation

$$y = 4x - x^2 ,\ x \in \mathbb{R} .$$

The point M is the maximum point of the curve and the point A is one of the x intercepts of the curve.

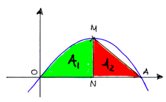
Find the exact area of the shaded region, bounded by the curve and the straight line segment joining A and M.

$\boxed{}$, $\boxed{\text{area} = \frac{4}{3}}$

LOOKING AT THE DIAGRAM

$y = 4x - x^2$
$y = x(4 - x)$

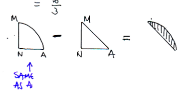
$\therefore A(4,0)$ BY INSPECTION
$N(2,0)$ BY SYMMETRY
$M(2,4) \leftarrow y = 4 \times 2 - 2^2$
$y = 8 - 4$
$y = 4$



AREA OF TRIANGLE $= A_2 = \frac{1}{2} \times |NA||MN|$
$= \frac{1}{2} \times 2 \times 4$
$= 4$

AREA UNDER CURVE $= A_1 = \int_0^2 4x - x^2 \, dx$
$= \left[2x^2 - \frac{1}{3}x^3\right]_0^2$
$= \left(8 - \frac{8}{3}\right) - 0$
$= \frac{16}{3}$

REQUIRED AREA =



$= \frac{16}{3} - 4$
$= \frac{4}{3}$

Question 139 (***+)

The curve C has equation

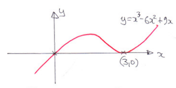
$$y = x^3 - 6x^2 + 9x.$$

a) Sketch the graph of C, indicating clearly the coordinates of the points where the graph meets the coordinate axes.

b) Determine the exact value of the area of the finite region bounded by the curve and the x axis.

$\boxed{(0,0)\ \&\ (3,0)}$, $\boxed{\text{area} = \frac{27}{4}}$

(a) $y = x^3 - 6x^2 + 9x$
$y = x(x^2 - 6x + 9)$
$y = x(x-3)(x-3)$
$y = x(x-3)^2$



(b) AREA $= \int_0^3 x^3 - 6x^2 + 9x \, dx = \left[\frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2\right]_0^3$
$= \left(\frac{1}{4}\times 3^4 - 2\times 3^3 + \frac{9}{2}\times 3^2\right) - (0) = \frac{81}{4} - 54 + \frac{81}{2} = \frac{27}{4}$

Question 140 (***+) **non calculator**

The curve C has equation

$$y = x + \sqrt{x^3}, \quad x \neq 0$$

Find the exact coordinates of the point on the curve where the gradient is 2.

$\boxed{\left(\frac{4}{9}, \frac{20}{27}\right)}$

$y = x + \sqrt{x^3}$
$\Rightarrow y = x + x^{\frac{3}{2}}$
$\Rightarrow \frac{dy}{dx} = 1 + \frac{3}{2}x^{\frac{1}{2}}$
$\Rightarrow 2 = 1 + \frac{3}{2}x^{\frac{1}{2}}$
$\Rightarrow 1 = \frac{3}{2}x^{\frac{1}{2}}$
$\Rightarrow \frac{2}{3} = x^{\frac{1}{2}}$
$\Rightarrow x = \frac{4}{9}$
$\therefore y = \frac{4}{9} + \left(\frac{4}{9}\right)^{\frac{3}{2}}$
$\Rightarrow y = \frac{4}{9} + \frac{4^{3/2}}{9^{3/2}}$
$\Rightarrow y = \frac{4}{9} + \frac{8}{27}$
$\Rightarrow y = \frac{12}{27} + \frac{8}{27}$
$\Rightarrow y = \frac{20}{27}$
$\therefore \left(\frac{4}{9}, \frac{20}{27}\right)$

Question 141 (***+)

The curve C has equation

$$y = 4x^3 - 7x - 1,\ x \in \mathbb{R}.$$

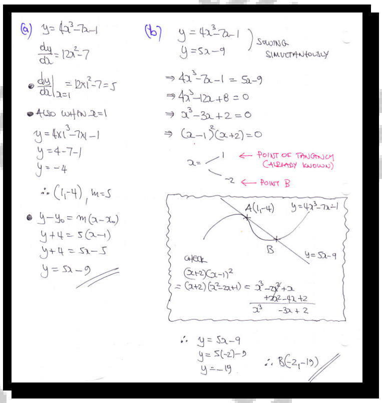
The point A lies on C where $x = 1$.

a) Find an equation of the tangent to C at A.

The tangent to C at A meets C again at the point B.

b) Find the coordinates of B.

$\boxed{}$, $\boxed{y = 5x - 9}$, $\boxed{B(-2, -19)}$



(a) $y = 4x^3 - 7x - 1$

$\frac{dy}{dx} = 12x^2 - 7$

• $\left.\frac{dy}{dx}\right|_{x=1} = 12 \times 1^2 - 7 = 5$

• Also when $x = 1$

$y = 4 \times 1^3 - 7 \times 1 - 1$

$y = 4 - 7 - 1$

$y = -4$

$\therefore (1, -4),\ m = 5$

• $y - y_0 = m(x - x_0)$

$y + 4 = 5(x - 1)$

$y + 4 = 5x - 5$

$y = 5x - 9$

(b) $y = 4x^3 - 7x - 1$, $y = 5x - 9$ } SOLVING SIMULTANEOUSLY

$\Rightarrow 4x^3 - 7x - 1 = 5x - 9$

$\Rightarrow 4x^3 - 12x + 8 = 0$

$\Rightarrow x^3 - 3x + 2 = 0$

$\Rightarrow (x - 1)^2(x + 2) = 0$

$x = 1$ ← POINT OF TANGENCY (ALREADY KNOWN)

$x = -2$ ← POINT B

CHECK

$(x+2)(x-1)^2$

$= (x+2)(x^2-2x+1) = x^3 - \cancel{2x^2} + x$

$\qquad + \cancel{2x^2} - 4x + 2$

$\qquad x^3 \quad - 3x + 2$

$\therefore y = 5x - 9$

$y = 5(-2) - 9$

$y = -19$

$\therefore B(-2, -19)$

Question 142 (***+)

The curve C has equation

$$y = x^3 - 3x^2 + 3x + 5.$$

Show that C has only one stationary point and determine its nature.

$\boxed{\text{MP1}}$, $\boxed{\text{point of inflection at } (1,6)}$

$$\boxed{y = x^3 - 3x^2 + 3x + 5}$$

DIFFERENTIATE & SET TO ZERO

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 6x + 3$$
$$\Rightarrow 0 = 3x^2 - 6x + 3$$
$$\Rightarrow 0 = x^2 - 2x + 1$$
$$\Rightarrow 0 = (x-1)^2$$

AT $x = 1$ IS THE ONLY STATIONARY POINT. $(y = 6)$

DETERMINING THE NATURE

$$\frac{d^2y}{dx^2} = 6x - 6$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 6\times1 - 6 = 0 \quad \leftarrow \text{EITHER A POINT OF INFLEXION OR THE TEST FAILS}$$

PROCEED WITH $\frac{d^3y}{dx^3}$

$$\frac{d^3y}{dx^3} = 6$$
$$\left.\frac{d^3y}{dx^3}\right|_{x=1} = 6 \neq 0 \quad \therefore \text{ A POINT OF INFLEXION}$$

Question 143 (***+)

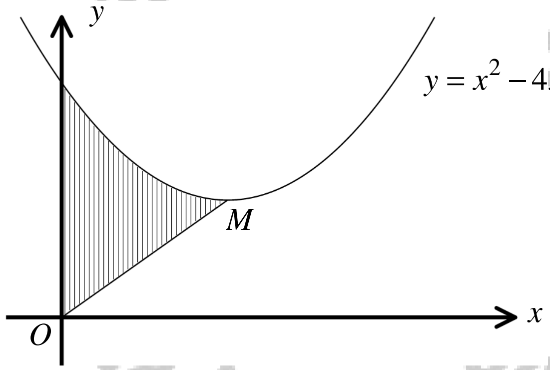
Prove by showing all the parts in the calculation that

$$\int_a^{a+h} (x-a)^2 \; dx = \frac{1}{3}h^3.$$

$\boxed{\text{proof}}$

$$\int_a^{a+h} (x-a)^2\,dx = \int_a^{a+h} x^2 - 2ax + a^2\,dx = \left[\tfrac{1}{3}x^3 - ax^2 + a^2x\right]_a^{a+h}$$
$$= \left[\tfrac{1}{3}(a+h)^3 - a(a+h)^2 + a^2(a+h)\right] - \left[\tfrac{1}{3}a^3 \cancel{- a^3} \cancel{+ a^3}\right]$$
$$= \left[\tfrac{1}{3}(a^3 + 3a^2h + 3ah^2 + h^3) - a(a^2 + 2ah + h^2) + a^3 + a^2h\right] - \tfrac{1}{3}a^3$$
$$= \cancel{\tfrac{1}{3}a^3} + \cancel{a^2h} + \cancel{ah^2} + \tfrac{1}{3}h^3 \cancel{- a^3} \cancel{- 2a^2h} \cancel{- ah^2} \cancel{+ a^3} \cancel{+ a^2h} \cancel{- \tfrac{1}{3}a^3}$$
$$= \tfrac{1}{3}h^3$$

Question 144 (***+)



The figure above shows the curve with equation

$$y = x^2 - 4x + 8, \; x \in \mathbb{R}.$$

The point M is the minimum point of the curve.

Find the area of the shaded region, bounded by the curve, the y axis and the straight line segment from O to M.

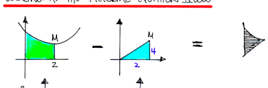
$\boxed{}$, $\boxed{\text{area} = \frac{20}{3}}$

START BY FINDING THE CO-ORDINATES OF M

$\Rightarrow y = x^2 - 4x + 8$
$\Rightarrow y = (x-2)^2 - 4 + 8$
$\Rightarrow y = (x-2)^2 + 4$

$\therefore M(2,4)$

LOOKING AT THE PICTORIAL EQUATION BELOW



$\int_0^2 x^2 - 4x + 8 \, dx$

$= \left[\frac{1}{3}x^3 - 2x^2 + 8x\right]_0^2$

$= \left(\frac{8}{3} - 8 + 16\right) - (0)$

$= \frac{32}{3}$

$\frac{1}{2} \times 2 \times 4 = 4$

REQUIRED AREA $= \frac{32}{3} - 4 = \frac{20}{3}$

Question 145 (***+)

The curve C has equation

$$y = x^3 + 2x^2 - 7, \; x \in \mathbb{R}.$$

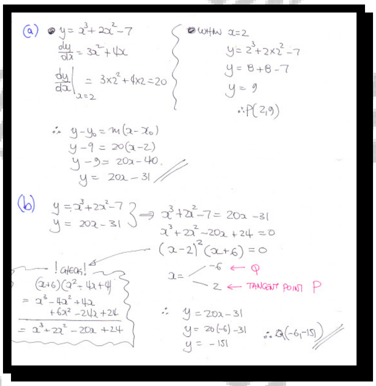
The point P lies on C where $x = 2$.

a) Find an equation of the tangent to C at P.

The tangent to C at P meets C again at the point Q.

b) Show that the coordinates of Q are $(-6, -151)$.

$\boxed{y = 20x - 31}$



(a) • $y = x^3 + 2x^2 - 7$

$\frac{dy}{dx} = 3x^2 + 4x$

$\left.\frac{dy}{dx}\right|_{x=2} = 3 \times 2^2 + 4 \times 2 = 20$

• WHEN $x = 2$

$y = 2^3 + 2 \times 2^2 - 7$

$y = 8 + 8 - 7$

$y = 9$

$\therefore P(2, 9)$

$\therefore y - y_0 = m(x - x_0)$

$y - 9 = 20(x - 2)$

$y - 9 = 20x - 40$

$y = 20x - 31$

(b) $\left.\begin{array}{l} y = x^3 + 2x^2 - 7 \\ y = 20x - 31 \end{array}\right\} \Rightarrow x^3 + 2x^2 - 7 = 20x - 31$

$x^3 + 2x^2 - 20x + 24 = 0$

$(x-2)^2(x+6) = 0$

$x = \begin{cases} -6 \leftarrow Q \\ 2 \leftarrow \text{TANGENT POINT } P \end{cases}$

!CHECK!

$(x+6)(x^2 - 4x + 4)$

$= x^3 - 4x^2 + 4x$
$\quad + 6x^2 - 24x + 24$

$= x^3 + 2x^2 - 20x + 24$

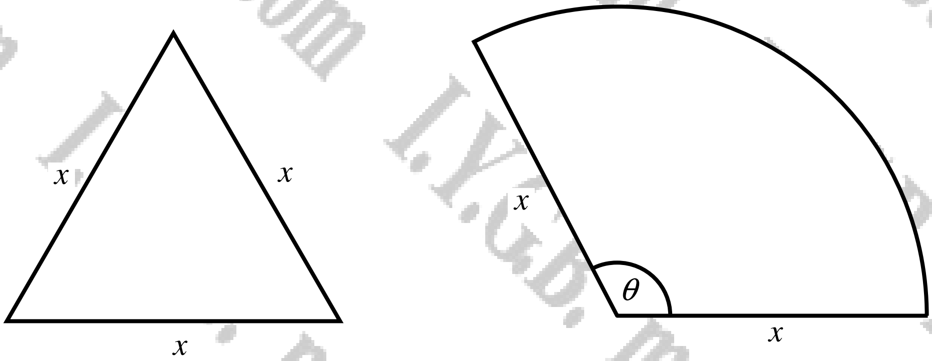
$\therefore y = 20x - 31$

$y = 20(-6) - 31$

$y = -151$

$\therefore Q(-6, -151)$

Question 146 (***+)



A wire of total length 60 cm is to be cut into two pieces. The first piece is bent to form an equilateral triangle of side length x cm and the second piece is bent to form a circular sector of radius x cm. The circular sector subtends an angle of θ radians at the centre.

a) Show that

$$x\theta = 60 - 5x\,.$$

The **total** area of the **two** shapes is A cm^2.

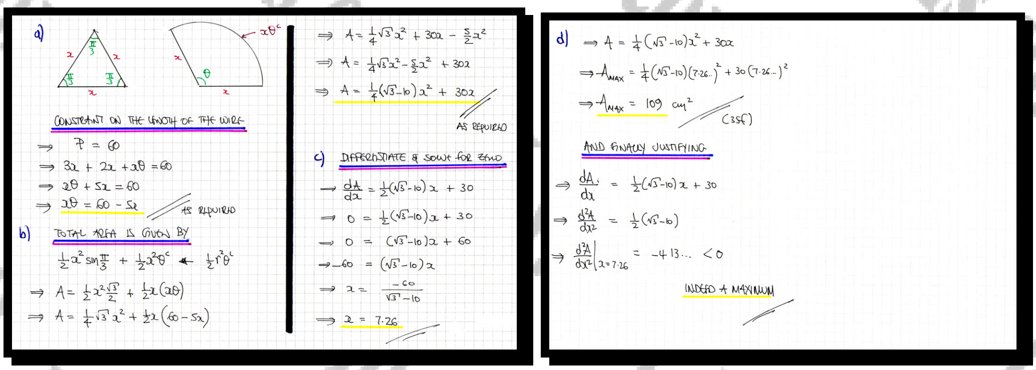
b) Show clearly that

$$A = \frac{1}{4}\left(\sqrt{3} - 10\right)x^2 + 30x\,.$$

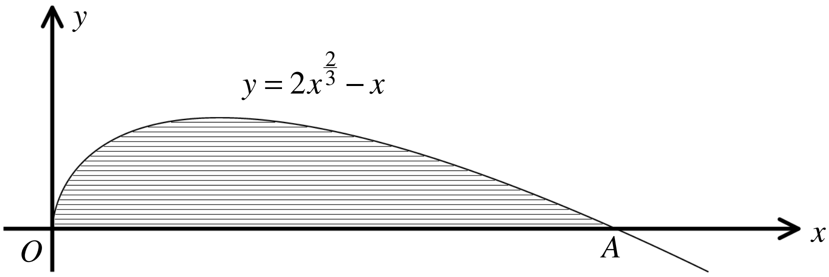
c) Use differentiation to find the value of x for which A is stationary.

d) Find, correct to three significant figures, the maximum value of A, justifying the fact that it is indeed the maximum value of A.

$\boxed{\text{SP}}$, $\boxed{x \approx 7.26}$, $\boxed{A_{\text{max}} \approx 109}$



Question 147 (***+)



The figure above shows part of the curve with equation

$$y = 2x^{\frac{2}{3}} - x, \ x \geq 0 .$$

The curve meets the x axis at the point A.

a) Show that the coordinates of A are $(8,0)$.

b) Find the exact area of the finite region bounded by the curve and the x axis.

$\boxed{\text{area} = \frac{32}{5}}$

(a) $y = 2x^{\frac{2}{3}} - x$

when $y = 0$

$0 = 2x^{\frac{2}{3}} - x$

$0 = x\left(2x^{-\frac{1}{3}} - 1\right)$

EITHER $x = 0$ OR $2x^{-\frac{1}{3}} - 1 = 0$

$\frac{2}{\sqrt[3]{x}} = 1$

$2 = \sqrt[3]{x}$

$x = 8$

$\therefore A(8,0)$

ALTERNATIVE

$2x^{\frac{2}{3}} - x = 0$

$2x^{\frac{2}{3}} = x$

$\frac{2x^{\frac{2}{3}}}{x} = 1$ SINCE $x \neq 0$ AT A

$2x^{-\frac{1}{3}} = 1$

$x^{-\frac{1}{3}} = \frac{1}{2}$

$\frac{1}{\sqrt[3]{x}} = \frac{1}{2}$

$\sqrt[3]{x} = 2$

$x = 8$ (AS BEFORE)

(b) AREA $= \int_{x_1}^{x_2} f(x)\,dx = \int_0^8 2x^{\frac{2}{3}} - x \, dx$

$= \left[\frac{2}{\frac{5}{3}}x^{\frac{5}{3}} - \frac{1}{2}x^2\right]_0^8 = \left[\frac{6}{5}x^{\frac{5}{3}} - \frac{1}{2}x^2\right]_0^8$

$= \left(\frac{6}{5} \times 8^{\frac{5}{3}} - \frac{1}{2} \times 8^2\right) - \cancel{(0-0)}$

$= \frac{6}{5} \times 32 - 32 = \frac{32}{5}$

Question 148 (*+) non calculator**

$$y = \frac{2 - 3x\sqrt{x}}{3x} + \frac{1}{\sqrt{2}}, \quad x > 0.$$

a) Find an expression for $\frac{dy}{dx}$.

b) Show that the value of $\frac{d^2y}{dx^2}$ where $x = 4$, is $\frac{5}{96}$.

$$\boxed{\frac{dy}{dx} = -\frac{2}{3}x^{-2} - \frac{1}{2}x^{-\frac{1}{2}}}$$

(a) $y = \frac{2-3x\sqrt{x}}{3x} + \frac{1}{\sqrt{2}} = \frac{2}{3x} - \frac{3x \times x^{\frac{1}{2}}}{3x} + \frac{1}{\sqrt{2}}$

$y = \frac{2}{3}x^{-1} - x^{\frac{1}{2}} + \frac{1}{\sqrt{2}}$

$\frac{dy}{dx} = -\frac{2}{3}x^{-2} - \frac{1}{2}x^{-\frac{1}{2}}$

(b) $\frac{d^2y}{dx^2} = \frac{4}{3}x^{-3} + \frac{1}{4}x^{-\frac{3}{2}} = \frac{4}{3x^3} + \frac{1}{4x^{\frac{3}{2}}}$

$\left.\frac{d^2y}{dx^2}\right|_{x=4} = \frac{4}{3 \times 4^3} + \frac{1}{4 \times 4^{\frac{3}{2}}} = \frac{1}{3 \times 4^2} + \frac{1}{4 \times 8} = \frac{1}{48} + \frac{1}{32}$

$= \frac{2}{96} + \frac{3}{96} = \frac{5}{96}$ AS REQUIRED

Question 149 (***+)

$$f(x)=\sqrt{x}\left(18x^2+35x-45\right),\ x\geq 0\,.$$

a) Find a simplified expression for $f'(x)$.

b) Show clearly that

$$f'(x)=\frac{15(3x-1)(2x-3)}{2\sqrt{x}}\,.$$

$$\boxed{f'(x)=45x^{\frac{3}{2}}+\frac{105}{2}x^{\frac{1}{2}}-\frac{45}{2}x^{-\frac{1}{2}}}$$

a) $f(x)=\sqrt{x}\left(18x^2+35x-45\right)$

$f(x)=x^{\frac{1}{2}}\left(18x^2+35x-45\right)$

$f(x)=18x^{\frac{5}{2}}+35x^{\frac{3}{2}}-45x^{\frac{1}{2}}$

$f'(x)=45x^{\frac{3}{2}}+\frac{105}{2}x^{\frac{1}{2}}-\frac{45}{2}x^{-\frac{1}{2}}$

b) $f'(x)=45x^{\frac{3}{2}}+\frac{105}{2}x^{\frac{1}{2}}-\frac{45}{2}x^{-\frac{1}{2}}$

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}\left[90x^2+105x-45\right]$

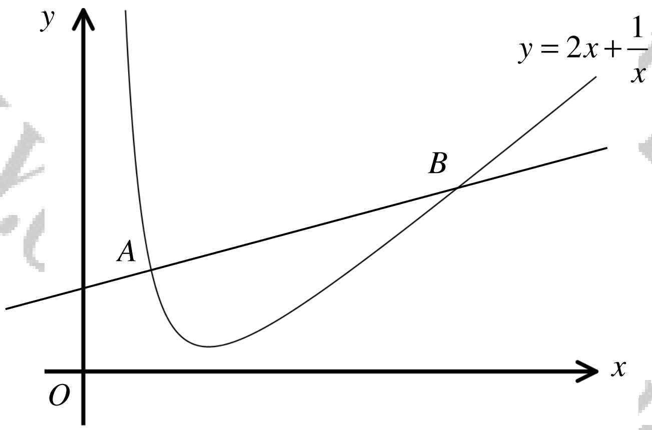
$f'(x)=\frac{15}{2}x^{-\frac{1}{2}}\left[6x^2-7x-3\right]$

$f'(x)=\frac{15}{2\sqrt{x}}(3x-1)(2x-3)$

$f'(x)=\frac{15(3x-1)(2x-3)}{2\sqrt{x}}$

AS REQUIRED

Question 150 (***+)



The figure above shows part of the curve C, with equation

$$y = 2x + \frac{1}{x}, \quad x \neq 0$$

The point A lies on C where $x = \frac{1}{2}$.

a) Find an equation of the normal to C at A.

The normal meets the curve again at the point B.

b) Determine the exact coordinates of B.

$\boxed{}$, $\boxed{2x - 4y + 11 = 0}$, $\boxed{B\left(\frac{4}{3}, \frac{41}{12}\right)}$

(a) $y = 2x + \frac{1}{x} = 2x + x^{-1}$

$\frac{dy}{dx} = 2 - x^{-2} = 2 - \frac{1}{x^2}$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = 2 - \frac{1}{\left(\frac{1}{2}\right)^2} = 2 - \frac{1}{\frac{1}{4}}$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = 2 - 4 = -2$

$\therefore$ NORMAL GRADIENT IS $\frac{1}{2}$

• WHEN $x = \frac{1}{2}$ $y = 2 \times \frac{1}{2} + \frac{1}{\frac{1}{2}} = 1 + 2$

$\therefore A\left(\frac{1}{2}, 3\right)$

$y - y_0 = m(x - x_0)$

$y - 3 = \frac{1}{2}\left(x - \frac{1}{2}\right)$

$2y - 6 = x - \frac{1}{2}$

$4y - 12 = 2x - 1$

$4y - 2x - 11 = 0$

(b) $\left.\begin{array}{l} y = 2x + \frac{1}{x} \\ 4y - 2x - 11 = 0 \end{array}\right\} \Rightarrow$

$\Rightarrow 4\left(2x + \frac{1}{x}\right) - 2x - 11 = 0$

$\Rightarrow 8x + \frac{4}{x} - 2x - 11 = 0$

$\Rightarrow 6x + \frac{4}{x} - 11 = 0$

$\Rightarrow 6x^2 + 4 - 11x = 0$

$\Rightarrow 6x^2 - 11x + 4 = 0$

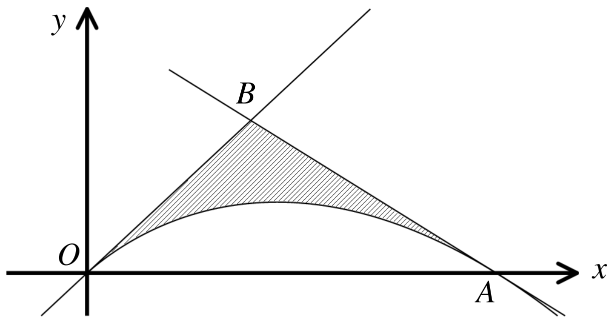
$\Rightarrow (2x - 1)(3x - 4) = 0$

$\therefore x = \frac{1}{2}$ ← POINT A, $\frac{4}{3}$ ← POINT B

& $y = 2\left(\frac{4}{3}\right) + \frac{1}{\frac{4}{3}} = \frac{8}{3} + \frac{3}{4} = \frac{32+9}{12}$

$\therefore B\left(\frac{4}{3}, \frac{41}{12}\right)$

Question 151 (***+)



The figure above show the graph of the curve C with equation

$$y = x\left(1 - x^{\frac{2}{3}}\right), \quad x \in \mathbb{R}, \ x \geq 0 .$$

The curve meets the coordinate axes at the origin and at the point $A(1,0)$.

The two tangents to C at the origin O and at the point A, meet at the point B.

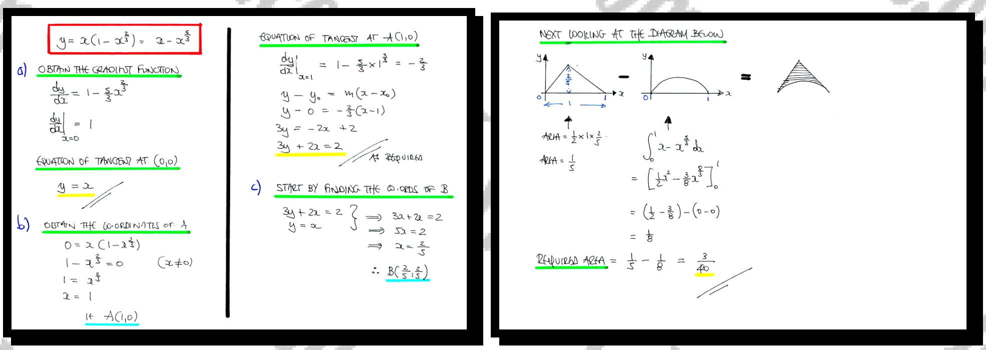
a) Calculate the value of $\frac{dy}{dx}$ at O, and hence write down an equation of the tangent to C at O.

b) Show that an equation of the tangent to C at A is

$$2x + 3y = 2$$

c) Determine the area of the finite region bounded by C and the tangents to C at O and at A.

$\boxed{}$, $\boxed{y = x}$, $\boxed{\text{area} = \frac{3}{40}}$



Question 152 **(***+)**

The point $P(3,k)$ lies on the curve with equation

$$y = x^2 + ax - 4,$$

where a and k are constants.

Given that the gradient at P is 3 determine the value of a and the value of k.

$\boxed{}$, $\boxed{a = -3, \quad k = -4}$

THE POINT $P(3,k)$ LIES ON THE CURVE

$$y = x^2 + ax - 4$$
$$k = 3^2 + a \times 3 - 4$$
$$k = 9 + 3a - 4$$
$$k = 3a + 5$$

DIFFERENTIATE & USE THE FACT THAT $\left.\frac{dy}{dx}\right|_{x=3} = 3$

$$\frac{dy}{dx} = 2x + a$$
$$3 = 2 \times 3 + a$$
$$3 = 6 + a$$
$$a = -3$$

& USING $k = 3a + 5$

$$k = 3(-3) + 5$$
$$k = -4$$

Question 153 (***+)

The curve C has equation

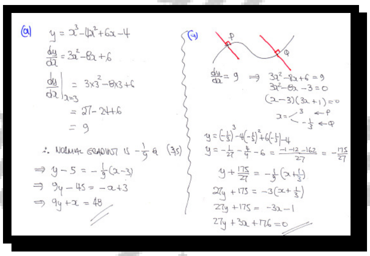
$$y = x^3 - 4x^2 + 6x - 4.$$

a) Find an equation of the normal to C at the point $P(3,5)$.

The point Q lies on C so that the normal to the curve at Q is parallel to the normal to the curve at P.

b) Find an equation of the normal to the curve at Q.

$\boxed{x + 9y = 48}$, $\boxed{3x + 27y + 176 = 0}$



Question 154 (***+)

The point $P(1,0)$ lies on the curve C with equation

$$y = x^3 - x, \quad x \in \mathbb{R}.$$

a) Find an equation of the tangent to C at P, giving the answer in the form $y = mx + c$, where m and c are constants.

The tangent to C at P meets C again at the point Q.

b) Find the coordinates of Q.

$\boxed{y = 2x - 2}$, $\boxed{Q(-2,-6)}$

(a) $y = x^3 - x$

$\frac{dy}{dx} = 3x^2 - 1$

$\left.\frac{dy}{dx}\right|_{x=1} = 3 \times 1^2 - 1 = 2$

HENCE $y - y_0 = m(x - x_0)$

$y - 0 = 2(x - 1)$

$y = 2x - 2$

(b) $\left.\begin{array}{l} y = x^3 - x \\ y = 2x - 2 \end{array}\right)$ SOLVE SIMULTANEOUSLY

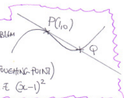
$\Rightarrow x^3 - x = 2x - 2$

$\Rightarrow x^3 - 3x + 2 = 0$

SINCE $P(1,0)$ MUST BE SOLUTION TO THE PROBLEM

$\Rightarrow x = 1$ IS A ROOT (TANGENT POINT)

$\Rightarrow$ FURTHERMORE IT MUST BE REPEATED (TOUCHING POINT) I.E. $(x-1)^2$



$\Rightarrow (x-1)^2(x+2) = 0$

POINT P POINT Q

CHECK

$(x+2)(x-1)^2$

$= (x+2)(x^2 - 2x + 1)$

$= x^3 - 2x^2 + x$

$\quad + 2x^2 - 4x + 2$

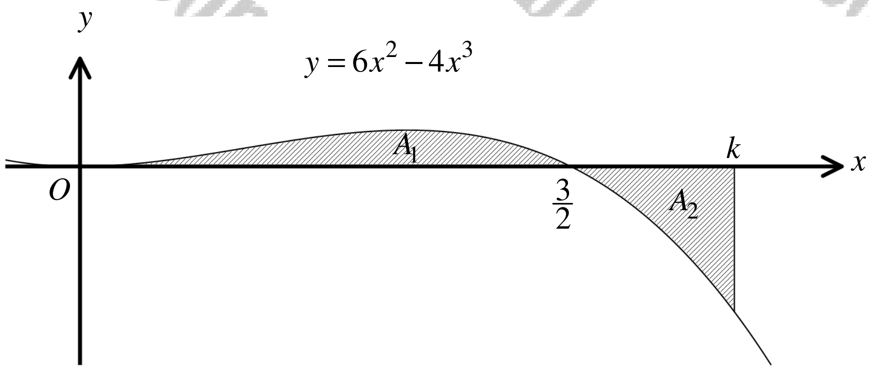
$= x^3 - 3x + 2$

$\therefore x = -2$

$y = 2x - 2 = 2(-2) - 2 = -6$

So $Q(-2,-6)$

Question 155 (***+)



The figure above shows the graph of the curve with equation

$$y = 6x^2 - 4x^3, \ x \in \mathbb{R}.$$

The curve meets x axis at the origin O and at the point $\left(\frac{3}{2}, 0\right)$.

The point $(k, 0)$, $k > \frac{3}{2}$ is such so that, the area A_1 of the region between the curve and the x axis for which $0 \le x \le \frac{3}{2}$, **is equal** to the area A_2 of the region between the curve and the x axis for which $\frac{3}{2} \le x \le k$.

Determine the value of k.

$\boxed{}$, $\boxed{k = 2}$

START BY FINDING THE AREA OF A_1

$$A_1 = \int_0^{\frac{3}{2}} 6x^2 - 4x^3 \, dx = \left[2x^3 - x^4\right]_0^{\frac{3}{2}}$$

$$= \left[2\left(\tfrac{3}{2}\right)^3 - \left(\tfrac{3}{2}\right)^4\right] - [0] = \frac{27}{4} - \frac{81}{16} = \frac{27}{16}$$

NOW THIS MUST BE THE SAME AREA AS A_2, BUT NOT THAT THE "SIGN" MUST BE NEGATIVE, AS IT IS BELOW THE x AXIS

$$\Rightarrow A_2 = -\frac{27}{16}$$

$$\Rightarrow \int_{\frac{3}{2}}^{k} 6x^2 - 4x^3 \, dx = -\frac{27}{16}$$

$$\Rightarrow \left[2x^3 - x^4\right]_{\frac{3}{2}}^{k} = -\frac{27}{16}$$

$$\Rightarrow (2k^3 - k^4) - \left(\frac{27}{4} - \frac{81}{16}\right) = -\frac{27}{16}$$

$$\Rightarrow 2k^3 - k^4 - \frac{27}{4} + \frac{81}{16} = -\frac{27}{16}$$

$$\Rightarrow 2k^3 - k^4 = 0$$

$$\Rightarrow k^3(2 - k) = 0$$

$$\Rightarrow k = \begin{cases} 0 \\ 2 \end{cases}$$

$$\therefore k = 2$$

Question 156 (***+)

The curve C has equation

$$y = 5x + \frac{4}{x} - 3, \ x \neq 0$$

Show that the straight line with equation

$$y = 4x + 1$$

is a tangent to C, and find the coordinates of the point of tangency.

$\boxed{}$, $\boxed{(2,9)}$

METHOD A – USING THE DISCRIMINANT

$$\left.\begin{array}{l} y = 5x + \frac{4}{x} - 3 \\ y = 4x + 1 \end{array}\right\} \Rightarrow 5x + \frac{4}{x} - 3 = 4x + 1$$

$\Rightarrow x + \frac{4}{x} - 4 = 0$

$\Rightarrow x^2 + 4 - 4x = 0$

$\Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow (x-2)^2 = 0$

REPEATED ROOT $\therefore$ A TANGENT AT $x = 2$

$\Rightarrow y = 4 \times 2 + 1$

$\therefore (2,9)$

METHOD B – USING DIFFERENTIATION

- THE LINE HAS GRADIENT 4 $(y = 4x+1)$
- DIFFERENTIATE THE EQUATION OF THE CURVE

$y = 5x + 4x^{-1} - 3$

$\frac{dy}{dx} = 5 - 4x^{-2}$

- SET THE GRADIENT FUNCTION EQUAL TO 4

$4 = 5 - 4x^{-2}$

$4x^{-2} = 1$

$\frac{4}{x^2} = 1$

$x^2 = 4$

$x = \begin{cases} 2 \\ -2 \end{cases}$ $\quad y = \begin{cases} 5 \times 2 + \frac{4}{2} - 3 = 9 \\ 5(-2) + \frac{4}{-2} - 3 = -15 \end{cases}$

- CHECK EACH POINT

$y - y_0 = m(x - x_0)$

$y - 9 = 4(x - 2)$

$y - 9 = 4x - 8$

$y = 4x + 1$

INDEED A TANGENT AT $(2,9)$

$y - y_0 = m(x - x_0)$

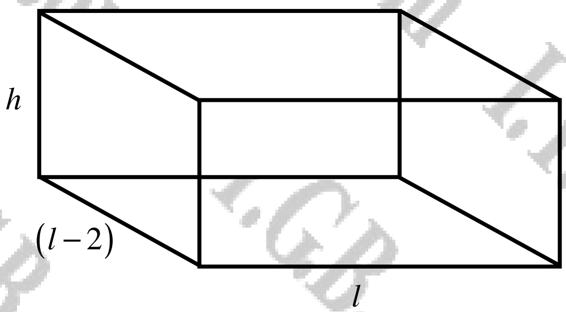
$y + 15 = 4(x + 2)$

$y + 15 = 4x + 8$

$y = 4x - 7$

A COMPLETELY DIFFERENT TANGENT AT $(-2,-15)$

Question 157 (***+)



The figure above shows 12 rigid rods, joined together to form the framework of a storage container, which in the shape of a cuboid.

Each of the four upright rods has height h m. Each of the longer horizontal rods has length l m and each of the shorter horizontal rods have length $(l-2)$ m.

a) Given that the total length of the 12 rods is 36 m show that the volume, V m^3, of the container satisfies

$$V = -2l^3 + 15l^2 - 22l .$$

b) Find, correct to 3 decimal places, the value of l which make V stationary.

c) Justify that the value of l found in part (b) maximizes the value of V, and find this maximum value of V, correct to the nearest m^3.

d) State the three measurements of the container when its volume is maximum.

$\boxed{l = 4.107}$, $\boxed{V_{\text{max}} \approx 24}$, $\boxed{4.11 \times 2.11 \times 2.79}$

a) FIRSTLY TOTAL LENGTH = 36

$[h + l + (l-2)] \times 4 = 36$

$h + 2l - 2 = 9$

$\boxed{h + 2l = 11}$

- $V = l(l-2)h$

$\Rightarrow V = l(l-2)(11-2l)$

$\Rightarrow V = -l(l-2)(2l-11)$

$\Rightarrow V = -l(2l^2 - 15l + 22)$

$\Rightarrow V = -2l^3 + 15l^2 - 22l$

b) $\frac{dV}{dl} = -6l^2 + 30l - 22$

SOLVE FOR ZERO $\Rightarrow -6l^2 + 30l - 22 = 0$

$3l^2 - 15l + 11 = 0$

$l = \frac{15 \pm \sqrt{225 - 4\times3\times11}}{6}$

$l = \frac{15 + \sqrt{93}}{6}$ $\quad l > 2$

$l \simeq 4.107$

c) - $\frac{d^2V}{dl^2} = -12l + 30$

$\left.\frac{d^2V}{dl^2}\right|_{l \simeq 4.107\ldots} = -2\sqrt{93} < 0$ INDEED A MAX

- $V_{MAX} = -2(4.107\ldots)^3 + 15(4.107\ldots)^2 - 22(4.107\ldots)$

$V_{MAX} = 24.1085\ldots$

$V_{MAX} \simeq 24 \text{ m}^3$

d) $l = 4.107$

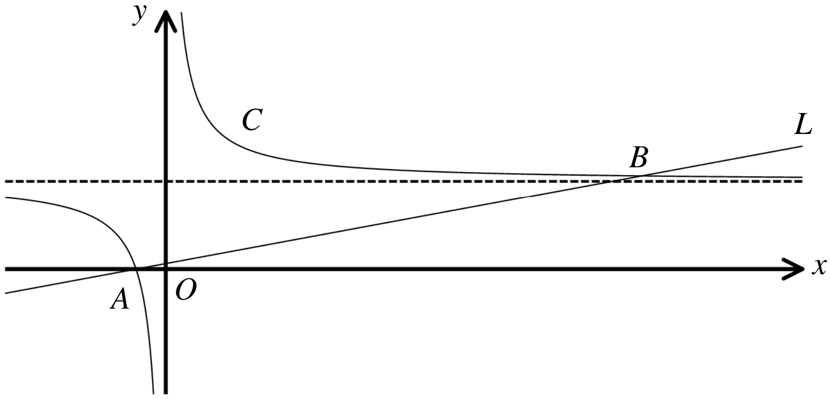
$l - 2 = 2.107$

$h = 11 - 2(4.107)$

$\therefore 4.11 \times 2.11 \times 2.79$

UNITS ARE IN m

Question 158 (***+)



The figure above shows a sketch of the curve C with equation

$$y = 2 + \frac{1}{x}, \quad x \neq 0 .$$

The dotted line represents an asymptote to C and the point A is the point where C crosses the x axis.

a) State the equations of the two asymptotes to the C.

b) Find the coordinates of A.

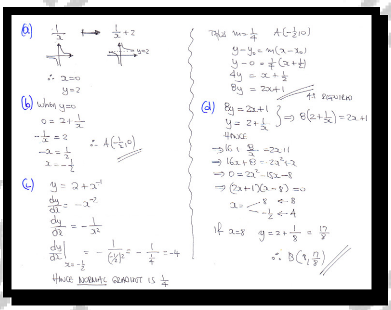
The straight line L is the normal to C at A, and B is the point where L meets C again.

c) Show that an equation for L is

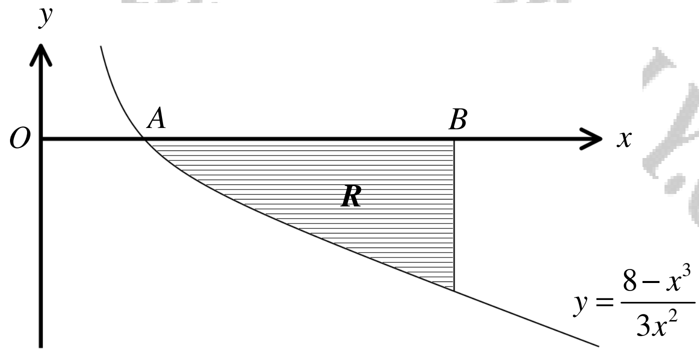
$$8y = 2x + 1 .$$

d) Determine the coordinates of B.

$\boxed{}$, $\boxed{x=0, y=2}$, $\boxed{A\left(-\frac{1}{2},0\right)}$



Question 159 (***+)



The figure below shows part of the curve with equation

$$y = \frac{8-x^3}{3x^2}, \quad x > 0.$$

The curve meets the x axis at the point A and B has coordinates $(4,0)$.

a) Find the coordinates of A.

b) Determine the exact area of the finite region R, bounded by the curve, the vertical line through B and the x axis.

$\boxed{C(4,0)}$, $\boxed{\text{area} = \frac{4}{3}}$

(a) $y = \frac{8-x^3}{3x^2}$ when $y=0$, $8-x^3=0$
$x^3 = 8$
$x = 2$ $\therefore A(2,0)$

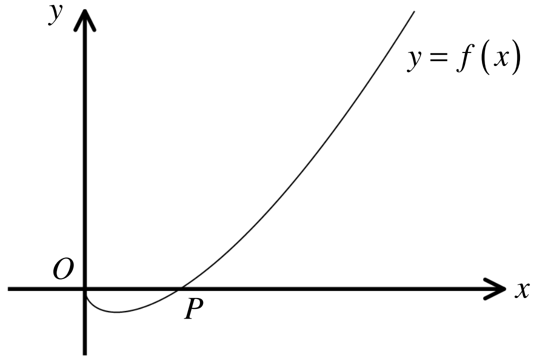
(b) $R = \int_2^4 \frac{8-x^3}{3x^2}\,dx = \int_2^4 \frac{8}{3x^2} - \frac{x^3}{3x^2}\,dx = \int_2^4 \frac{8}{3}x^{-2} - \frac{1}{3}x\,dx$

$= \left[-\frac{8}{3}x^{-1} - \frac{1}{6}x^2\right]_2^4 = \left[\frac{8}{3x} + \frac{1}{6}x^2\right]_4^2$

$= \left(\frac{8}{6} + \frac{4}{6}\right) - \left(\frac{8}{12} + \frac{16}{6}\right) = -\frac{4}{3}$

$\therefore$ AREA $= \frac{4}{3}$

Question 160 (***+)



The figure above shows a curve with equation $y = f(x)$ which meets the x axis at the origin O and at the point P.

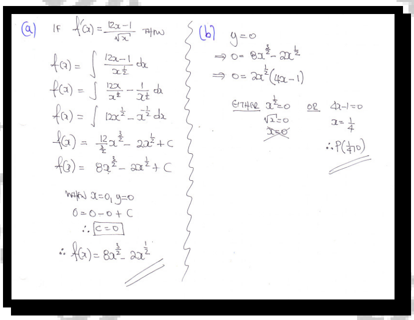
The gradient function of the curve is given by

$$f'(x) = \frac{12x-1}{\sqrt{x}}, \quad x > 0.$$

a) Find an equation of the curve.

b) Determine the coordinates of P.

$\boxed{f(x) = 8x^{\frac{3}{2}} - 2\sqrt{x}}$, $\boxed{P\left(\frac{1}{4}, 0\right)}$



(a) IF $f'(x) = \frac{12x-1}{\sqrt{x}}$ THEN

$f(x) = \int \frac{12x-1}{x^{\frac{1}{2}}} \, dx$

$f(x) = \int \frac{12x}{x^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}} \, dx$

$f(x) = \int 12x^{\frac{1}{2}} - x^{-\frac{1}{2}} \, dx$

$f(x) = \frac{12}{\frac{3}{2}}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + C$

$f(x) = 8x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + C$

WHEN $x=0$, $y=0$

$0 = 0 - 0 + C$

$\therefore \boxed{C=0}$

$\therefore f(x) = 8x^{\frac{3}{2}} - 2x^{\frac{1}{2}}$

(b) $y=0$

$\Rightarrow 0 = 8x^{\frac{3}{2}} - 2x^{\frac{1}{2}}$

$\Rightarrow 0 = 2x^{\frac{1}{2}}(4x-1)$

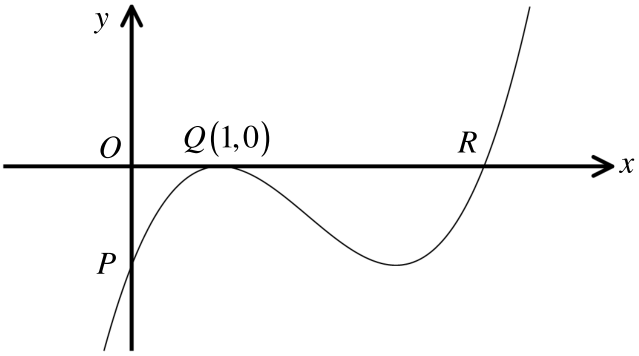
EITHER $x^{\frac{1}{2}}=0$ OR $4x-1=0$

$\sqrt{x}=0$ $\quad x = \frac{1}{4}$

$\cancel{x=0}$

$\therefore P(\frac{1}{4}, 0)$

Question 161 **(***+)**



The figure above shows the graph of a cubic curve, which touches the x axis at the point $Q(1,0)$.

a) Determine an equation for the cubic curve , given its gradient function is

$$\frac{dy}{dx} = 3x^2 - 12x + 9 .$$

The cubic curve crosses the x axis and the y axis at the points R and P, respectively.

b) Determine the coordinates …

i. … of the point P.

ii. … of the point R.

$\boxed{}$, $\boxed{y = x^3 - 6x^2 + 9x - 4}$, $\boxed{P(0,-4)}$, $\boxed{R(4,0)}$

(a) IF $\frac{dy}{dx} = 3x^2 - 12x + 9$

THEN $y = \int 3x^2 - 12x + 9 \, dx$

$\boxed{y = x^3 - 6x^2 + 9x + C}$

APPLY CONDITION $(1,0)$

$0 = 1 - 6 + 9 + C$

$C = -4$

$\therefore y = x^3 - 6x^2 + 9x - 4$

(b) (I) WHEN $x = 0$

$y = -4$

$\therefore P(0,-4)$

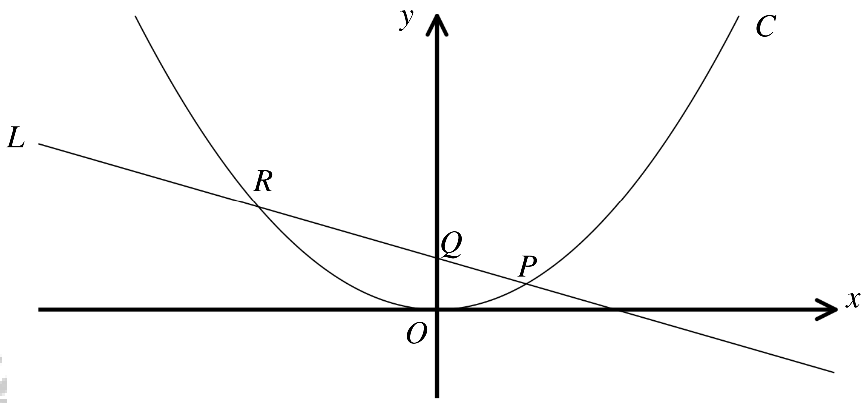
(II) $y = x^3 - 6x^2 + 9x - 4$

$y = (x-1)^2(x-4)$

TOUCHES AT $Q(1,0)$

$\therefore R(4,0)$

Question 162 **(***+)**



The figure above shows a sketch of the curve C with equation

$$y = 2x^2, \ x \in \mathbb{R}.$$

The straight line L passes through the points $P\left(\frac{1}{2}, \frac{1}{2}\right)$ and $Q(0,1)$, where the point P lies on C.

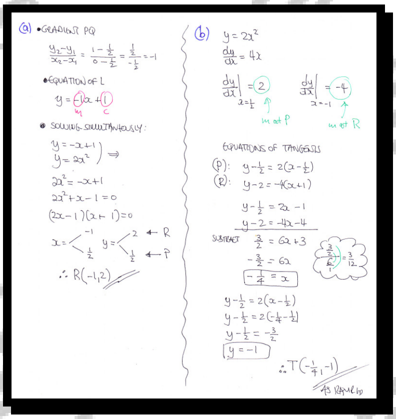
The straight line L meets C again, at the point R.

a) Find the coordinates of R.

The tangents to C at the points P and R meet at the point T.

b) Show that the coordinates of T are $\left(-\frac{1}{4}, -1\right)$

$\boxed{R(-1,2)}$



(a) • GRADIENT PQ

$\frac{y_2-y_1}{x_2-x_1} = \frac{1-\frac{1}{2}}{0-\frac{1}{2}} = \frac{\frac{1}{2}}{-\frac{1}{2}} = -1$

• EQUATION OF L

$y = \underset{m}{-1}x + \underset{c}{1}$

• SOLVING SIMULTANEOUSLY:

$\left.\begin{array}{l} y = -x+1 \\ y = 2x^2 \end{array}\right) \Rightarrow$

$2x^2 = -x+1$

$2x^2+x-1=0$

$(2x-1)(x+1)=0$

$x = \begin{cases} -1 \\ \frac{1}{2} \end{cases} \quad y = \begin{cases} 2 & \leftarrow R \\ \frac{1}{2} & \leftarrow P \end{cases}$

$\therefore R(-1,2)$

(b) $y = 2x^2$

$\frac{dy}{dx} = 4x$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = 2$ m at P $\qquad \left.\frac{dy}{dx}\right|_{x=-1} = -4$ m at R

EQUATIONS OF TANGENTS

(P): $y - \frac{1}{2} = 2(x - \frac{1}{2})$

(R): $y - 2 = -4(x+1)$

$y - \frac{1}{2} = 2x - 1$

$y - 2 = -4x - 4$

SUBTRACT $\frac{3}{2} = 6x + 3$

$-\frac{3}{2} = 6x$

$\boxed{-\frac{1}{4} = x}$

$\frac{\frac{3}{2}}{\frac{6}{1}} = \frac{3}{12}$

$y - \frac{1}{2} = 2(x - \frac{1}{2})$

$y - \frac{1}{2} = 2(-\frac{1}{4} - \frac{1}{2})$

$y - \frac{1}{2} = -\frac{3}{2}$

$\boxed{y = -1}$

$\therefore T(-\frac{1}{4}, -1)$

Question 163 (***+)

A cubic curve has equation

$$y = x^3 - x^2 + 5, \ x \in \mathbb{R}.$$

The point P lies on the curve where $x = 1$.

Show that the normal to the curve at P does not meet the curve again.

$\boxed{}$, $\boxed{\text{proof}}$

START BY OBTAINING THE EQUATION OF THE NORMAL AT P

$\Rightarrow \quad y = x^3 - x^2 + 5$

$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 2x$

$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=1} = 3 \times 1^2 - 2 \times 1 = 1 \quad \& \quad y\big|_{x=1} = 1^3 - 1^2 + 5 = 5$

$\therefore$ NORMAL GRADIENT IS -1 & $P(1,5)$

$\Rightarrow \quad y - y_0 = m(x - x_0)$

$\Rightarrow \quad y - 5 = -1(x - 1)$

$\Rightarrow \quad y - 5 = -x + 1$

$\Rightarrow \quad y = -x + 6$

$\Rightarrow \quad y + x = 6$

SUPPOSE THIS NORMAL MEETS THE CURVE AGAIN

$$\left.\begin{array}{l} y = 6 - x \\ y = x^3 - x^2 + 5 \end{array}\right\} \Rightarrow x^3 - x^2 + 5 = 6 - x$$

$\Rightarrow \quad x^3 - x^2 + x - 1 = 0$

FACTORIZE BY FACTORIZATION IN PAIRS

$\Rightarrow \quad x^2(x - 1) + (x - 1) = 0$

$\Rightarrow \quad (x - 1)(x^2 + 1) = 0$

ONLY SOLUTION $x = 1$, AS $x^2 + 1 \neq 0$

$\therefore$ ONLY INTERSECTION IS THE POINT OF NORMALITY $(1,5)$ & NO MORE !!

Question 164 (***+)

The curve C has equation

$$y = ax^3 + bx^2 - 10,$$

where a and b are constants.

The point $A(2,2)$ lies on C.

Given that the gradient at A is 4, determine the value of a and the value of b.

$\boxed{a = -2}$, $\boxed{b = 7}$

$\boxed{y = ax^3 + bx^2 - 10} \implies \boxed{\frac{dy}{dx} = 3ax^2 + 2bx}$

- (2,2) LIES ON THE CURVE

$2 = 8a + 4b - 10$

$12 = 8a + 4b$

$\boxed{3 = 2a + b}$

- WHEN $x = 2$, $\frac{dy}{dx} = 4$

$4 = 12a + 4b$

$\boxed{1 = 3a + b}$

$\left.\begin{array}{l} b = 3 - 2a \\ b = 1 - 3a \end{array}\right\} \implies 3 - 2a = 1 - 3a$

$a = -2$

& $b = 7$

Question 165 (***+)

The curve C has equation

$$y = 2x^3 - 6x^2 + 3x + 5.$$

The point $P(2,3)$ lies on C and the straight line L_1 is the tangent to C at P.

a) Find an equation of L_1.

The straight lines L_2 and L_3 are parallel to L_1, and they are the respective normals to C at the points Q and R.

b) Determine the x coordinate of Q and the x coordinate of R.

$\boxed{}$, $\boxed{y = 3x - 3}$, $\boxed{x = \frac{1}{3}, \frac{5}{3}}$

(a) $y = 2x^3 - 6x^2 + 3x + 5$

$\frac{dy}{dx} = 6x^2 - 12x + 3$

$\left.\frac{dy}{dx}\right|_{x=2} = 6 \times 2^2 - 12 \times 2 + 3$

$= \cancel{24 - 24} = 3$

• TANGENT AT $(2,3)$, GRADIENT 3

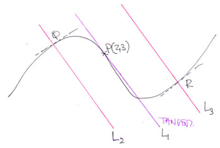
$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 3 = 3(x - 2)$

$\Rightarrow y - 3 = 3x - 6$

$\Rightarrow y = 3x - 3$

(b)



$\frac{dy}{dx} = 6x^2 - 12x + 3$

$\Rightarrow -\frac{1}{3} = 6x^2 - 12x + 3$

$\Rightarrow -1 = 18x^2 - 36x + 9$

$\Rightarrow 0 = 18x^2 - 36x + 10$

$\Rightarrow 9x^2 - 18x + 5 = 0$

$\Rightarrow (3x - 5)(3x - 1)$

$\Rightarrow x = \frac{1}{3}, \frac{5}{3}$

IN ANY ORDER

Question 166 (***+)

The point $P(2,9)$ lies on the curve C with equation

$$y = x^3 - 3x^2 + 2x + 9,\ x \in \mathbb{R},\ x \geq 1.$$

a) Find an equation of the tangent to C at P, giving the answer in the form $y = mx + c$, where m and c are constants.

The point Q also lies on C so that the tangent to C at Q is perpendicular to the tangent to C at P.

b) Show that the x coordinate of Q is

$$\frac{6+\sqrt{6}}{6}.$$

$\boxed{}$, $\boxed{y = 2x + 5}$

(a) $y = x^3 - 3x^2 + 2x + 9$

$\frac{dy}{dx} = 3x^2 - 6x + 2$

$\left.\frac{dy}{dx}\right|_{x=2} = 3\times2^2 - 6\times2 + 2$

$= \cancel{12 - 12} + 2 = 2$

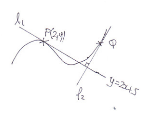
SO $y - y_0 = m(x - x_0)$

$\Rightarrow y - 9 = 2(x - 2)$

$\Rightarrow y - 9 = 2x - 4$

$\Rightarrow y = 2x + 5$

(b)



GRADIENT OF l_1 IS 2 (FOUND)

$\therefore$ GRADIENT OF l_2 MUST BE $-\frac{1}{2}$

$\therefore \frac{dy}{dx} = -\frac{1}{2}$

$\Rightarrow 3x^2 - 6x + 2 = -\frac{1}{2}$

$\Rightarrow 6x^2 - 12x + 4 = -1$

$\Rightarrow 6x^2 - 12x + 5 = 0$

BY QUADRATIC FORMULA OR BY COMPLETING THE SQUARE

$x = \frac{12 \pm \sqrt{144 - 4\times6\times5}}{2\times6}$

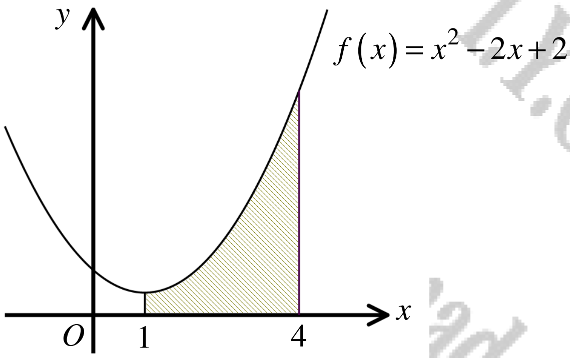
$x = \frac{12 \pm \sqrt{24}}{12}$

$x = \frac{12 \pm 2\sqrt{6}}{12}$

$x = \frac{6 \pm \sqrt{6}}{6}$

$x = \frac{6 + \sqrt{6}}{6}$ $(x \geq 1)$ AS REQUIRED

Question 167 (***+)



The curve C has equation

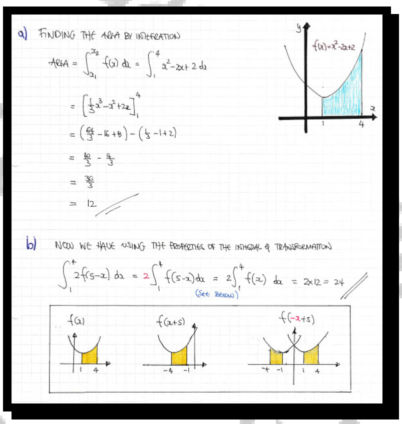
$$f(x) = x^2 - 2x + 2, \quad x \in \mathbb{R}.$$

a) Find the area of the finite region bounded by C, the x axis and the straight lines with equations $x = 1$ and $x = 4$, shown shaded in the figure above.

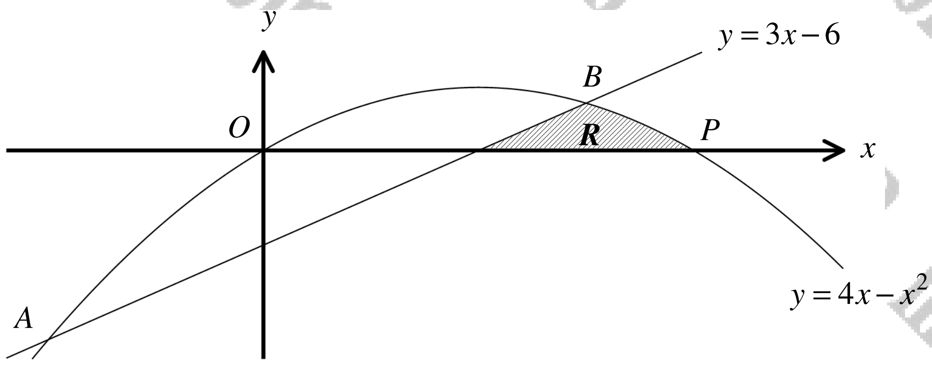
b) Hence evaluate

$$\int_1^4 2f(5-x)\ dx.$$

$\boxed{}$, $\boxed{12}$, $\boxed{24}$



Question 168 (****)



The figure above shows the graph of the curve C with equation

$$y = 4x - x^2, \; x \in \mathbb{R},$$

intersected by the straight line L with equation

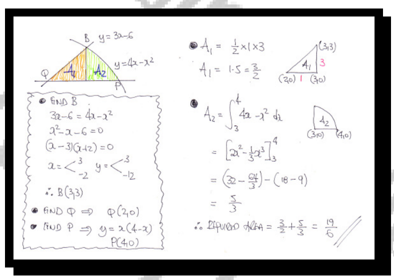
$$y = 3x - 6, \; x \in \mathbb{R}.$$

As shown in the above figure, C meets L at the points A and B, and crosses the x axis at the origin O and at the point P.

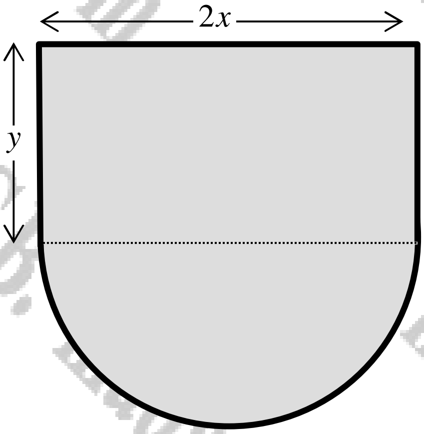
The finite region R is bounded by C, L and the x axis.

Show that the area of R, shown shaded in the figure, is $\frac{19}{6}$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 169 **(****)**



The figure above shows the design of a theatre stage which is the shape of a semicircle attached to rectangle. The diameter of the semicircle is $2x$ m and is attached to one side of the rectangle also measuring $2x$ m. The other side of the rectangle is y m.

The perimeter of the stage is 60 m.

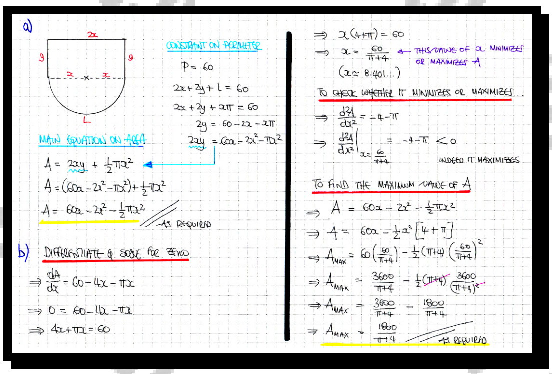
a) Show that the total area of the stage, A m^2, is given by

$$A = 60x - 2x^2 - \frac{1}{2}\pi x^2 .$$

b) Show further, by using a **differentiation** method, that the maximum area of the stage is

$$\frac{1800}{\pi + 4} \text{ m}^2 .$$

SYN-F, proof



a)

CONSTRAINT ON PERIMETER

$P = 60$

$2x + 2y + L = 60$

$2x + 2y + x\pi = 60$

$2y = 60 - 2x - x\pi$

$2xy = 60x - 2x^2 - \pi x^2$

MAIN EQUATION ON AREA

$A = 2xy + \frac{1}{2}\pi x^2$

$A = (60x - 2x^2 - \pi x^2) + \frac{1}{2}\pi x^2$

$A = 60x - 2x^2 - \frac{1}{2}\pi x^2$ // AS REQUIRED

b) DIFFERENTIATE & SOLVE FOR ZERO

$\Rightarrow \frac{dA}{dx} = 60 - 4x - \pi x$

$\Rightarrow 0 = 60 - 4x - \pi x$

$\Rightarrow 4x + \pi x = 60$

$\Rightarrow x(4 + \pi) = 60$

$\Rightarrow x = \frac{60}{\pi + 4}$ ← THIS VALUE OF x MINIMIZES OR MAXIMIZES A

$(x \simeq 8.401\ldots)$

TO CHECK WHETHER IT MINIMIZES OR MAXIMIZES...

$\Rightarrow \frac{d^2A}{dx^2} = -4 - \pi$

$\Rightarrow \left.\frac{d^2A}{dx^2}\right|_{x = \frac{60}{\pi + 4}} = -4 - \pi < 0$

INDEED IT MAXIMIZES

TO FIND THE MAXIMUM VALUE OF A

$\Rightarrow A = 60x - 2x^2 - \frac{1}{2}\pi x^2$

$\Rightarrow A = 60x - \frac{1}{2}x^2[4 + \pi]$

$\Rightarrow A_{MAX} = 60\left(\frac{60}{\pi + 4}\right) - \frac{1}{2}(\pi + 4)\left(\frac{60}{\pi + 4}\right)^2$

$\Rightarrow A_{MAX} = \frac{3600}{\pi + 4} - \frac{1}{2}\cancel{(\pi + 4)}\frac{3600}{(\pi + 4)^{\cancel{2}}}$

$\Rightarrow A_{MAX} = \frac{3600}{\pi + 4} - \frac{1800}{\pi + 4}$

$\Rightarrow A_{MAX} = \frac{1800}{\pi + 4}$ // AS REQUIRED

Question 170 (****)

The curve C and the straight line L have respective equations

$$y = 2x^2 - 6x + 5 \quad \text{and} \quad 2y + x = 4.$$

a) Find the coordinates of the points of intersection between C and L.

b) Show that L is a normal to C.

The tangent to C at the point P is parallel to L.

c) Determine the x coordinate of P.

$\boxed{}$, $\boxed{(2,1), \left(\frac{3}{4}, \frac{13}{8}\right)}$, $\boxed{x_P = \frac{11}{8}}$

(a) $y = 2x^2 - 6x + 5$, $2y + x = 4$ $\Rightarrow 2(2x^2 - 6x + 5) + x = 4$

$\Rightarrow 4x^2 - 12x + 10 + x = 4$

$\Rightarrow 4x^2 - 11x + 6 = 0$

$\Rightarrow (4x - 3)(x - 2)$

$x = \frac{4-2y}{2}$

$\Rightarrow x = 2$ or $\frac{3}{4}$, $y = \frac{4-2}{2} = 1$ or $\frac{4-\frac{3}{4}}{2} = \frac{16-3}{8} = \frac{13}{8}$

$\therefore (2,1)$ & $\left(\frac{3}{4}, \frac{13}{8}\right)$

(b) CURVE $\frac{dy}{dx} = 4x - 6$ & LINE $2y + x = 4$

$2y = -x + 4$

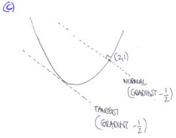
$y = -\frac{1}{2}x + 2$

$\left.\frac{dy}{dx}\right|_{x=2} = 4 \times 2 - 6 = 2$ ← TANGENT AT $(2,1)$ HAS GRADIENT 2

$\therefore$ NORMAL AT $(2,1)$ HAS GRADIENT $-\frac{1}{2}$

$\therefore$ $2y + x = 4$ IS A NORMAL TO THE CURVE AT $(2,1)$

(c)



$\therefore \frac{dy}{dx}$ OF TANGENT IS $-\frac{1}{2}$

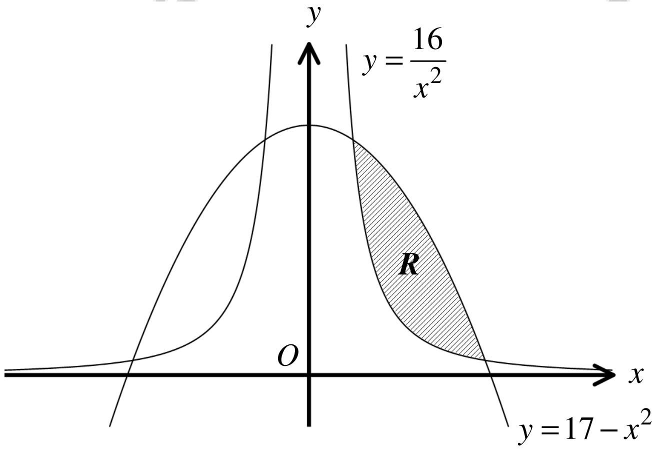
$-\frac{1}{2} = 4x - 6$

$-1 = 8x - 12$

$11 = 8x$

$x = \frac{11}{8}$

Question 171 (****)



The figure above shows the graphs of the curves with equations

$$y = \frac{16}{x^2} \quad \text{and} \quad y = 17 - x^2 .$$

The finite region R, shown shaded in the figure above, is bounded by the two curves in the first quadrant.

Find the area of R.

$\boxed{}$, $\boxed{18}$

START BY FINDING THE INTERSECTIONS OF THE TWO CURVES

$$\left.\begin{array}{l} y = \frac{16}{x^2} \\ y = 17 - x^2 \end{array}\right\} \Rightarrow \frac{16}{x^2} = 17 - x^2$$

$$\Rightarrow x^2 - 17 + \frac{16}{x^2} = 0$$

$$\Rightarrow x^4 - 17x^2 + 16 = 0$$

$$\Rightarrow (x^2 - 1)(x^2 - 16) = 0$$

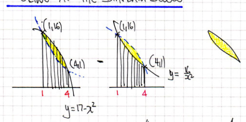
$$\Rightarrow x^2 = \begin{cases} 1 \\ 16 \end{cases}$$

$$\Rightarrow x = \begin{cases} +1 \\ +4 \end{cases} \quad \text{(FIRST QUADRANT)}$$

$$\Rightarrow y = \begin{cases} 16 \\ 1 \end{cases}$$

$$\therefore \ (1,16) \ \& \ (4,1)$$

LOOKING AT THE DIAGRAM BELOW



$$\text{AREA} = \int_1^4 17 - x^2 \, dx - \int_1^4 \frac{16}{x^2} \, dx = \int_1^4 17 - x^2 - \frac{16}{x^2} \, dx$$

$$\text{AREA} = \left[17x - \frac{1}{3}x^3 + \frac{16}{x} \right]_1^4$$

$$= \left(68 - \frac{64}{3} + 4\right) - \left(17 - \frac{1}{3} + 16\right)$$

$$= \frac{152}{3} - \frac{98}{3}$$

$$= 18$$

Question 172 (****)

The curve C and the straight line L have equations

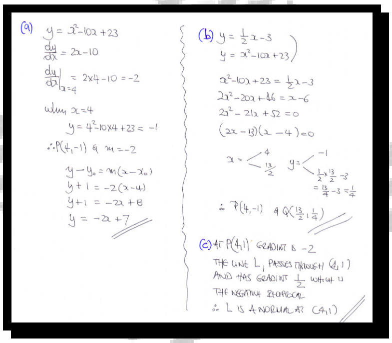
$$C: \ y = x^2 - 10x + 23 \quad \text{and} \quad L: \ y = \tfrac{1}{2}x - 3 .$$

a) Find an equation of the tangent to C at the point P, where $x = 4$.

b) Determine the coordinates of the points of intersection between L and C.

c) Show that L is a normal to C.

$\boxed{}$, $\boxed{y = -2x + 7}$, $\boxed{P(4,-1), \ Q\left(\frac{13}{2}, \frac{1}{4}\right)}$


(a) $y = x^2 - 10x + 23$

$\frac{dy}{dx} = 2x - 10$

$\left.\frac{dy}{dx}\right|_{x=4} = 2 \times 4 - 10 = -2$

when $x = 4$

$y = 4^2 - 10 \times 4 + 23 = -1$

$\therefore P(4,-1)$ & $m = -2$

$y - y_0 = m(x - x_0)$

$y + 1 = -2(x - 4)$

$y + 1 = -2x + 8$

$y = -2x + 7$

(b) $y = \frac{1}{2}x - 3$

$y = x^2 - 10x + 23$

$x^2 - 10x + 23 = \frac{1}{2}x - 3$

$2x^2 - 20x + 46 = x - 6$

$2x^2 - 21x + 52 = 0$

$(2x - 13)(x - 4) = 0$

$x = 4$ or $\frac{13}{2}$, $y = -1$ or $\frac{1}{2} \times \frac{13}{2} - 3 = \frac{13}{4} - 3 = \frac{1}{4}$

$\therefore P(4,-1)$ & $Q\left(\frac{13}{2}, \frac{1}{4}\right)$

(c) AT P(4,1) GRADIENT IS -2

THE LINE L, PASSES THROUGH (4,1)

AND HAS GRADIENT $\frac{1}{2}$ WHICH IS

THE NEGATIVE RECIPROCAL

$\therefore$ L IS A NORMAL AT (4,1)

Question 173 (****)

$$y = \sqrt[3]{x} + \frac{27}{x}, \ x \neq 0.$$

Find the range of values of x for which y is increasing.

[illegible], $\boxed{x > 27}$

$\boxed{y = \sqrt[3]{x^1} + \frac{27}{x} \quad x \neq 0}$

START BY WRITING IN INDEX NOTATION, BEFORE DIFFERENTIATING

$$y = x^{\frac{1}{3}} + 27x^{-1}$$

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} - 27x^{-2}$$

INCREASING $\Rightarrow \frac{dy}{dx} > 0$

$$\Rightarrow \frac{1}{3}x^{-\frac{2}{3}} - 27x^{-2} > 0$$

$$\Rightarrow \frac{1}{3x^{\frac{2}{3}}} - \frac{27}{x^2} > 0$$

$$\Rightarrow \frac{1}{3x^{\frac{2}{3}}} > \frac{27}{x^2} \quad (x^2 > 0)$$

$$\Rightarrow \frac{x^2}{3x^{\frac{2}{3}}} > 27$$

$$\Rightarrow x^{\frac{4}{3}} > 81$$

$$\Rightarrow \left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} > 81^{\frac{3}{4}}$$

$$\Rightarrow x^1 > \left(\sqrt[4]{81}\right)^3$$

$$\Rightarrow x > 27$$

Question 174 (****)

The points $(-2,-1)$ and $(1,-4)$ lie on the curve C with equation $y = f(x)$.

The gradient function of C is given by

$$\frac{dy}{dx} = 3x^2 + 4x + k \ ,$$

where k is a constant.

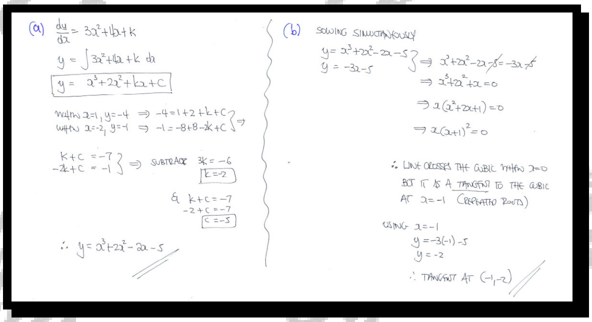
a) Find an equation of C, in the form $y = f(x)$.

The straight line L has equation

$$y = -3x - 5 .$$

b) Show that L is a tangent to C and determine further the coordinates of the point of tangency.

$\boxed{}$, $\boxed{y = x^3 + 2x^2 - 2x - 5}$, $\boxed{(-1,-2)}$



(a) $\frac{dy}{dx} = 3x^2 + 4x + k$

$y = \int 3x^2 + 4x + k \, dx$

$\boxed{y = x^3 + 2x^2 + kx + C}$

WHEN $x=1, y=-4 \Rightarrow -4 = 1 + 2 + k + C$
WHEN $x=-2, y=-1 \Rightarrow -1 = -8 + 8 - 2k + C$

$\left.\begin{array}{l} k + C = -7 \\ -2k + C = -1 \end{array}\right\} \Rightarrow$ SUBTRACT $3k = -6$
$\boxed{k = -2}$

& $k + C = -7$
$-2 + C = -7$
$\boxed{C = -5}$

$\therefore y = x^3 + 2x^2 - 2x - 5$

(b) SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l} y = x^3 + 2x^2 - 2x - 5 \\ y = -3x - 5 \end{array}\right\} \Rightarrow x^3 + 2x^2 - 2x - \cancel{5} = -3x - \cancel{5}$

$\Rightarrow x^3 + 2x^2 + x = 0$

$\Rightarrow x(x^2 + 2x + 1) = 0$

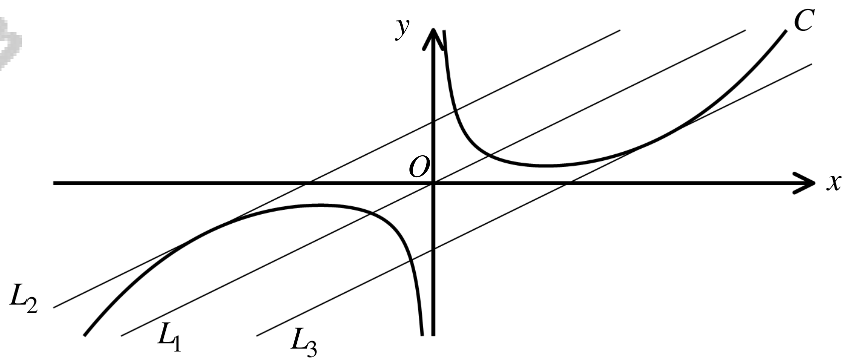
$\Rightarrow x(x+1)^2 = 0$

$\therefore$ LINE CROSSES THE CUBIC WHEN $x=0$ BUT IT IS A TANGENT TO THE CUBIC AT $x=-1$ (REPEATED ROOTS)

USING $x = -1$
$y = -3(-1) - 5$
$y = -2$

$\therefore$ TANGENT AT $(-1,-2)$

Question 175 (****)



The figure above shows the curve C with equation

$$y = \frac{2x^4 - x + 6}{6x}, \quad x \neq 0.$$

The straight line L_1 has equation $4y = 15x$.

The straight lines L_2 and L_3 are tangents to C.

Given that L_1, L_2 and L_3 are parallel to one another, determine an equation of L_2 and an equation of L_3.

$\boxed{}$, $\boxed{4y = 15x - 18 \quad \text{or} \quad y = \frac{15}{4}x - \frac{9}{2}}$, $\boxed{12y = 45x + 50 \quad \text{or} \quad y = \frac{15}{4}x + \frac{25}{6}}$

- FIRSTLY $y = \frac{15}{4}x$ IMPLIES GRADIENT IS $\frac{15}{4}$
- $y = \frac{2x^4 - x + 6}{6x} = \frac{2x^4}{6x} - \frac{x}{6x} + \frac{6}{6x} = \frac{1}{3}x^3 - \frac{1}{6} + x^{-1}$

  $\frac{dy}{dx} = x^2 - x^{-2} = x^2 - \frac{1}{x^2}$
- NOW $x^2 - \frac{1}{x^2} = \frac{15}{4}$

  $\Rightarrow x^4 - 1 = \frac{15}{4}x^2$

  $\Rightarrow 4x^4 - 4 = 15x^2$

  $\Rightarrow 4x^4 - 15x^2 - 4 = 0$

  $\Rightarrow (4x^2 + 1)(x^2 - 4) = 0$

  $\Rightarrow x^2 = 4$ or $\cancel{-\frac{1}{4}}$

  $\Rightarrow x = 2$ or -2

IF $x=2$ $y = \frac{2 \times 2^4 - 2 + 6}{6 \times 2} = \frac{36}{12} = 3$

IF $x=-2$ $y = \frac{2(-2)^4 - (-2) + 6}{6 \times (-2)} = \frac{40}{-12} = -\frac{10}{3}$

$\therefore (2, 3)$ & $(-2, -\frac{10}{3})$

THUS $\boxed{y - y_0 = m(x - x_0)}$

HENCE $y - 3 = \frac{15}{4}(x - 2)$

$4y - 12 = 15x - 30$

$4y = 15x - 18$

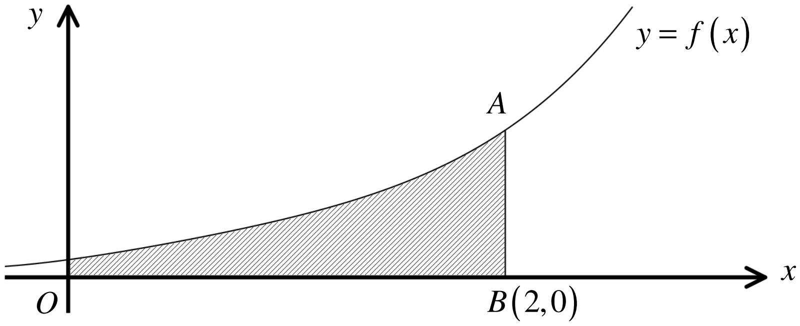
AND $y + \frac{10}{3} = \frac{15}{4}(x + 2)$

$y + \frac{10}{3} = \frac{15}{4}x + \frac{15}{2}$

$12y + 40 = 45x + 90$

$12y = 45x + 50$

Question 176 **(****)**



The figure above shows part the curve C with equation $y = f(x)$.

The gradient function of this curve is given by

$$\frac{dy}{dx} = 12x^2 - 12x + 6.$$

The point A lies on C and the point $B(2,0)$ lies on the x axis, so that the straight line segment AB is parallel to the y axis.

The area of the finite region bounded by C, the coordinate axes and the straight line segment AB, shown shaded in the figure, is 22 square units.

Find an equation of C.

$\boxed{}$, $\boxed{y = x^4 - 2x^3 + 3x^2 + 5x + 5}$

$\frac{dy}{dx} = 12x^2 - 12x + 6$

$y = \int 12x^2 - 12x + 6 \, dx$

$\boxed{y = 4x^3 - 6x^2 + 6x + C}$

Now

$\int_0^2 4x^3 - 6x^2 + 6x + C \, dx = 22$

$\left[x^4 - 2x^3 + 3x^2 + Cx\right]_0^2 = 22$

$(\cancel{16} - \cancel{16} + 12 + 2C) - (0) = 22$

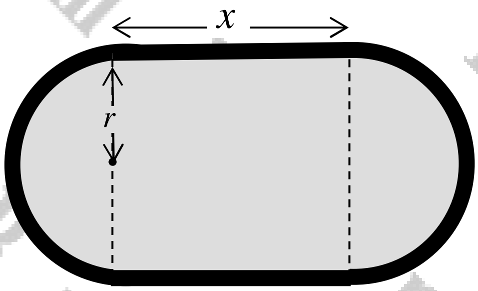
$2C + 12 = 22$

$2C = 10$

$C = 5$

$\therefore y = 4x^3 - 6x^2 + 6x + 5$

Question 177 (****)



The figure above shows the design of an athletics track inside a stadium.

The track consists of two semicircles, each of radius r m, joined up to a rectangular section of length x metres.

The total length of the track is 400 m and encloses an area of A m^2.

a) By obtaining and manipulating expressions for the total length of the track and the area enclosed by the track, show that

$$A = 400r - \pi r^2 .$$

In order to hold field events safely, it is required for the area enclosed by the track to be as large as possible.

b) Determine by **differentiation** an exact value of r for which A is stationary.

c) Show that the value of r found in part **(b)** gives the maximum value for A.

d) Show further that the maximum area the area enclosed by the track is

$$\frac{40000}{\pi} \text{ m}^2 .$$

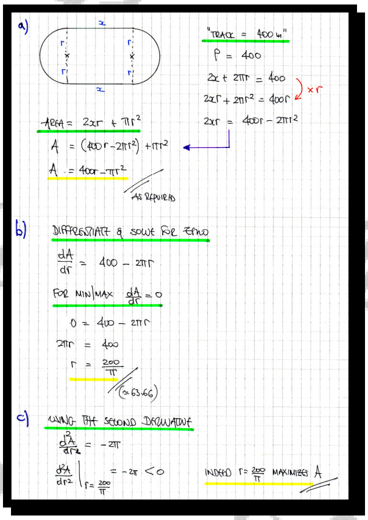
[continues overleaf]

[continued from overleaf]

The calculations for maximizing the area of the field within the track are shown to a mathematician. The mathematician agrees that the calculations are correct but he feels the resulting shape of the track might not be suitable.

e) Explain, by calculations, the mathematician's reasoning.

$\boxed{}$, $\boxed{r = \dfrac{200}{\pi} \approx 63.66}$



a) "TRACK = 400 m"

$P = 400$

$2x + 2\pi r = 400$

$2xr + 2\pi r^2 = 400r$ ($\times r$)

$2xr = 400r - 2\pi r^2$

AREA $= 2xr + \pi r^2$

$A = (400r - 2\pi r^2) + \pi r^2$

$A = 400r - \pi r^2$ // AS REQUIRED

b) DIFFERENTIATE & SOLVE FOR ZERO

$\dfrac{dA}{dr} = 400 - 2\pi r$

FOR MIN/MAX $\dfrac{dA}{dr} = 0$

$0 = 400 - 2\pi r$

$2\pi r = 400$

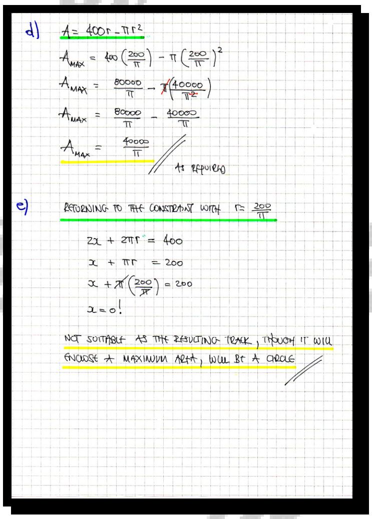
$r = \dfrac{200}{\pi}$ // (≈ 63.66)

c) USING THE SECOND DERIVATIVE

$\dfrac{d^2A}{dr^2} = -2\pi$

$\left.\dfrac{d^2A}{dr^2}\right|_{r=\frac{200}{\pi}} = -2\pi < 0$

INDEED $r = \frac{200}{\pi}$ MAXIMIZES A //



d) $A = 400r - \pi r^2$

$A_{MAX} = 400\left(\dfrac{200}{\pi}\right) - \pi\left(\dfrac{200}{\pi}\right)^2$

$A_{MAX} = \dfrac{80000}{\pi} - \cancel{\pi}\left(\dfrac{40000}{\pi^{\cancel{2}}}\right)$

$A_{MAX} = \dfrac{80000}{\pi} - \dfrac{40000}{\pi}$

$A_{MAX} = \dfrac{40000}{\pi}$ // AS REQUIRED

e) RETURNING TO THE CONSTRAINT WITH $r = \frac{200}{\pi}$

$2x + 2\pi r = 400$

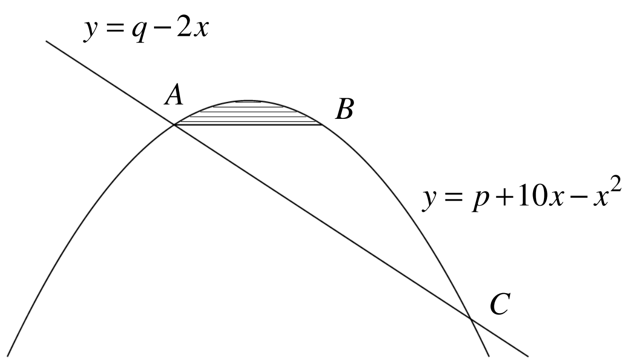
$x + \pi r = 200$

$x + \cancel{\pi}\left(\dfrac{200}{\cancel{\pi}}\right) = 200$

$x = 0$!

NOT SUITABLE AS THE RESULTING TRACK, THOUGH IT WILL ENCLOSE A MAXIMUM AREA, WILL BE A CIRCLE //

Question 178 (****)



The figure above shows a curve and a straight line with respective equations

$$y = p + 10x - x^2 \quad \text{and} \quad y = q - 2x,$$

where p and q are constants.

The curve meets the straight line at the points A and C, and the point B lies on the curve so that AB is parallel to the x axis.

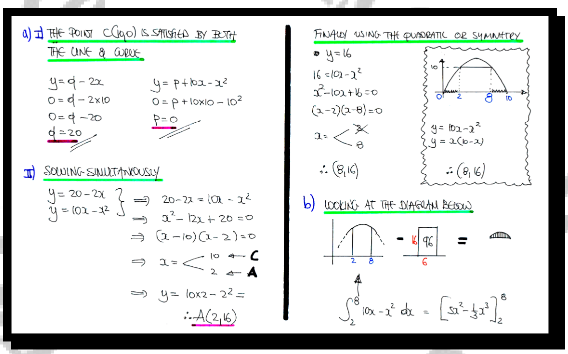
a) Given the coordinates of C are $(10,0)$ find …

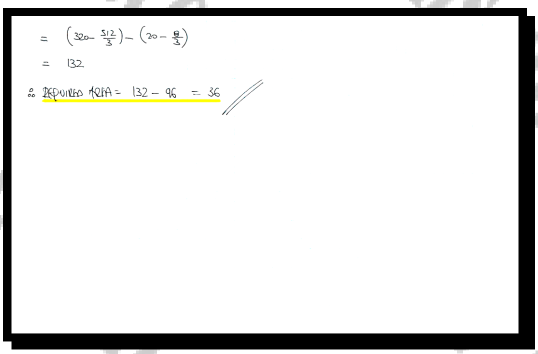
i. … the value of p and the value of q.

ii. … the coordinates of A and B.

a) Determine the value of the area of the finite region bounded by the curve and the straight line segment AB.

$\boxed{}$, $\boxed{p=0}$, $\boxed{q=20}$, $\boxed{A(2,16)}$, $\boxed{B(8,16)}$, $\boxed{\text{area}=36}$





Question 179 **(****)**

$$y = 6 + 6\sqrt{x} + 5x, \quad x \geq 0.$$

Show clearly that

$$\int \left(y^2 - x^2 \right) dx = 36x + Px^{\frac{3}{2}} + 48x^2\sqrt{x} + Qx^{\frac{5}{2}} + Rx^3 + C,$$

where P, Q and R are constants to be found, and C is an arbitrary constant.

$\boxed{P = 48}$, $\boxed{Q = 24}$, $\boxed{R = 8}$

$$y^2 = \left(6 + 6\sqrt{x} + 5x\right)^2 = \left(6 + 6x^{\frac{1}{2}} + 5x\right) \times \left(6 + 6x^{\frac{1}{2}} + 5x\right)$$

$$= 36 + 36x^{\frac{1}{2}} + 30x$$
$$36x^{\frac{1}{2}} + 36x + 30x^{\frac{3}{2}}$$
$$+ 30x + 30x^{\frac{3}{2}} + 25x^2$$
$$= 36 + 72x^{\frac{1}{2}} + 96x + 60x^{\frac{3}{2}} + 25x^2$$

$$\therefore \int y^2 - x^2 \, dx = \int \left(36 + 72x^{\frac{1}{2}} + 96x + 60x^{\frac{3}{2}} + 25x^2\right) - x^2 \, dx$$

$$= \int 36 + 72x^{\frac{1}{2}} + 96x + 60x^{\frac{3}{2}} + 24x^2 \, dx$$

$$= 36x + \frac{72}{\frac{3}{2}}x^{\frac{3}{2}} + \frac{96}{2}x^2 + \frac{60}{\frac{5}{2}}x^{\frac{5}{2}} + \frac{24}{3}x^3 + C$$

$$= 36x + \underset{P}{48}x^{\frac{3}{2}} + 48x^2 + \underset{Q}{24}x^{\frac{5}{2}} + \underset{R}{8}x^3 + C$$

$$\frac{\frac{72}{1}}{\frac{3}{2}} = \frac{144}{3} = 48 \qquad \frac{\frac{60}{1}}{\frac{5}{2}} = \frac{120}{5} = 24$$

Question 180 (****)

$$y = \frac{1}{3\sqrt{x}}\left[\frac{2}{x} - 3\right], \ x > 0.$$

Find the range of values of x for which y is decreasing.

$\boxed{\phantom{0<x}}$, $\boxed{0 < x < 2}$

TIDY THE EXPRESSION INTO INDICIAL FORM

$\Rightarrow y = \frac{1}{3\sqrt{x}}\left[\frac{2}{x} - 3\right]$

$\Rightarrow y = \frac{1}{3}x^{-\frac{1}{2}}\left[2x^{-1} - 3\right]$

$\Rightarrow y = \frac{2}{3}x^{-\frac{3}{2}} - x^{-\frac{1}{2}}$

FIND THE GRADIENT FUNCTION

$\Rightarrow \frac{dy}{dx} = -x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$

DECREASING $\Rightarrow \frac{dy}{dx} < 0$

$\Rightarrow -x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{3}{2}} < 0$

$\Rightarrow \frac{1}{2}x^{-\frac{3}{2}} < x^{-\frac{5}{2}}$

$\Rightarrow x^{-\frac{3}{2}} < 2x^{-\frac{5}{2}}$

$\Rightarrow \frac{1}{x^{3/2}} < \frac{2}{x^{5/2}}$

AS $x > 0$ WE MAY MULTIPLY THE INEQUALITY THROUGH WITHOUT REVERSING DIRECTION

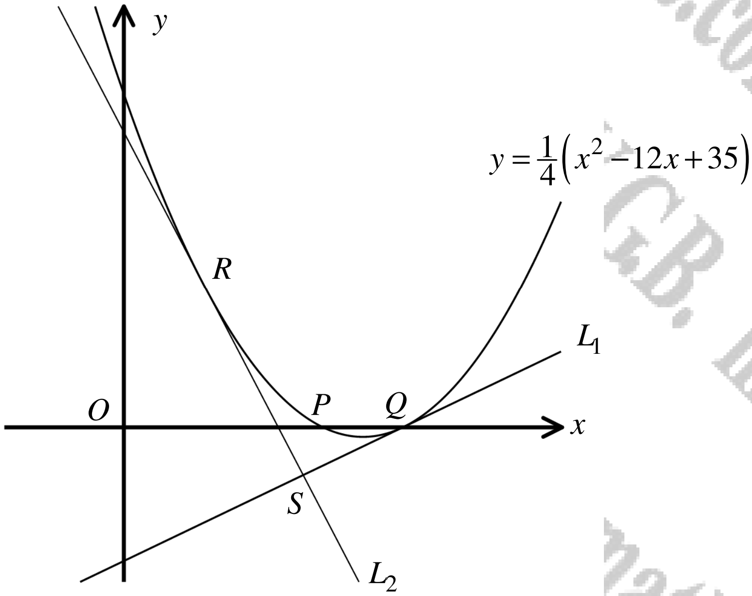
$\Rightarrow \frac{x^{5/2}}{x^{3/2}} < 2$

$\Rightarrow x < 2$

BUT SINCE $x > 0$

$0 < x < 2$

Question 181 (****)



The figure above shows the curve with equation

$$y=\frac{1}{4}\left(x^2-12x+35\right).$$

The curve crosses the x axis at the points $P(x_1,0)$ and $Q(x_2,0)$, where $x_2 > x_1$.

The tangent to the curve at Q is the straight line L_1.

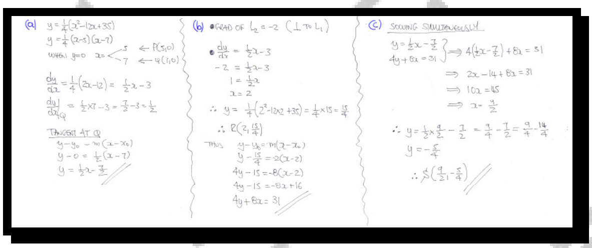
a) Find an equation of L_1.

The tangent to the curve at the point R is denoted by L_2. It is further given that L_2 meets L_1 at right angles, at the point S.

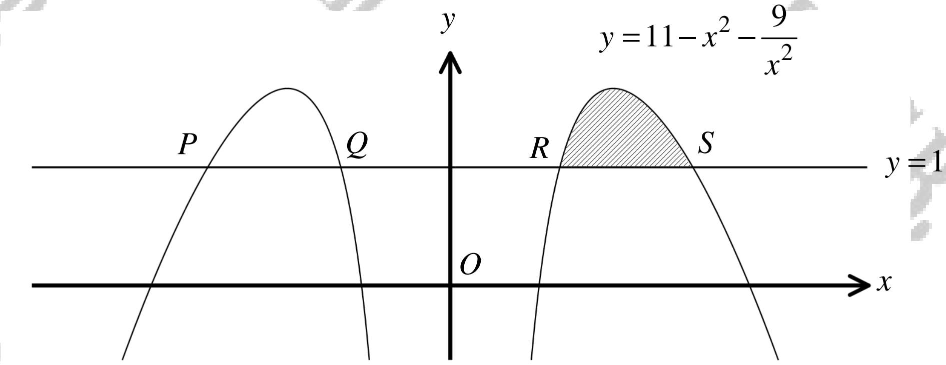
b) Find an equation of L_2.

c) Determine the exact coordinates of S.

$\boxed{}$, $\boxed{y=\frac{1}{2}x-\frac{7}{2}}$, $\boxed{4y+8x=31}$, $\boxed{S\left(\frac{9}{2},-\frac{5}{4}\right)}$



Question 182 (****)



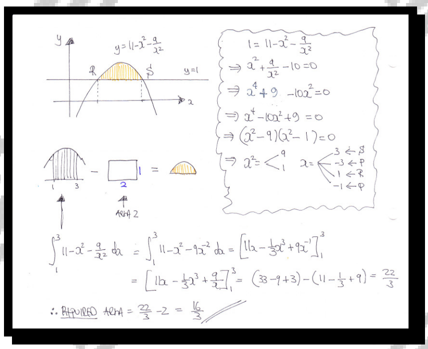
The figure above shows the curve C with equation

$$y = 11 - x^2 - \frac{9}{x^2}, \quad x \neq 0 .$$

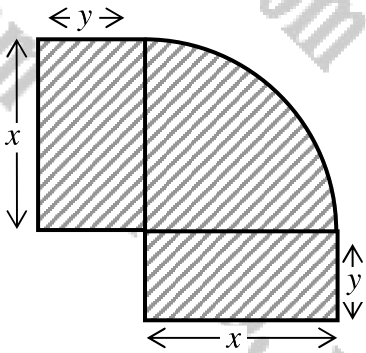
The straight line with equation $y = 1$ meets C at the points P, Q, R and S, where R and S have positive x coordinates, as shown in the figure.

Find the area of the finite region bounded by C and the line segment RS.

$\boxed{\frac{16}{3}}$



Question 183 (****)



The figure above shows the design for an earring consisting of a quarter circle with two identical rectangles attached to either straight edge of the quarter circle. The quarter circle has radius x cm and the each of the rectangles measure x cm by y cm.

The earring is assumed to have negligible thickness and treated as a two dimensional object with area 12.25 cm^2.

a) Show that the perimeter, P cm, of the earring is given by

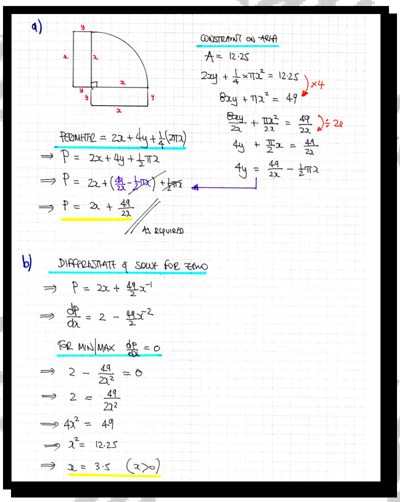
$$P = 2x + \frac{49}{2x}.$$

b) Find the value of x that makes the perimeter of the earring minimum, fully justifying that this value of x produces a minimum perimeter.

c) Show that for the value of x found in part **(b)**, the corresponding value of y is

$$\frac{7}{16}(4-\pi).$$

$\boxed{\text{MP}}$, $\boxed{x = 3.5}$



a)

CONSTRAINT ON AREA

$A = 12.25$

$2xy + \frac{1}{4} \times \pi x^2 = 12.25$ $\times 4$

$8xy + \pi x^2 = 49$

$\frac{8xy}{2x} + \frac{\pi x^2}{2x} = \frac{49}{2x}$ $\div 2x$

$4y + \frac{\pi}{2}x = \frac{49}{2x}$

$4y = \frac{49}{2x} - \frac{1}{2}\pi x$

PERIMETER $= 2x + 4y + \frac{1}{4}(2\pi x)$

$\Rightarrow P = 2x + 4y + \frac{1}{2}\pi x$

$\Rightarrow P = 2x + \left(\frac{49}{2x} - \cancel{\frac{1}{2}\pi x}\right) + \cancel{\frac{1}{2}\pi x}$

$\Rightarrow P = 2x + \frac{49}{2x}$ // AS REQUIRED

b) DIFFERENTIATE & SOLVE FOR ZERO

$\Rightarrow P = 2x + \frac{49}{2}x^{-1}$

$\Rightarrow \frac{dP}{dx} = 2 - \frac{49}{2}x^{-2}$

FOR MIN/MAX $\frac{dP}{dx} = 0$

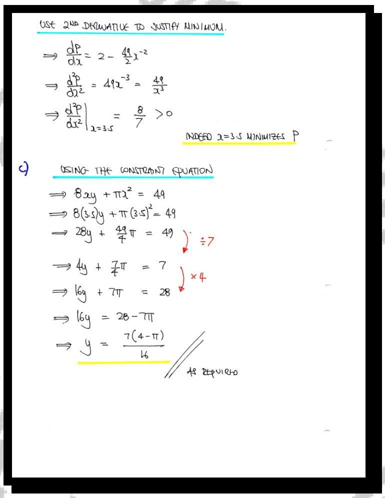
$\Rightarrow 2 - \frac{49}{2x^2} = 0$

$\Rightarrow 2 = \frac{49}{2x^2}$

$\Rightarrow 4x^2 = 49$

$\Rightarrow x^2 = 12.25$

$\Rightarrow x = 3.5 \quad (x > 0)$



USE 2ND DERIVATIVE TO JUSTIFY MINIMUM.

$\Rightarrow \frac{dP}{dx} = 2 - \frac{49}{2}x^{-2}$

$\Rightarrow \frac{d^2P}{dx^2} = 49x^{-3} = \frac{49}{x^3}$

$\Rightarrow \left.\frac{d^2P}{dx^2}\right|_{x=3.5} = \frac{8}{7} > 0$

INDEED $x = 3.5$ MINIMIZES P

c) USING THE CONSTRAINT EQUATION

$\Rightarrow 8xy + \pi x^2 = 49$

$\Rightarrow 8(3.5)y + \pi(3.5)^2 = 49$

$\Rightarrow 28y + \frac{49}{4}\pi = 49$ $\div 7$

$\Rightarrow 4y + \frac{7}{4}\pi = 7$ $\times 4$

$\Rightarrow 16y + 7\pi = 28$

$\Rightarrow 16y = 28 - 7\pi$

$\Rightarrow y = \frac{7(4-\pi)}{16}$ // AS REQUIRED

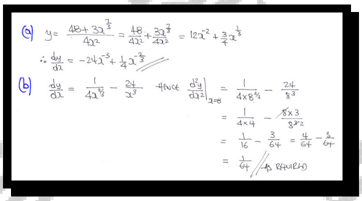
Question 184 (****) **non calculator**

$$y = \frac{48 + 3x^{\frac{7}{3}}}{4x^2}, \quad x > 0.$$

a) Find an expression for $\frac{dy}{dx}$.

b) Show that the value of $\frac{dy}{dx}$ where $x = 8$, is $\frac{1}{64}$.

$$\boxed{\frac{dy}{dx} = \frac{1}{4}x^{-\frac{2}{3}} - 24x^{-3}}$$

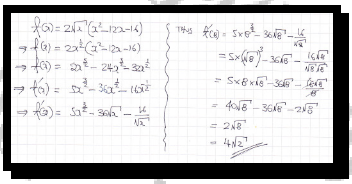


Question 185 (****) **non calculator**

$$f(x) = 2\sqrt{x}\left(x^2 - 12x - 16\right), \quad x > 0$$

Find the exact value of $f'(8)$.

$$\boxed{f'(8) = 4\sqrt{2}}$$



Question 186 (****)

A curve C has equation

$$y = \frac{3\sqrt{x} - 2}{x^{\frac{3}{2}}}, \quad x > 0.$$

a) Find the coordinates of the stationary point of C, and determine its nature.

The curve has a non stationary turning point, P.

b) Determine the coordinates of P.

$\boxed{\text{max at } (1,1)}$, $\boxed{P\left(\frac{25}{16}, \frac{112}{125}\right)}$

a) $y = \frac{3\sqrt{x} - 2}{x^{\frac{3}{2}}} = \frac{3x^{\frac{1}{2}} - 2}{x^{\frac{3}{2}}} = 3x^{-1} - 2x^{-\frac{3}{2}}$

$\frac{dy}{dx} = -3x^{-2} + 3x^{-\frac{5}{2}} = \frac{3}{x^{\frac{5}{2}}} - \frac{3}{x^2}$

$\frac{d^2y}{dx^2} = 6x^{-3} - \frac{15}{2}x^{-\frac{7}{2}} = \frac{6}{x^3} - \frac{15}{2x^{\frac{7}{2}}}$

• FOR STATIONARY POINTS $\frac{dy}{dx} = 0$

$\frac{3}{x^{\frac{5}{2}}} - \frac{3}{x^2} = 0$

$\Rightarrow \frac{3}{x^{\frac{5}{2}}} = \frac{3}{x^2}$

$\Rightarrow \frac{x^2}{x^{\frac{5}{2}}} = 1$

$\Rightarrow \frac{1}{x^{\frac{1}{2}}} = 1$

$\Rightarrow x^{\frac{1}{2}} = 1$

$\boxed{x = 1}$

$\therefore y = \frac{3\sqrt{1} - 2}{1^{3/2}} = \frac{3-2}{1} = 1$

• TO TEST THE NATURE

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{6}{1^3} - \frac{15}{2 \times 1^{\frac{7}{2}}}$

$= 6 - \frac{15}{2}$

$= -\frac{3}{2} < 0$

$\therefore (1,1)$ IS A MAX

b) FOR POINTS OF INFLEXION

$\frac{d^2y}{dx^2} = 0$

$\Rightarrow \frac{6}{x^3} - \frac{15}{2x^{\frac{7}{2}}} = 0$

$\Rightarrow \frac{6}{x^3} = \frac{15}{2x^{\frac{7}{2}}}$

$\Rightarrow 12x^{\frac{7}{2}} = 15x^3$

$\Rightarrow 4x^{\frac{7}{2}} = 5x^3$ DIVIDE BY x^3 NOTE $x>0$

$\Rightarrow 4x^{\frac{1}{2}} = 5$

$\Rightarrow x^{\frac{1}{2}} = \frac{5}{4}$

$\Rightarrow x = \frac{25}{16}$

$y = \frac{3x^{\frac{1}{2}} - 2}{x^{\frac{3}{2}}}$

$x^{\frac{1}{2}} = \frac{5}{4}$

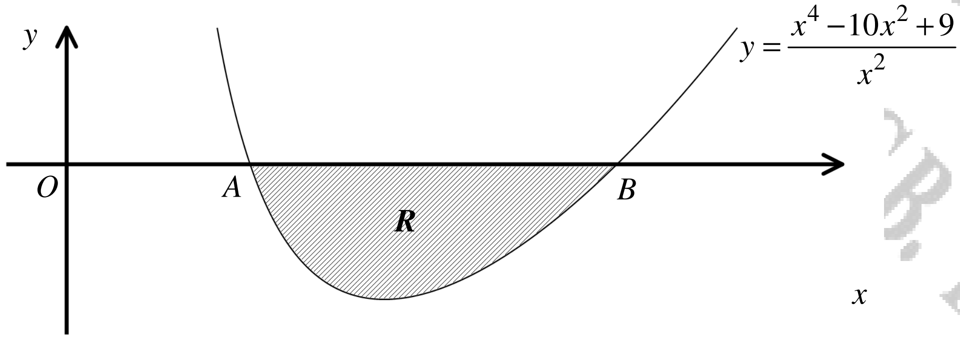
$y = \frac{3 \times \frac{5}{4} - 2}{\left(\frac{5}{4}\right)^3}$

$y = \frac{\frac{15}{4} - 2}{\frac{125}{64}}$

$y = \frac{240 - 128}{125} = \frac{112}{125}$

$\therefore P\left(\frac{25}{16}, \frac{112}{125}\right)$

Question 187 (****)



The figure above shows the graph of the curve with equation

$$y = \frac{x^4 - 10x^2 + 9}{x^2}, \quad x > 0 .$$

The curve meets the x axis at the points A and B.

The finite region R, shown shaded in the figure above, is bounded by the curve and the x axis.

Find the exact area of R.

$\boxed{\frac{16}{3}}$

$y = 0$

$\frac{x^4 - 10x^2 + 9}{x^2} = 0$

$x^4 - 10x^2 + 9 = 0$

$(x^2 - 9)(x^2 - 1) = 0$

$x^2 = 9, 1$

$x = 3, \cancel{-3}, \cancel{-1}, 1$

$\therefore A(1,0) \quad B(3,0)$

$R = \int_1^3 \frac{x^4 - 10x^2 + 9}{x^2}\, dx = \int_1^3 \frac{x^4}{x^2} - \frac{10x^2}{x^2} + \frac{9}{x^2}\, dx$

$= \int_1^3 x^2 - 10 + 9x^{-2}\, dx = \left[\frac{1}{3}x^3 - 10x - 9x^{-1}\right]_1^3$

$= \left[\frac{1}{3}x^3 - 10x - \frac{9}{x}\right]_1^3$

$= (9 - 30 - 3) - \left(\frac{1}{3} - 10 - 9\right)$

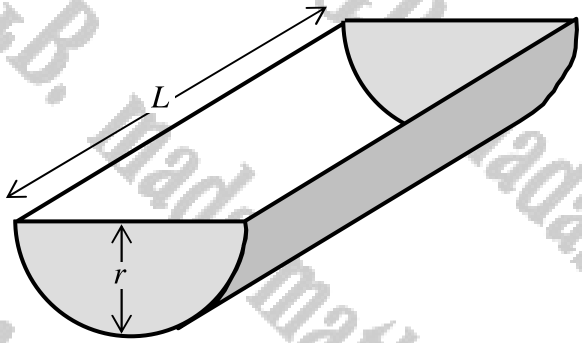
$= -\frac{16}{3}$

$\therefore$ AREA IS $\frac{16}{3}$ // MINUS INDICATES AREA IS ENTIRELY BELOW THE x AXIS

Question 188 (****)

The figure below shows the design of an animal feeder which in the shape of a hollow, open topped half cylinder, made of thin sheet metal. The radius of the semicircular ends is r cm and the length of the feeder is L cm.

The metal used in the construction of the feeder is $600\pi\ \text{cm}^2$.



a) Show that the capacity, $V\ \text{cm}^3$, of the feeder is given by

$$V = 300\pi r - \tfrac{1}{2}\pi r^3 .$$

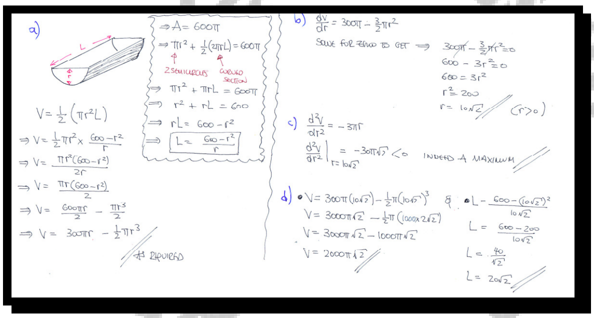
The design of the feeder is such so its capacity is maximum.

b) Determine the exact value of r for which V is stationary.

c) Show that the value of r found in part (b) gives the maximum value for V.

d) Find, in exact form, the capacity and the length of the feeder.

$\boxed{r = 10\sqrt{2} \approx 14.14}$, $\boxed{L = 20\sqrt{2} \approx 28.28}$, $\boxed{V_{\text{max}} = 2000\pi\sqrt{2} \approx 8886}$



a)

$\Rightarrow A = 600\pi$

$\Rightarrow \pi r^2 + \frac{1}{2}(2\pi r L) = 600\pi$

2 SEMICIRCLES CURVED SECTION

$\Rightarrow \pi r^2 + \pi r L = 600\pi$

$\Rightarrow r^2 + rL = 600$

$\Rightarrow rL = 600 - r^2$

$\Rightarrow \boxed{L = \frac{600 - r^2}{r}}$

$V = \frac{1}{2}(\pi r^2 L)$

$\Rightarrow V = \frac{1}{2}\pi r^2 \times \frac{600 - r^2}{r}$

$\Rightarrow V = \frac{\pi r^2(600 - r^2)}{2r}$

$\Rightarrow V = \frac{\pi r(600 - r^2)}{2}$

$\Rightarrow V = \frac{600\pi r}{2} - \frac{\pi r^3}{2}$

$\Rightarrow V = 300\pi r - \frac{1}{2}\pi r^3$

AS REQUIRED

b) $\frac{dV}{dr} = 300\pi - \frac{3}{2}\pi r^2$

SOLVE FOR ZERO TO GET $\Rightarrow$ $300\pi - \frac{3}{2}\pi r^2 = 0$

$600 - 3r^2 = 0$

$600 = 3r^2$

$r^2 = 200$

$r = 10\sqrt{2}$ $(r>0)$

c) $\frac{d^2V}{dr^2} = -3\pi r$

$\left.\frac{d^2V}{dr^2}\right|_{r=10\sqrt{2}} = -30\pi\sqrt{2} < 0$ INDEED A MAXIMUM

d) • $V = 300\pi(10\sqrt{2}) - \frac{1}{2}\pi(10\sqrt{2})^3$

$V = 3000\pi\sqrt{2} - \frac{1}{2}\pi(1000 \times 2\sqrt{2})$

$V = 3000\pi\sqrt{2} - 1000\pi\sqrt{2}$

$V = 2000\pi\sqrt{2}$

& • $L = \frac{600 - (10\sqrt{2})^2}{10\sqrt{2}}$

$L = \frac{600 - 200}{10\sqrt{2}}$

$L = \frac{40}{\sqrt{2}}$

$L = 20\sqrt{2}$

Question 189 (****) **non calculator**

A quartic curve has equation

$$f(x) = x^4 - 2x^3.$$

This curve has three turning points of whom two are stationary.

Find the coordinates of these three turning points and determine their nature.

$\boxed{\min\left(\frac{3}{2}, -\frac{27}{16}\right)}$, $\boxed{\text{point of inflection } (0,0)}$, $\boxed{\text{point of inflection } (1,-1)}$

$$\boxed{\begin{aligned} f(x) &= x^4 - 2x^3 \\ f'(x) &= 4x^3 - 6x^2 \\ f''(x) &= 12x^2 - 12x \\ f'''(x) &= 24x - 12 \end{aligned}}$$

- STATIONARY $\Rightarrow f'(x) = 0$

$4x^3 - 6x^2 = 0$

$2x^2(2x-3) = 0$

$x = \begin{cases} 0 \\ \frac{3}{2} \end{cases} \quad y = \begin{cases} 0 \\ \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 = \frac{81}{16} - \frac{27}{4} = \frac{81}{16} - \frac{108}{16} = -\frac{27}{16} \end{cases}$

$\therefore (0,0)$ & $\left(\frac{3}{2}, -\frac{27}{16}\right)$

- $f''(0) = 0 \leftarrow$ POSSIBLY INFLECTION IF $f'''(0) \neq 0$

$f'''(0) = -12 \neq 0$

$\therefore (0,0)$ STATIONARY POINT OF INFLECTION

- $f''\left(\frac{3}{2}\right) = 12 \times \frac{9}{4} - 12 \times \frac{3}{2} = 27 - 18 = 9 > 0$

$\therefore \left(\frac{3}{2}, -\frac{27}{16}\right)$ IS A MIN

- NON STATIONARY POINTS OF INFLECTION $\Longrightarrow f''(x) = 0$ & $f'''(x) \neq 0$

$12x^2 - 12x = 0$

$12x(x-1)$

$x = \begin{cases} 0 \leftarrow \text{ALREADY INVESTIGATED} \\ 1 \leftarrow y = 1 - 2 = -1 \end{cases}$

$f'''(1) = 12 \neq 0$ $\therefore (1,-1)$ IS A NON STATIONARY POINT OF INFLECTION

Question 190 **(****)** **non calculator**

A cubic curve has equation

$$y = -x^3 + 7x^2 - 8x - 10.$$

a) Find the range of values of x, for which y is increasing.

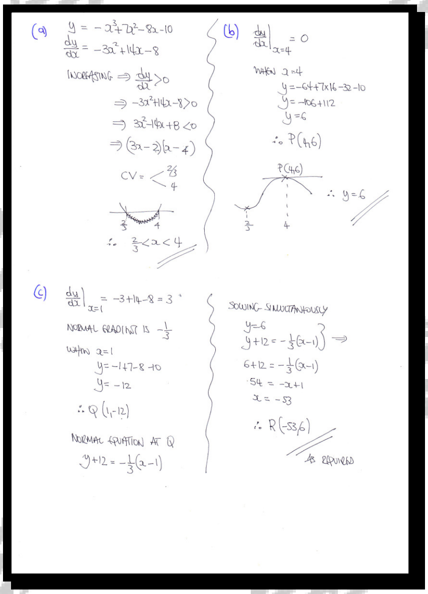
b) Find an equation of the tangent to the curve at the point P, where $x = 4$.

The point Q also lies on the curve, where $x = 1$.

The normal to the curve at Q meets the tangent to the curve at P, at the point R.

c) Show that the coordinates of R are $(-53, 6)$.

$\boxed{\frac{2}{3} < x < 4}$, $\boxed{y = 6}$



(a) $y = -x^3 + 7x^2 - 8x - 10$

$\frac{dy}{dx} = -3x^2 + 14x - 8$

INCREASING $\Rightarrow \frac{dy}{dx} > 0$

$\Rightarrow -3x^2 + 14x - 8 > 0$

$\Rightarrow 3x^2 - 14x + 8 < 0$

$\Rightarrow (3x-2)(x-4)$

CV $= \begin{cases} 2/3 \\ 4 \end{cases}$

$\frac{2}{3}$ 4

$\therefore \frac{2}{3} < x < 4$

(b) $\left.\frac{dy}{dx}\right|_{x=4} = 0$

WHEN $x = 4$

$y = -64 + 7 \times 16 - 32 - 10$

$y = -106 + 112$

$y = 6$

$\therefore P(4,6)$

P(4,6)

$\frac{2}{3}$ 4

$\therefore y = 6$

(c) $\left.\frac{dy}{dx}\right|_{x=1} = -3 + 14 - 8 = 3$

NORMAL GRADIENT IS $-\frac{1}{3}$

WHEN $x = 1$

$y = -1 + 7 - 8 - 10$

$y = -12$

$\therefore Q(1,-12)$

NORMAL EQUATION AT Q

$y + 12 = -\frac{1}{3}(x-1)$

SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l} y = 6 \\ y + 12 = -\frac{1}{3}(x-1) \end{array}\right\} \Rightarrow$

$6 + 12 = -\frac{1}{3}(x-1)$

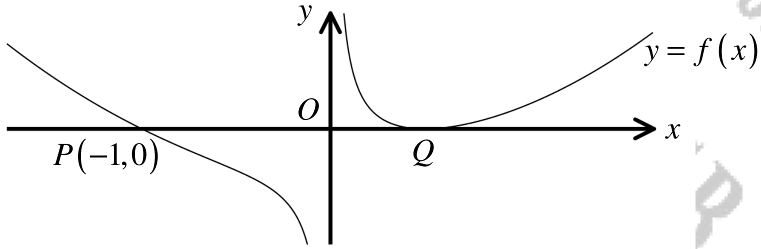
$54 = -x + 1$

$x = -53$

$\therefore R(-53,6)$

AS REQUIRED

Question 191 (****)



The figure above shows a curve with equation $y = f(x)$.

The curve meets the x axis at the points $P(-1,0)$ and Q, and its gradient function is given by

$$f'(x) = \frac{8x^3 - 1}{x^2}, \quad x \neq 0.$$

a) Find an equation of the tangent to the curve at P.

b) Find an expression for $f''(x)$.

c) Determine …

  i. … an equation of the curve.

  ii. … the coordinates of Q.

$\boxed{}$, $\boxed{y = -9x - 9}$, $\boxed{f''(x) = 8 + 2x^{-3}}$, $\boxed{y = 4x^2 + \frac{1}{x} - 3}$, $\boxed{Q\left(\frac{1}{2}, 0\right)}$

(a) AT P, $x = -1$

$f'(-1) = \frac{8(-1)^3 - 1}{(-1)^2} = \frac{-8-1}{1} = -9$

∴ TANGENT

$y - y_0 = m(x - x_0)$

$y - 0 = -9(x+1)$

$y = -9x - 9$

(b) $f'(x) = \frac{8x^3 - 1}{x^2}$

$f'(x) = \frac{8x^3}{x^2} - \frac{1}{x^2}$

$f'(x) = 8x - x^{-2}$

$\therefore f''(x) = 8 + 2x^{-3}$

OR $f''(x) = 8 + \frac{2}{x^3}$

(c)(I) IF $f'(x) = 8x - x^{-2}$

$f(x) = \int 8x - x^{-2} \, dx$

$f(x) = 4x^2 + x^{-1} + C$

$\boxed{f(x) = 4x^2 + \frac{1}{x} + C}$

APPLY CONDITION $(-1,0)$

$0 = 4(-1)^2 + \frac{1}{-1} + C$

$0 = 4 - 1 + C$

$C = -3$

$\therefore f(x) = 4x^2 + \frac{1}{x} - 3$

(II) $y = 0$ $\quad 4x^2 + \frac{1}{x} - 3 = 0$

$4x^3 + 1 - 3x = 0$

$4x^3 - 3x + 1 = 0$

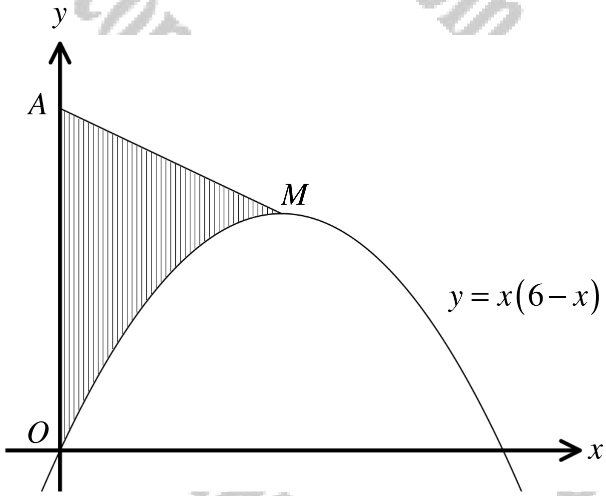
$(x+1)(4x^2 + Ax + 1) = 0$

POINT P $\quad$ BY INSPECTION $A = -4$

$(x+1)(4x^2 - 4x + 1) = 0$

$(x+1)(2x-1)^2 = 0$ $\quad \therefore Q\left(\frac{1}{2}, 0\right)$

Question 192 (****)



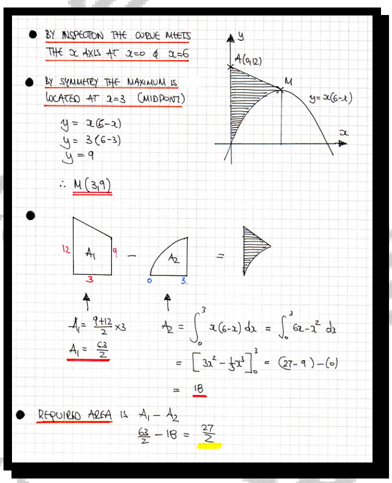
The figure above shows the curve C with equation

$$y = x(6-x), \; x \in \mathbb{R}.$$

The point M is the maximum point of C and the point A has coordinates $(0,12)$.

Find the exact area of the shaded region, bounded by the curve, the y axis and the straight line segment from A to M.

$\boxed{}$, $\boxed{\text{area} = \frac{27}{2}}$



- BY INSPECTION THE CURVE MEETS THE x AXIS AT $x=0$ & $x=6$
- BY SYMMETRY THE MAXIMUM IS LOCATED AT $x=3$ (MIDPOINT)

$y = x(6-x)$
$y = 3(6-3)$
$y = 9$

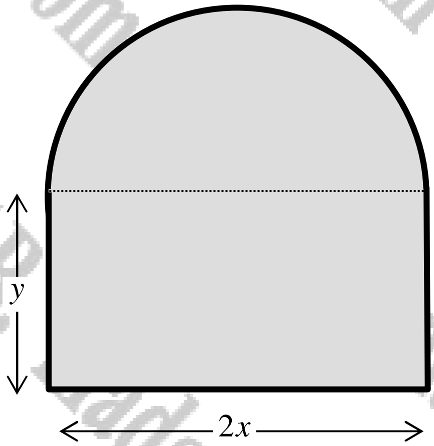
$\therefore M(3,9)$

$A_1 = \frac{9+12}{2} \times 3$
$A_1 = \frac{63}{2}$

$A_2 = \int_0^3 x(6-x)\,dx = \int_0^3 6x - x^2\,dx$
$= \left[3x^2 - \frac{1}{3}x^3\right]_0^3 = (27-9)-(0)$
$= 18$

- REQUIRED AREA IS $A_1 - A_2$
$\frac{63}{2} - 18 = \frac{27}{2}$

Question 193 (****)



The figure above shows the design of a window which is the shape of a semicircle attached to rectangle. The diameter of the semicircle is $2x$ m and is attached to one side of the rectangle also measuring $2x$ m. The other side of the rectangle is y m.

The **perimeter** of the window is 6 m.

a) Show that the total area of the window, A m^2, is given by

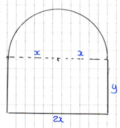
$$A = 6x - \tfrac{1}{2}(4+\pi)x^2 .$$

b) Given that the measurements of the window are such so that A is maximum, show by a method involving differentiation that this maximum value of A is

$$\frac{18}{4+\pi}.$$

proof

a)



CONSTRAINT ON PERIMETER

$P = 6$

$2x + 2y + \frac{1}{2}(2\pi x) = 6$

$2x + 2y + \pi x = 6$

$2y = 6 - 2x - \pi x$) $\times x$

$2xy = 6x - 2x^2 - \pi x^2$

$A = 2xy + \frac{1}{2}\pi x^2$

$A = (6x - 2x^2 - \pi x^2) + \frac{1}{2}\pi x^2$

$A = 6x - 2x^2 - \frac{1}{2}\pi x^2$

$A = 6x - \frac{1}{2}(4+\pi)x^2$ // AS REQUIRED

b) TO MAXIMIZE USE DIFFERENTIATION

$\frac{dA}{dx} = 6 - (4+\pi)x$

FOR MIN/MAX $\frac{dA}{dx} = 0$

$6 - (4+\pi)x = 0$

$6 = (4+\pi)x$

$x = \frac{6}{4+\pi} \approx 0.84$ m

- TO JUSTIFY THAT THIS VALUE OF x YIELDS A MAX

$\frac{dA}{dx} = 6 - (4+\pi)x$

$\frac{d^2A}{dx^2} = -(4+\pi)$

$\left.\frac{d^2A}{dx^2}\right|_{x=\frac{6}{4+\pi}} = -(4+\pi) < 0$

INDEED $x = \frac{6}{4+\pi}$ MAXIMIZES A //

- $A = 6x - \frac{1}{2}(4+\pi)x^2$

$A_{MAX} = 6\left(\frac{6}{4+\pi}\right) - \frac{1}{2}(4+\pi)\left(\frac{6}{4+\pi}\right)^2$

$A_{MAX} = \frac{36}{4+\pi} - \frac{1}{2}\cancel{(4+\pi)} \times \frac{36}{(4+\pi)^{\cancel{2}}}$

$A_{MAX} = \frac{36}{4+\pi} - \frac{18}{4+\pi}$

$A_{MAX} = \frac{18}{4+\pi}$ // AS REQUIRED

Question 194 (****)

The point $A(p,q)$ lies on the curve with equation

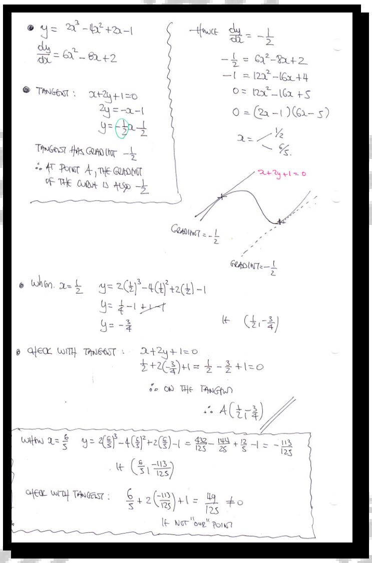
$$y = 2x^3 - 4x^2 + 2x - 1.$$

The tangent to the curve at A has equation

$$x + 2y + 1 = 0.$$

Find the coordinates of A.

$\boxed{}$, $\boxed{A\left(\frac{1}{2}, -\frac{3}{4}\right)}$



- $y = 2x^3 - 4x^2 + 2x - 1$

$\frac{dy}{dx} = 6x^2 - 8x + 2$

- TANGENT: $x + 2y + 1 = 0$

$2y = -x - 1$

$y = -\frac{1}{2}x - \frac{1}{2}$

TANGENT HAS GRADIENT $-\frac{1}{2}$

$\therefore$ AT POINT A, THE GRADIENT OF THE CURVE IS ALSO $-\frac{1}{2}$

HENCE $\frac{dy}{dx} = -\frac{1}{2}$

$-\frac{1}{2} = 6x^2 - 8x + 2$

$-1 = 12x^2 - 16x + 4$

$0 = 12x^2 - 16x + 5$

$0 = (2x-1)(6x-5)$

$x = \frac{1}{2}$ or $\frac{6}{5}$.

- WHEN $x = \frac{1}{2}$ $\quad y = 2\left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 1$

$y = \frac{1}{4} - 1 \cancel{+1-1}$

$y = -\frac{3}{4}$ $\quad$ i.e. $\left(\frac{1}{2}, -\frac{3}{4}\right)$

- CHECK WITH TANGENT: $x + 2y + 1 = 0$

$\frac{1}{2} + 2\left(-\frac{3}{4}\right) + 1 = \frac{1}{2} - \frac{3}{2} + 1 = 0$

$\therefore$ ON THE TANGENT

$\therefore A\left(\frac{1}{2}, -\frac{3}{4}\right)$

WHEN $x = \frac{6}{5}$ $\quad y = 2\left(\frac{6}{5}\right)^3 - 4\left(\frac{6}{5}\right)^2 + 2\left(\frac{6}{5}\right) - 1 = \frac{432}{125} - \frac{144}{25} + \frac{12}{5} - 1 = -\frac{113}{125}$

i.e. $\left(\frac{6}{5}, \frac{-113}{125}\right)$

CHECK WITH TANGENT: $\frac{6}{5} + 2\left(\frac{-113}{125}\right) + 1 = \frac{49}{125} \neq 0$

i.e. NOT "OUR" POINT

Question 195 (****) **non calculator**

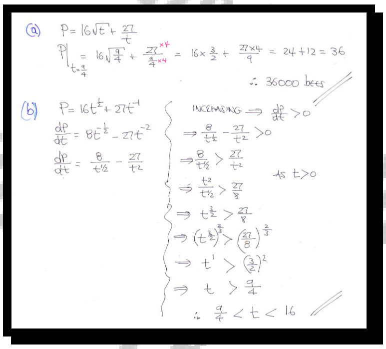
The population, P in thousands, of a colony of wild bees, t weeks after a certain instant, is given by the formula

$$P = 16\sqrt{t} + \frac{27}{t}, \quad 0 < t < 16.$$

a) Calculate the number of bees in the colony after $2\frac{1}{4}$ weeks.

b) Find the range of values of t during which the bees' population is increasing.

$\boxed{}$, $\boxed{P = 36000}$, $\boxed{\frac{9}{4} < t < 16}$



(a) $P = 16\sqrt{t} + \frac{27}{t}$

$P\big|_{t=\frac{9}{4}} = 16\sqrt{\frac{9}{4}} + \frac{27^{\times 4}}{\frac{9}{4}\times 4} = 16 \times \frac{3}{2} + \frac{27\times 4}{9} = 24 + 12 = 36$

$\therefore$ 36000 bees

(b) $P = 16t^{\frac{1}{2}} + 27t^{-1}$

$\frac{dP}{dt} = 8t^{-\frac{1}{2}} - 27t^{-2}$

$\frac{dP}{dt} = \frac{8}{t^{1/2}} - \frac{27}{t^2}$

INCREASING $\Rightarrow \frac{dP}{dt} > 0$

$\Rightarrow \frac{8}{t^{\frac{1}{2}}} - \frac{27}{t^2} > 0$

$\Rightarrow \frac{8}{t^{1/2}} > \frac{27}{t^2}$

As $t > 0$

$\Rightarrow \frac{t^2}{t^{1/2}} > \frac{27}{8}$

$\Rightarrow t^{\frac{3}{2}} > \frac{27}{8}$

$\Rightarrow \left(t^{\frac{3}{2}}\right)^{\frac{2}{3}} > \left(\frac{27}{8}\right)^{\frac{2}{3}}$

$\Rightarrow t^1 > \left(\frac{3}{2}\right)^2$

$\Rightarrow t > \frac{9}{4}$

$\therefore \frac{9}{4} < t < 16$

Question 196 **(****)**

A curve has equation

$$y = x\left(x^2 - 128\sqrt{x}\right), \qquad x \in \mathbb{R}, \qquad x > 0.$$

The curve has a single stationary point with coordinates $\left(2^{\alpha}, -2^{\beta}\right)$, where α and β are positive integers.

Find the value of β and justify that the stationary point is a local minimum.

$\boxed{}$, $\boxed{\beta = 12}$

REWRITE THE EQUATION IN INDICIAL FORM & DIFFERENTIATE

$\Rightarrow y = x\left(x^2 - 128\sqrt{x}\right)$

$\Rightarrow y = x\left(x^2 - 128x^{\frac{1}{2}}\right)$

$\Rightarrow y = x^3 - 128x^{\frac{3}{2}}$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 192x^{\frac{1}{2}}$

FOR STATIONARY POINTS SET $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 192x^{\frac{1}{2}} = 0$

$\Rightarrow x^2 - 64x^{\frac{1}{2}} = 0$

$\Rightarrow x^2 = 64x^{\frac{1}{2}}$

$\Rightarrow \frac{x^2}{x^{\frac{1}{2}}} = 64$ (WE ARE NOT CONCERNED WITH $x=0$)

$\Rightarrow x^{\frac{3}{2}} = 64$

$\Rightarrow \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = (64)^{\frac{2}{3}}$

$\Rightarrow x^1 = \left(\sqrt[3]{64}\right)^2$

$\Rightarrow x = 16$

CHECK THE NATURE OF THE POINT BY THE SECOND DERIVATIVE TEST

$\Rightarrow \frac{d^2y}{dx^2} = 6x - 96x^{-\frac{1}{2}}$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=16} = 6 \times 16 - 96 \times 16^{-\frac{1}{2}} = 96 - 96 \times \frac{1}{4} = 96 - 24 = 72 > 0$

LOCAL MINIMUM

FINALLY TO FIND THE y CO-ORDINATE IN THE REQUIRED FORM

$y = x\left(x^2 - 128\sqrt{x}\right)$

$y = 16\left(16^2 - 128\sqrt{16}\right)$

$y = -4096$

$y = -2^{12}$ (TRIAL & ERROR OF POWER OF 2)

$\therefore$ LOCAL MINIMUM AT $\left(16, -2^{12}\right)$

Question 197 **(****)**

The cubic curve with equation

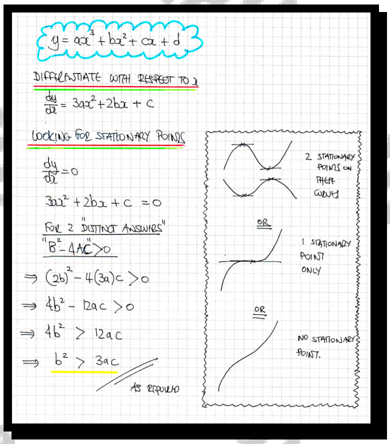
$$y = ax^3 + bx^2 + cx + d \, ,$$

where a, b, c are non zero constants and d is a constant, has one local maximum and one local minimum.

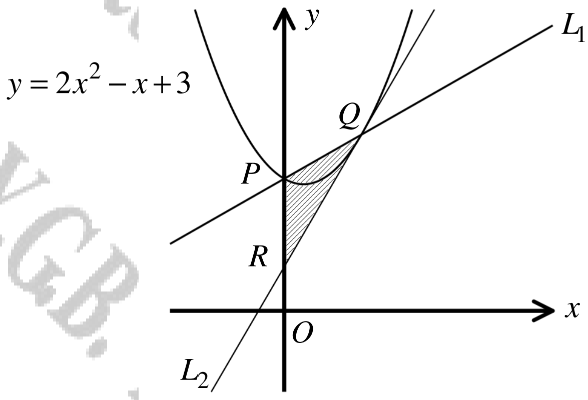
Show clearly that

$$b^2 > 3ac$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 198 (****)



The figure above shows the curve C with equation

$$y = 2x^2 - x + 3.$$

C crosses the y axis at the point P. The normal to C at P is the straight line L_1.

a) Find an equation of L_1.

L_1 meets the curve again at the point Q.

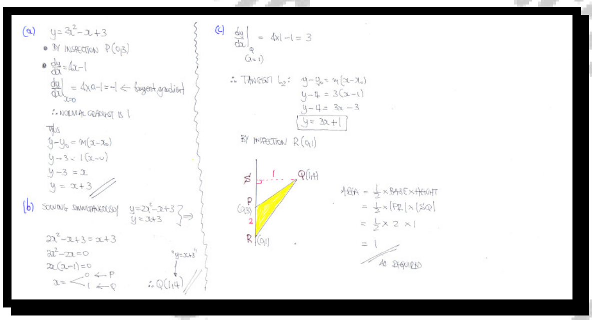
b) Determine the coordinates of Q.

The tangent to C at Q is the straight line L_2.

L_2 meets the y axis at the point R.

c) Show that the area of the triangle PQR is one square unit.

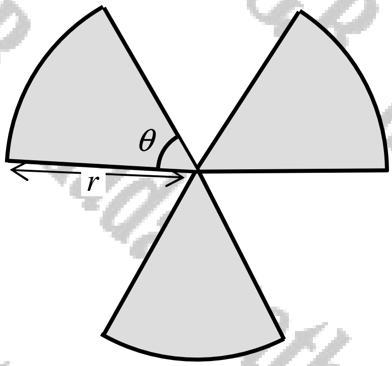
$\boxed{}$, $\boxed{y = x + 3}$, $\boxed{Q(1,4)}$



Question 199 (****)

The figure below shows the design of a hazard warning logo which consists of three identical sectors of radius r cm, joined together at the centre.

Each sector subtends an angle θ radians at the centre and the sectors are equally spaced so that the logo has rotational symmetry of order 3.



The area of the logo is 75 cm^2.

a) Show that the perimeter P cm of the logo is given by

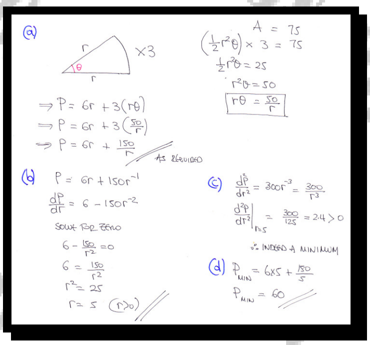
$$P = 6r + \frac{150}{r}.$$

b) Determine by differentiation the value of r for which P is stationary.

c) Show that the value of r found in part (b) gives the minimum value for P.

d) Find the minimum perimeter of the feeder.

$\boxed{r = 5}$, $\boxed{P_{\text{min}} = 60}$



Question 200 (****)

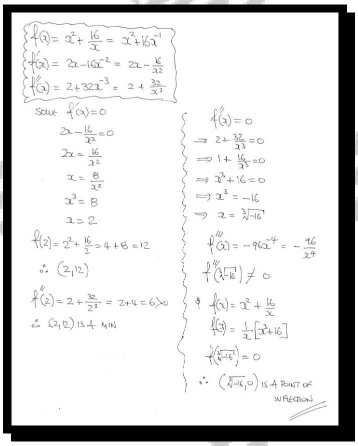
$$f(x) = x^2 + \frac{16}{x}, \ x \neq 0.$$

The curve C has equation $y = f(x)$.

Show that C has two turning points, of which one is stationary, and the other is a non stationary point of inflection.

Determine the exact coordinates of each point.

$\boxed{\text{point of inflection at } \left(\sqrt[3]{-16}, 0\right), \min(2,12)}$



$f(x) = x^2 + \frac{16}{x} = x^2 + 16x^{-1}$

$f'(x) = 2x - 16x^{-2} = 2x - \frac{16}{x^2}$

$f''(x) = 2 + 32x^{-3} = 2 + \frac{32}{x^3}$

SOLVE $f'(x) = 0$

$2x - \frac{16}{x^2} = 0$

$2x = \frac{16}{x^2}$

$x = \frac{8}{x^2}$

$x^3 = 8$

$x = 2$

$f(2) = 2^2 + \frac{16}{2} = 4 + 8 = 12$

$\therefore (2, 12)$

$f''(2) = 2 + \frac{32}{2^3} = 2 + 4 = 6 > 0$

$\therefore (2,12)$ IS A MIN

$f''(x) = 0$

$\Rightarrow 2 + \frac{32}{x^3} = 0$

$\Rightarrow 1 + \frac{16}{x^3} = 0$

$\Rightarrow x^3 + 16 = 0$

$\Rightarrow x^3 = -16$

$\Rightarrow x = \sqrt[3]{-16}$

$f'''(x) = -96x^{-4} = -\frac{96}{x^4}$

$f'''(\sqrt[3]{-16}) \neq 0$

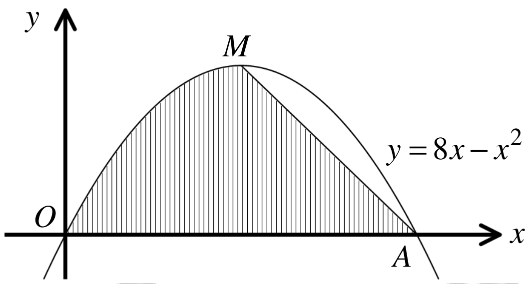
& $f(x) = x^2 + \frac{16}{x}$

$f(x) = \frac{1}{x}\left[x^3 + 16\right]$

$f(\sqrt[3]{-16}) = 0$

$\therefore (\sqrt[3]{-16}, 0)$ IS A POINT OF INFLECTION

Question 201 (****)



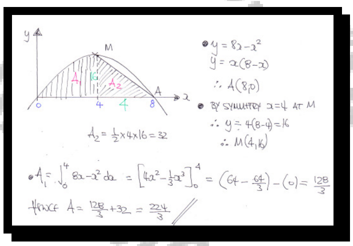
The figure above shows the quadratic curve with equation

$$y = 8x - x^2, \ x \in \mathbb{R}.$$

The point M is the maximum point of the curve and the point A is one of the curve's x intercepts.

Find the exact area of the shaded region, bounded by the curve, the x axis and the straight line segment from A to M.

$\boxed{}$, $\boxed{\text{area} = \frac{224}{3}}$



Question 202 (****)

The point P, whose x coordinate is $\frac{1}{4}$, lies on the curve with equation

$$y=\frac{k+4x\sqrt{x}}{7x}, \quad x\in\mathbb{R}, \quad x>0,$$

where k is a non zero constant.

a) Determine, in terms of k, the gradient of the curve at P.

The tangent to the curve at P is parallel to the straight line with equation

$$44x+7y-5=0.$$

b) Find an equation of the tangent to the curve at P.

$\boxed{}$, $\boxed{\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}}=\frac{4-16k}{7}}$, $\boxed{44x+7y=25}$

a) TIDY THE EQUATION INTO INDICIAL FORM AND DIFFERENTIATE

- $y=\frac{4x\sqrt{x}+k}{7x}=\frac{4x\sqrt{x}}{7x}+\frac{k}{7x}=\frac{4}{7}x^{\frac{1}{2}}+\frac{k}{7}x^{-1}$
- $\frac{dy}{dx}=\frac{2}{7}x^{-\frac{1}{2}}-\frac{k}{7}x^{-2}$
- $\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}}=\frac{2}{7}\left(\frac{1}{4}\right)^{-\frac{1}{2}}-\frac{k}{7}\left(\frac{1}{4}\right)^{-2}=\frac{2}{7}\times 2-\frac{k}{7}\times 16=\frac{4}{7}-\frac{16}{7}k$

b) REARRANGE THE EQUATION OF THE LINE TO "READ" THE GRADIENT

$\Rightarrow 7y+44x-5=0$
$\Rightarrow 7y=-44x+5$
$\Rightarrow y=-\frac{44}{7}x+\frac{5}{7}$

HENCE THE GRADIENT AT P WILL BE $-\frac{44}{7}$ (PARALLEL)

$\Rightarrow \frac{4}{7}-\frac{16}{7}k=-\frac{44}{7}$
$\Rightarrow 4-16k=-44$
$\Rightarrow 48=16k$
$\Rightarrow k=3$

FIND THE y CO-ORDINATE OF P

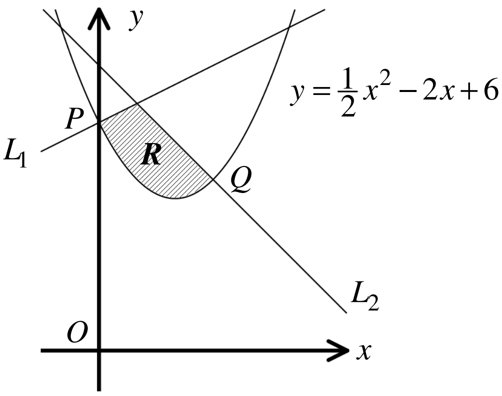
$y=\frac{4x\sqrt{x}+3}{7x}=\frac{4\times\frac{1}{4}\times\sqrt{\frac{1}{4}}+3}{7\times\frac{1}{4}}=\frac{\left(\frac{1}{2}+3\right)^{\times 4}}{\left(\frac{7}{4}\right)^{\times 4}}=\frac{2+12}{7}=2$

EQUATION OF TANGENT AT $P\left(\frac{1}{4},2\right)$

$y-y_0=m(x-x_0)$
$y-2=-\frac{44}{7}\left(x-\frac{1}{4}\right)$
$7y-14=-44x+11$

$\therefore 44x+7x=25$

Question 203 (****)



The figure above shows the graph of the curve C with equation

$$y=\frac{1}{2}x^2-2x+6.$$

The point P is the point where C meets the y axis so that the straight line L_1 is the normal to C at P.

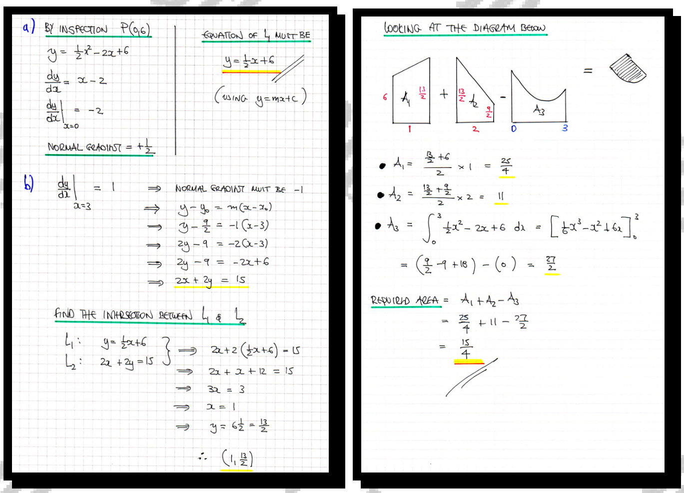
a) Find an equation for L_1.

The point $Q\left(3,\frac{9}{2}\right)$ lies on C and the straight line L_2 is the normal to C at Q.

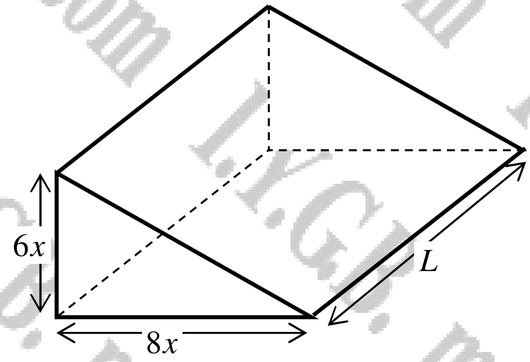
The finite region R, shown shaded in the figure above, is bounded by L_1, L_2 and C.

b) Find the area of R.

$\boxed{}$, $\boxed{y=\frac{1}{2}x+6}$, $\boxed{\frac{15}{4}}$



Question 204 (****)



The figure above shows a triangular prism with a volume of 960 cm^3.

The triangular faces of the prism are right angled with a base $8x$ cm and a height of $6x$ cm. The length of the prism is L cm.

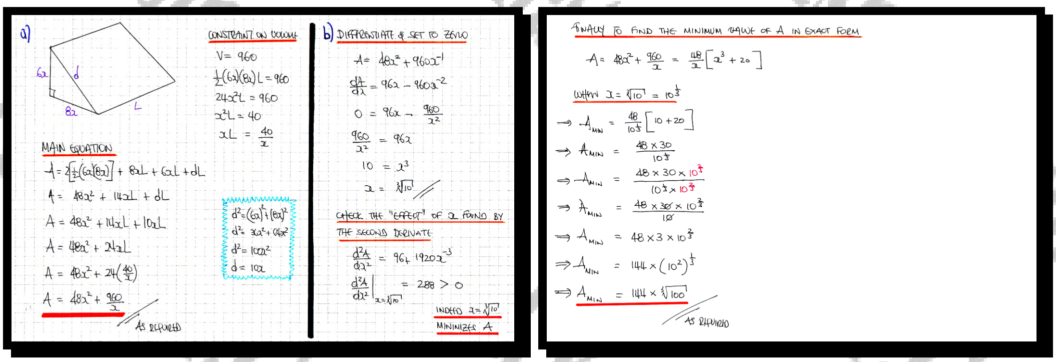
a) Show that the surface area of the prism, $A \text{ cm}^2$, is given by

$$A = 48x^2 + \frac{960}{x}.$$

b) Determine an exact value of x for which A is stationary and show that this value of x minimizes A.

c) Show further that the minimum surface area of the prism is $144\sqrt[3]{100} \text{ cm}^2$.

SYN-Z , $\boxed{x = \sqrt[3]{10} \approx 2.15}$



Question 205 (****) non calculator

$$y = \frac{1}{3\sqrt{x}}\left(\frac{2}{x} - 3\right), \; x \in \mathbb{R}, \; x > 0 .$$

Show that the value of $\dfrac{d^2y}{dx^2}$ where $x = 2$, is $\dfrac{\sqrt{2}}{16}$.

$\boxed{\text{proof}}$

$y = \frac{1}{3\sqrt{x}}\left[\frac{2}{x} - 3\right]$

$y = \frac{1}{3}x^{-\frac{1}{2}}\left[2x^{-1} - 3\right]$

$y = \frac{2}{3}x^{-\frac{3}{2}} - x^{-\frac{1}{2}}$

$\frac{dy}{dx} = -x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$

$\frac{d^2y}{dx^2} = \frac{5}{2}x^{-\frac{7}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

$\frac{d^2y}{dx} = \frac{5}{2x^{\frac{7}{2}}} - \frac{3}{4x^{\frac{5}{2}}}$

Thus $\left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{5}{2 \times 2^{\frac{7}{2}}} - \frac{3}{4 \times 2^{\frac{5}{2}}}$

$= \frac{5 \times 2^{\frac{1}{2}}}{2 \times 2^{\frac{7}{2}} \times 2^{\frac{1}{2}}} - \frac{3 \times 2^{\frac{1}{2}}}{4 \times 2^{\frac{5}{2}} \times 2^{\frac{1}{2}}}$

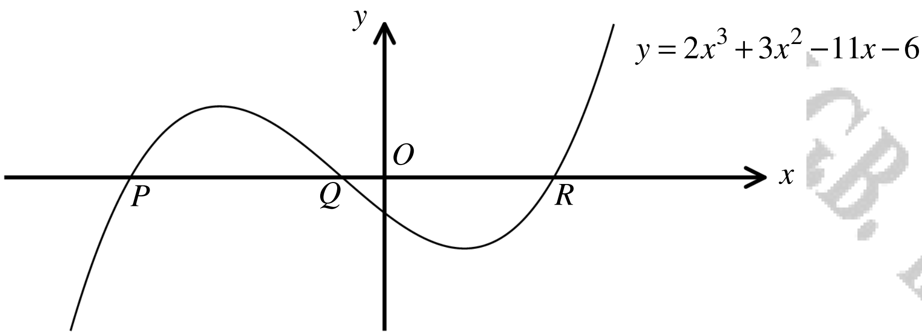
$= \frac{5 \times 2^{\frac{1}{2}}}{2 \times 2^{4}} - \frac{3 \times 2^{\frac{1}{2}}}{4 \times 2^{3}}$

$= \frac{5\sqrt{2}}{32} - \frac{3\sqrt{2}}{32}$

$= \frac{2\sqrt{2}}{32}$

$= \frac{\sqrt{2}}{16}$ // AS REQUIRED

Question 206 (****)



The figure above shows the curve with equation

$$y = 2x^3 + 3x^2 - 11x - 6.$$

The curve crosses the x axis at the points P, Q and $R(2,0)$.

The tangent to the curve at R is the straight line L_1.

a) Find an equation of L_1.

The normal to the curve at P is the straight line L_2.

The point S is the point of intersection between L_1 and L_2.

b) Show that $\measuredangle PSR = 90°$.

$\boxed{}$, $\boxed{y = 25x - 50}$

a) OBTAIN THE GRADIENT AT $R(2,0)$

$y = 2x^3 + 3x^2 - 11x - 6$

$\frac{dy}{dx} = 6x^2 + 6x - 11$

$\left.\frac{dy}{dx}\right|_{x=2} = 6 \times 2^2 + 6 \times 2 - 11 = 24 + 12 - 11 = 25$

EQUATION OF TANGENT AT R

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 0 = 25(x - 2)$

$\Rightarrow y = 25(x - 2)$

b) START BY OBTAINING THE CO-ORDINATES OF P

$$\begin{array}{r|l} & 2x^2 + 7x + 3 \\ \hline x-2 & 2x^3 + 3x^2 - 11x - 6 \\ & -2x^3 + 4x^2 \\ \hline & \quad 7x^2 - 11x - 6 \\ & \quad -7x^2 + 14x \\ \hline & \qquad 3x - 6 \\ & \qquad -3x + 6 \\ \hline & \qquad\quad 0 \end{array}$$

$\therefore y = (x-2)(2x^2 + 7x + 3)$

$y = (x-2)(2x+1)(x+3)$

$\therefore P(-3,0)$

FIND THE GRADIENT AT $P(-3,0)$

$\frac{dy}{dx} = 6x^2 + 6x - 11$

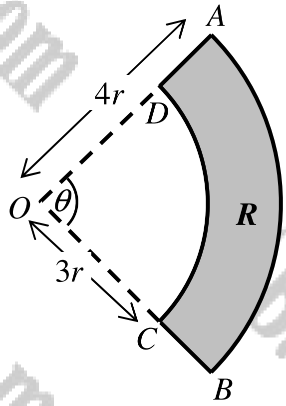
$\left.\frac{dy}{dx}\right|_{x=-3} = 6(-3)^2 + 6(-3) - 11 = 54 - 18 - 11 = 25$

NORMAL AT P HAS GRADIENT $-\frac{1}{25}$

$\therefore$ TANGENT AT R HAS GRADIENT 25
NORMAL AT P HAS GRADIENT $-\frac{1}{25}$) NEGATIVE RECIPROCALS

$\therefore$ THEY MEET AT RIGHT ANGLES $\Rightarrow P\hat{S}R = 90°$

Question 207 (****)



The figure above shows a circular sector OAB of radius $4r$ subtending an angle θ radians at the centre O. Another circular sector OCD of radius $3r$ also subtending an angle θ radians at the centre O is removed from the first sector leaving the shaded region R.

It is given that R has an area of 50 square units.

a) Show that the perimeter P, of the region R, is given by

$$P = 2r + \frac{100}{r}.$$

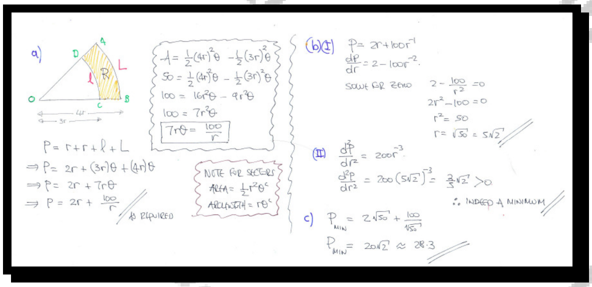
b) Given that the value of r can vary, …

i. … find an exact value of r for which P is stationary.

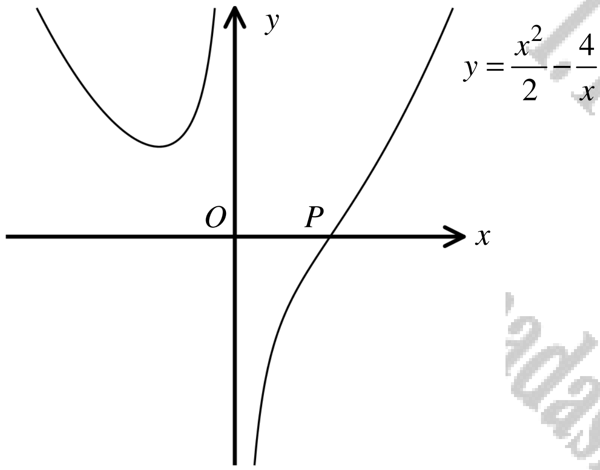
ii. … show clearly that the value of r found above gives the minimum value for P.

c) Calculate the minimum value of P.

$\boxed{}$, $\boxed{r = 5\sqrt{2} \approx 7.07}$, $\boxed{P_{\text{min}} = 20\sqrt{2} \approx 28.28}$



Question 208 (****)



The figure above shows the curve C with equation

$$y = \frac{x^2}{2} - \frac{4}{x}, \quad x \neq 0.$$

The curve crosses the x axis at the point P.

The straight line L is the normal to C at P.

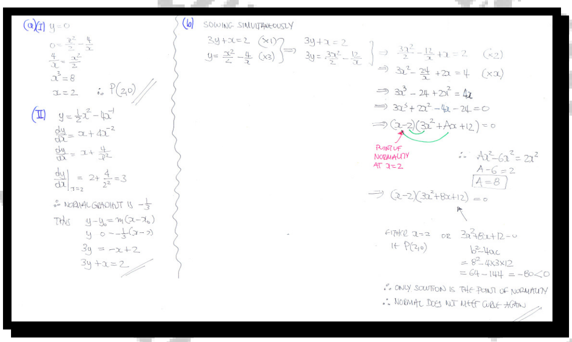
a) Find …

  i. … the coordinates of P.

  ii. … an equation of L.

b) Show that L does not meet C again.

$\boxed{}$, $\boxed{P(2,0)}$, $\boxed{x+3y=2}$



(a)(I) $y=0$

$0 = \frac{x^2}{2} - \frac{4}{x}$

$\frac{4}{x} = \frac{x^2}{2}$

$x^3 = 8$

$x = 2$ $\therefore P(2,0)$

(II) $y = \frac{1}{2}x^2 - 4x^{-1}$

$\frac{dy}{dx} = x + 4x^{-2}$

$\frac{dy}{dx} = x + \frac{4}{x^2}$

$\left.\frac{dy}{dx}\right|_{x=2} = 2 + \frac{4}{2^2} = 3$

$\therefore$ NORMAL GRADIENT IS $-\frac{1}{3}$

THUS $y - y_0 = m(x - x_0)$

$y - 0 = -\frac{1}{3}(x-2)$

$3y = -x + 2$

$3y + x = 2$

(b) SOLVING SIMULTANEOUSLY

$3y + x = 2$ $(\times 1)$, $y = \frac{x^2}{2} - \frac{4}{x}$ $(\times 3)$ $\Rightarrow$ $3y + x = 2$, $3y = \frac{3x^2}{2} - \frac{12}{x}$

$\Rightarrow \frac{3x^2}{2} - \frac{12}{x} + x = 2$ $(\times 2)$

$\Rightarrow 3x^2 - \frac{24}{x} + 2x = 4$ $(\times x)$

$\Rightarrow 3x^3 - 24 + 2x^2 = 4x$

$\Rightarrow 3x^3 + 2x^2 - 4x - 24 = 0$

$\Rightarrow (x-2)(3x^2 + Ax + 12) = 0$

POINT OF NORMALITY AT $x=2$

$\therefore Ax^2 - 6x^2 = 2x^2$

$A - 6 = 2$

$\boxed{A = 8}$

$\Rightarrow (x-2)(3x^2+8x+12) = 0$

EITHER $x=2$ IE $P(2,0)$ OR $3x^2+8x+12=0$

$b^2 - 4ac$

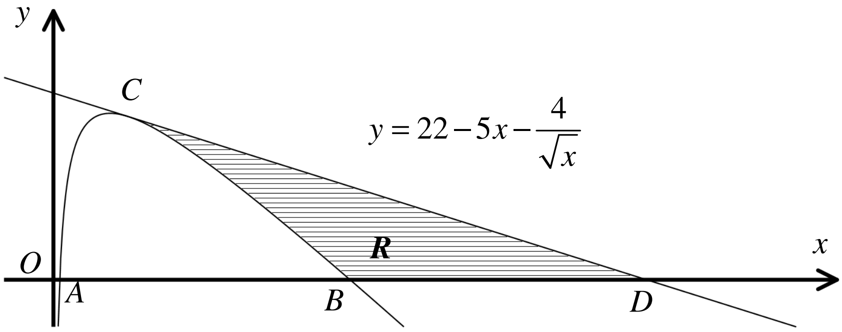
$= 8^2 - 4\times3\times12$

$= 64 - 144 = -80 < 0$

$\therefore$ ONLY SOLUTION IS THE POINT OF NORMALITY

$\therefore$ NORMAL DOES NOT MEET CURVE AGAIN

Question 209 (****)



The figure above shows part of the curve with equation

$$y = 22 - 5x - \frac{4}{\sqrt{x}}, \quad x > 0$$

The curve meets the x axis at A and B. The point C lies on the curve where $x = 1$.

a) Verify that the coordinates of B are $(4,0)$.

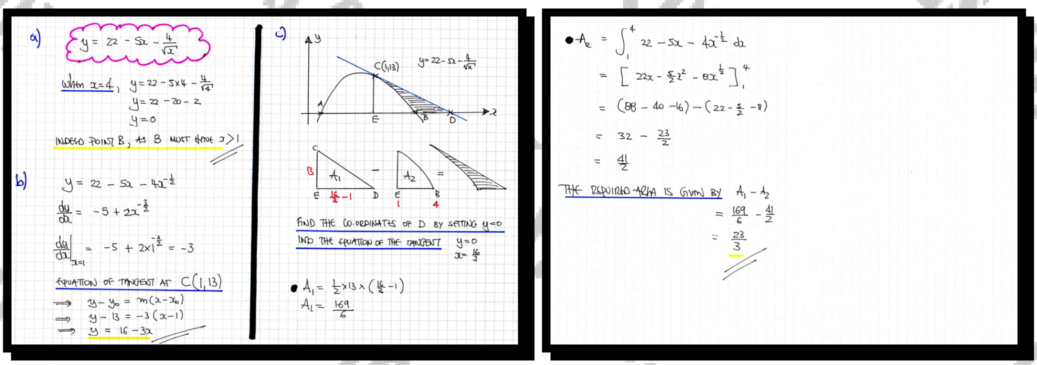
b) Find an equation of the tangent to the curve at C.

The tangent to the curve at C crosses the x axis at the point D.

The finite region R, shown shaded in the above figure, is bounded by the curve, the tangent to the curve at C and the x axis.

c) Determine the exact area of R.

$\boxed{}$, $\boxed{y + 3x = 16}$, $\boxed{D\left(\frac{16}{3}, 0\right)}$, $\boxed{\text{area} = \frac{23}{3}}$



Question 210 (****)

$$y = \frac{1-x^3}{x}, \; x \neq 0.$$

a) Find an expression for $\dfrac{d^2y}{dx^2}$.

b) Show that the value of $\dfrac{d^2y}{dx^2}$ where $\dfrac{dy}{dx} = 0$, is -6.

$$\boxed{\frac{d^2y}{dx^2} = 2x^{-3} - 2}$$

(a) $y = \frac{1-x^3}{x} = \frac{1}{x} - x^2 = x^{-1} - x^2$

$\frac{dy}{dx} = -x^{-2} - 2x = -2x - \frac{1}{x^2}$

$\frac{d^2y}{dx^2} = -2 + 2x^{-3} = -2 + \frac{2}{x^3}$

(b) $-2x - \frac{1}{x^2} = 0$

$-2x^3 = 1$

$x^3 = -\frac{1}{2}$

$x = \sqrt[3]{-\frac{1}{2}}$

HENCE

$\left.\frac{d^2y}{dx^2}\right|_{\frac{dy}{dx}=0} = \left.\frac{d^2y}{dx^2}\right|_{x^3=-\frac{1}{2}} = -2 + \frac{2}{-\frac{1}{2}}$

$= -2 - 4$

$= -6$

AS REQUIRED

Question 211 **(****)**

The curve C has equation

$$y = \sqrt{2x}, \quad x \geq 0$$

Find an equation of the normal to C at the point where $x = 2$, giving the answer in the form $y = mx + c$, where m and c are constants.

$\boxed{}$, $\boxed{y = 6 - 2x}$

WRITE IN INDEX NOTATION

$$y = \sqrt{2x} = \sqrt{2}\,x^{\frac{1}{2}}$$

DIFFERENTIATING W.R.T x & FINDING THE GRADIENT AT $x=2$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{2}\,x^{-\frac{1}{2}}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{2}\sqrt{2} \times 2^{-\frac{1}{2}} = \frac{1}{2}\cancel{\sqrt{2}} \times \frac{1}{\cancel{\sqrt{2}}} = \frac{1}{2}$$

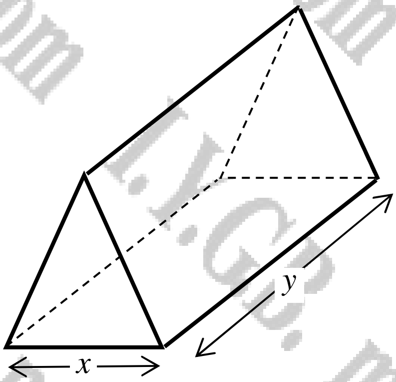
NORMAL GRADIENT IS -2

WHEN $x=2$, $y = \sqrt{2 \times 2} = 2$ IE $(2,2)$

EQUATION OF NORMAL SATISFIES

$$y - y_0 = m(x - x_0)$$
$$y - 2 = -2(x-2)$$
$$y - 2 = -2x + 4$$
$$y = -2x + 6$$

Question 212 (****)



The figure above shows a triangular prism whose triangular faces are parallel to each other and are in the shape of **equilateral** triangles of side length x cm.

The length of the prism is y.

a) Given that total surface area of the prism is exactly $54\sqrt{3}$ cm^2, show clearly that the volume of the prism, V cm^3, is given by

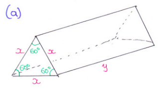
$$V = \frac{27}{2}x - \frac{1}{8}x^3.$$

b) Find the maximum value of V, fully justifying the fact that it is indeed the maximum value.

c) Determine the value of y when V takes this maximum value.

$\boxed{}$, $\boxed{V_{max} = 27}$, $\boxed{y = 2\sqrt{3}}$

(a)



$S = 54\sqrt{3}$

$2\left(\frac{1}{2}x^2\sin 60°\right) + 3(xy) = 54\sqrt{3}$

$x^2\frac{\sqrt{3}}{2} + 3xy = 54\sqrt{3}$

$\boxed{x^2\sqrt{3} + 6xy = 108\sqrt{3}}$

$6xy = 108\sqrt{3} - x^2\sqrt{3}$

$\boxed{xy = 18\sqrt{3} - \frac{1}{6}x^2\sqrt{3}}$

$V = \left(\frac{1}{2}x^2\sin 60°\right)y$

$V = \frac{1}{2}x^2\frac{\sqrt{3}}{2}y$

$V = \frac{1}{4}\sqrt{3}x^2y$

$V = \frac{1}{4}\sqrt{3}x(xy)$

$V = \frac{1}{4}\sqrt{3}x\left(18\sqrt{3} - \frac{1}{6}x^2\sqrt{3}\right)$

$V = \frac{27}{2}x - \frac{1}{8}x^3$

AS REQUIRED

(b) • $\frac{dV}{dx} = \frac{27}{2} - \frac{3}{8}x^2$

SOLVE FOR ZERO

$\frac{27}{2} - \frac{3}{8}x^2 = 0$

$\frac{27}{2} = \frac{3}{8}x^2$

$x^2 = 36$

$x = 6 \quad (x>0)$

• $\frac{d^2V}{dx^2} = -\frac{3}{4}x$

$\left.\frac{d^2V}{dx^2}\right|_{x=6} = -\frac{9}{2} < 0$

INDEED A MAXIMUM

• WHEN $x=6$

$V = \frac{27}{4}\times 6 - \frac{1}{16}\times 6^3$

$V = 27$

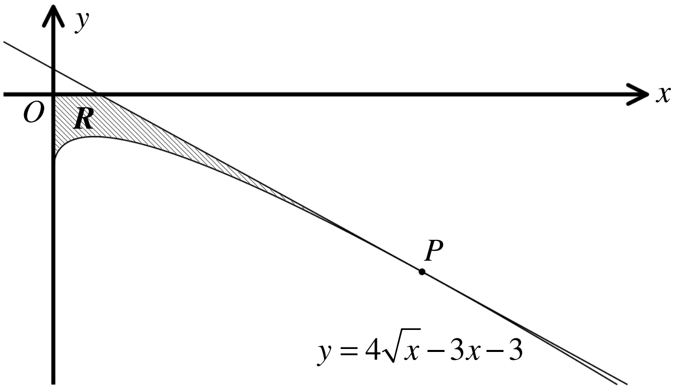
(c) NOW $xy = 18\sqrt{3} - \frac{1}{6}x^2\sqrt{3}$

$6y = 18\sqrt{3} - 6\sqrt{3}$

$6y = 12\sqrt{3}$

$y = 2\sqrt{3}$

Question 213 (****)



The figure above shows part of the curve with equation

$$y = 4\sqrt{x} - 3x - 3, \quad x > 0$$

a) Show that an equation of the tangent to the curve at the point P, where $x = 4$, is given by

$$y = 1 - 2x.$$

The finite region R is bounded by the curve, the tangent to the curve at P and the coordinate axes.

b) Determine the exact area of R.

$\boxed{}$, $\boxed{\text{area} = \frac{29}{12}}$

a) WRITE IN INDEX NOTATION AND DIFFERENTIATE

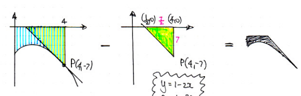
$\Rightarrow y = 4\sqrt{x} - 3x - 3$
$\Rightarrow y = 4x^{\frac{1}{2}} - 3x - 3$
$\Rightarrow \frac{dy}{dx} = 2x^{-\frac{1}{2}} - 3$
$\Rightarrow \left.\frac{dy}{dx}\right|_{x=4} = 2 \times 4^{-\frac{1}{2}} - 3 = -2 \leftarrow$ TANGENT GRADIENT

FIND THE y CO-ORDINATE OF P & THEN THE TANGENT

$y\big|_{x=4} = 4\sqrt{4} - 3\times4 - 3 = -7$

$\therefore \; y - y_0 = m(x - x_0)$
$y + 7 = -2(x - 4)$
$y + 7 = -2x + 8$
$y = 1 - 2x$ // AS REQUIRED

b) LOOKING AT THE PICTORIAL EQUATION BELOW



$y = 1 - 2x$
$0 = 1 - 2x$
$x = \frac{1}{2}$

AREA ABOVE IS GIVEN BY

$-\int_0^4 4x^{\frac{1}{2}} - 3x - 3 \, dx$

AREA OF TRIANGLE $= \frac{1}{2} \times 7 \times \frac{7}{2} = \frac{49}{4}$

$= -\int_0^4 4x^{\frac{1}{2}} - 3x - 3 \, dx = \int_4^0 4x^{\frac{1}{2}} - 3x - 3 \, dx$

$= \left[\frac{8}{3}x^{\frac{3}{2}} - \frac{3}{2}x^2 - 3x\right]_4^0 = 0 - \left(\frac{64}{3} - 24 - 12\right)$

$= \frac{44}{3}$

HENCE THE REQUIRED AREA CAN BE FOUND

$\frac{44}{3} - \frac{49}{4} = \frac{176}{12} - \frac{147}{12} = \frac{29}{12}$ //

Question 214 (****)

The straight line with equation

$$y = 2x + c$$

is a tangent to the curve with equation

$$y = x^2 + 6x + 7 .$$

Without using the discriminant, determine the value of the constant c and find the point of contact between the tangent and the curve.

$\boxed{c = 3}$, $\boxed{(-2,-1)}$

- IF $y = 2x + c$
  LINE HAS GRADIENT 2
- $y = x^2 + 6x + 7$
  $\frac{dy}{dx} = 2x + 6$
  WE REQUIRE $\frac{dy}{dx} = 2$
  $2x + 6 = 2$
  $2x = -4$
  $x = -2$
  $y = 4 - 12 + 7 = -1$
  $(-2, -1)$

LINE MUST BE A TANGENT.
SO IT MUST PASS THROUGH THE POINT $(-2,-1)$

$y = 2x + c$
$-1 = 2(-2) + c$
$-1 = -4 + c$
$c = 3$

Question 215 **(****)**

$$f(x) = ax^2 + bx + c, \ x \in \mathbb{R},$$

where a, b and c are constants.

It is given that $f(x)$ passes through the points $(2,3)$ and $(-1,9)$, and the curve has a stationary point where $x=1$.

Determine the value of a, b and c ,

$\boxed{a=2}$, $\boxed{b=-4}$, $\boxed{c=3}$

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

T.P. $x=1 \Rightarrow 0 = 2a+b$

$\boxed{b=-2a}$

Now

$\left.\begin{array}{l}(2,3) \Rightarrow 3 = a\times 2^2 + b\times 2 + c \\ (-1,9) \Rightarrow 9 = a\times(-1)^2 + b(-1) + c\end{array}\right\} \Rightarrow$

$\left\{\begin{array}{l}3 = 4a + 2b + c \\ 9 = a - b + c\end{array}\right\} \Rightarrow$

$\left\{\begin{array}{l}3 = 4a + 2(-2a) + c \\ 9 = a - (-2a) + c\end{array}\right\} \Rightarrow$

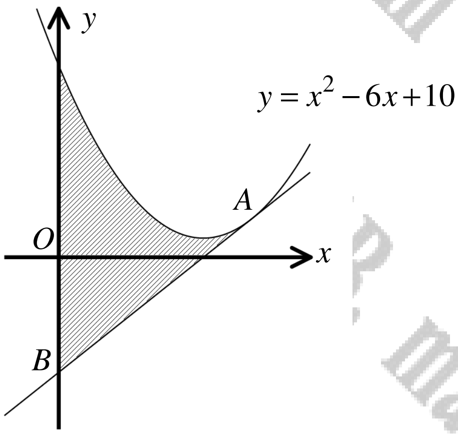
$\left\{\begin{array}{l}3 = c \\ 9 = 3a + c\end{array}\right\}$

$\left\{\begin{array}{l}c = 3 \\ 9 = 3a + 3\end{array}\right.$

Hence

$c=3$ $\quad a=2$ $\quad b=-2$

Question 216 (****)



The figure above shows the curve with equation

$$y = x^2 - 6x + 10, \ x \in \mathbb{R}.$$

The point A, where $x = 4$, lies on the curve.

The tangent to the curve at A, meets the y axis at B.

Determine the area of the finite region bounded by the curve, the tangent to the curve at A and the y axis.

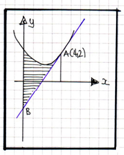
$\boxed{}, \boxed{\frac{64}{3}}$

START BY FINDING THE EQUATION OF THE TANGENT.

- WHEN $x=4$, $y = 4^2 - 6\times4 + 10$
  $y = 2$ $\quad \therefore A(4,2)$

- $y = x^2 - 6x + 10$
  $\frac{dy}{dx} = 2x - 6$
  $\left.\frac{dy}{dx}\right|_{x=4} = 2 \quad \leftarrow$ TANGENT GRADIENT

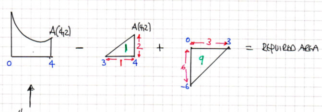
- $y - y_0 = m(x - x_0)$
  $y - 2 = 2(x - 4)$
  $y = 2x - 6$



FIND THE x & y INTERCEPT OF THE TANGENT.

$x=0 \quad y=-6 \quad (0,-6)$
$y=0 \quad x=3 \quad (3,0)$

LOOKING AT THE DIAGRAM BELOW



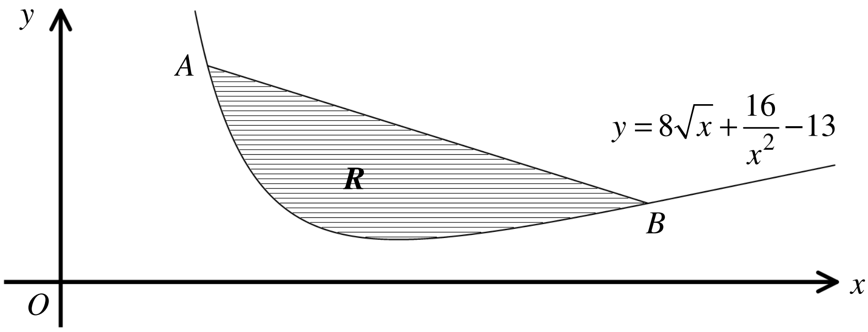
$$\int_0^4 x^2 - 6x + 10 \, dx = \left[\tfrac{1}{3}x^3 - 3x^2 + 10x\right]_0^4$$

$$= \left(\tfrac{64}{3} - 48 + 40\right) - (0)$$

$$= \tfrac{40}{3}$$

HENCE THE REQUIRED AREA IS $\frac{40}{3} - 1 + 9 = \frac{64}{3}$ //

Question 217 (****)



The figure above shows part of the curve with equation

$$y = 8\sqrt{x} + \frac{16}{x^2} - 13, \quad x > 0.$$

The points A and B lie on the curve where $x = 1$ and $x = 4$, respectively.

The finite region R is bounded by the curve, the straight line segment AB.

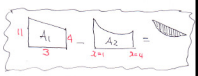
Determine the exact area of R.

$\boxed{}$, $\boxed{\text{area} = \frac{73}{6}}$

• $y = 8\sqrt{x} + \frac{16}{x^2} - 13$

WHEN $x=1$ $y = 8+16-13 = 11$

WHEN $x=4$ $y = 16+1-13 = 4$



• $A_1 = \frac{11+4}{2} \times 3 = 22.5$

• $A_2 = \int_1^4 8x^{\frac{1}{2}} + 16x^{-2} - 13 \, dx = \left[\frac{8}{\frac{3}{2}} x^{\frac{3}{2}} - 16x^{-1} - 13x \right]_1^4$

$= \left[\frac{16}{3} x^{\frac{3}{2}} - \frac{16}{x} - 13x \right]_1^4 = \left(\frac{128}{3} - 4 - 52 \right) - \left(\frac{16}{3} - 16 - 13 \right)$

$= -\frac{40}{3} - \left(-\frac{71}{3} \right) = \frac{31}{3}$

$\therefore R = 22.5 - \frac{31}{3} = \frac{73}{6}$

Question 218 (****)

The curve C has equation

$$y = 7\sqrt{x} - \frac{3}{\sqrt{x}}, \quad x > 0$$

Show clearly that

$$y = 4x\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right).$$

$\boxed{}$, $\boxed{\text{proof}}$

REWRITE IN INDEX NOTATION AND DIFFERENTIATE TWICE

$$\Rightarrow y = 7\sqrt{x} - \frac{3}{\sqrt{x}}$$
$$\Rightarrow y = 7x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$$
$$\Rightarrow \frac{dy}{dx} = \frac{7}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}$$
$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{7}{4}x^{-\frac{3}{2}} - \frac{9}{4}x^{-\frac{5}{2}}$$

SUBSTITUTE INTO THE R.H.S. OF THE REQUIRED EXPRESSION

$$\begin{aligned} 4x\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) &= 4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} \\ &= 4x^2\left[-\frac{7}{4}x^{-\frac{3}{2}} - \frac{9}{4}x^{-\frac{5}{2}}\right] + 4x\left[\frac{7}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}\right] \\ &= -7x^{\frac{1}{2}} - 9x^{-\frac{1}{2}} + 14x^{\frac{1}{2}} + 6x^{-\frac{1}{2}} \\ &= 7x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} \\ &= 7\sqrt{x} - \frac{3}{\sqrt{x}} \\ &= y \end{aligned}$$

AS REQUIRED

Question 219 (****) non calculator

$$\sqrt{y} = 2\sqrt[3]{x} - 3, \ \ x > 0 .$$

Show clearly that

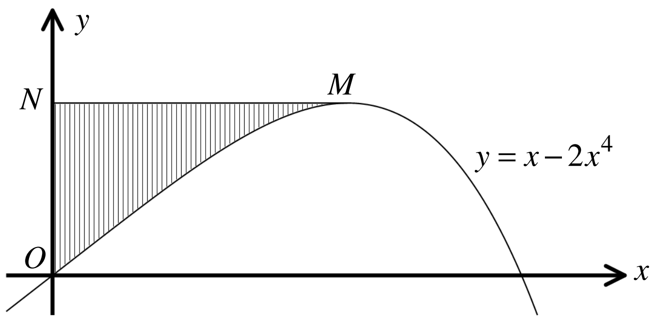
$$\int_1^8 y \ \ dx = \frac{12}{5} .$$

$\boxed{\text{proof}}$

$$\sqrt{y} = 2\sqrt[3]{x} - 3$$
$$\sqrt{y} = 2x^{\frac{1}{3}} - 3$$
$$y = \left(2x^{\frac{1}{3}} - 3\right)^2$$
$$y = 4x^{\frac{2}{3}} - 12x^{\frac{1}{3}} - 9$$

Hence
$$\int_1^8 y \, dx = \int_1^8 4x^{\frac{2}{3}} - 12x^{\frac{1}{3}} + 9 \, dx$$
$$= \left[\frac{12}{5}x^{\frac{5}{3}} - 9x^{\frac{4}{3}} + 9x\right]_1^8$$
$$= \left(\frac{384}{5} - 144 + 72\right) - \left(\frac{12}{5} \cancel{-9+9}\right)$$
$$= \frac{24}{5} - \frac{12}{5}$$
$$= \frac{24}{5} - \frac{12}{5}$$
$$= \frac{12}{5}$$

Question 220 (****)



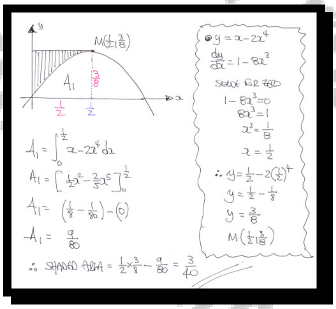
The diagram below shows the quartic curve with equation

$$y = x - 2x^4, \ x \in \mathbb{R}.$$

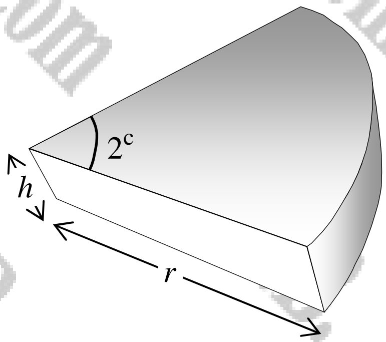
The point M is the maximum point on the curve and the point N lies on the y axis so that the straight line segment MN is parallel to the x axis.

Find the exact area of the shaded region, bounded by the curve, the y axis and the straight line segment from M to N.

$\boxed{}$, $\boxed{\text{area} = \frac{3}{40}}$



Question 221 (****)



The figure above shows solid right prism of height h cm.

The cross section of the prism is a circular sector of radius r cm, subtending an angle of 2 radians at the centre.

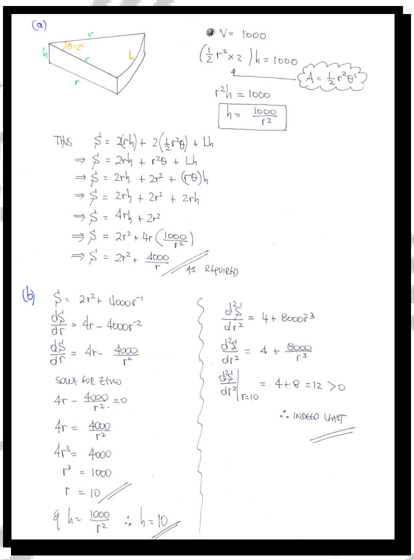
a) Given that the volume of the prism is 1000 cm^3, show clearly that

$$S = 2r^2 + \frac{4000}{r},$$

where $S \text{ cm}^2$ is the total surface area of the prism.

b) Hence determine the value of r and the value of h which make S **least**, fully justifying your answer.

$\boxed{}$, $\boxed{r = 10}$, $\boxed{h = 10}$



(a)

$V = 1000$

$\left(\frac{1}{2}r^2 \times 2\right)h = 1000$

$A = \frac{1}{2}r^2\theta^c$

$r^2 h = 1000$

$\boxed{h = \frac{1000}{r^2}}$

THUS $S = 2(rh) + 2\left(\frac{1}{2}r^2\theta\right) + Lh$

$\Rightarrow S = 2rh + r^2\theta + Lh$

$\Rightarrow S = 2rh + 2r^2 + (r\theta)h$

$\Rightarrow S = 2rh + 2r^2 + 2rh$

$\Rightarrow S = 4rh + 2r^2$

$\Rightarrow S = 2r^2 + 4r\left(\frac{1000}{r^2}\right)$

$\Rightarrow S = 2r^2 + \frac{4000}{r}$ AS REQUIRED

(b)

$S = 2r^2 + 4000r^{-1}$

$\frac{dS}{dr} = 4r - 4000r^{-2}$

$\frac{dS}{dr} = 4r - \frac{4000}{r^2}$

SOLVE FOR ZERO

$4r - \frac{4000}{r^2} = 0$

$4r = \frac{4000}{r^2}$

$4r^3 = 4000$

$r^3 = 1000$

$r = 10$

& $h = \frac{1000}{r^2}$ $\therefore h = 10$

$\frac{d^2S}{dr^2} = 4 + 8000r^{-3}$

$\frac{d^2S}{dr^2} = 4 + \frac{8000}{r^3}$

$\left.\frac{d^2S}{dr^2}\right|_{r=10} = 4 + 8 = 12 > 0$

$\therefore$ INDEED LEAST

Question 222 (****)

A curve has equation

$$y = x^2 - 6x\sqrt[3]{x} + 2, \qquad x \in \mathbb{R}, \qquad x \geq 0.$$

Find the coordinates of the stationary points of the curve and classify them as local maxima, local minima or a points of inflexion.

$\boxed{}$, $\boxed{\text{local minimum at } (8,-30)}$, $\boxed{\text{local maximum at } (0,2)}$

REWRITE THE EQUATION "IN INDICES" & DIFFERENTIATE

$\Rightarrow y = x^2 - 6x\sqrt[3]{x} + 2$

$\Rightarrow y = x^2 - 6x^1 x^{\frac{1}{3}} + 2$

$\Rightarrow y = x^2 - 6x^{\frac{4}{3}} + 2$

$\Rightarrow \frac{dy}{dx} = 2x - 8x^{\frac{1}{3}}$

SOLVING FOR ZERO, SEEKING STATIONARY POINTS

$\Rightarrow 2x - 8x^{\frac{1}{3}} = 0$

$\Rightarrow 2x = 8x^{\frac{1}{3}}$

$\Rightarrow x = 4x^{\frac{1}{3}}$

EITHER $x=0$ (BY INSPECTION) OR IF WE DIVIDE WE OBTAIN

$\Rightarrow x^{\frac{2}{3}} = 4$

$\Rightarrow (\sqrt[3]{x})^2 = 4$

$\Rightarrow \sqrt[3]{x} = \begin{cases} 2 \\ -2 \end{cases}$

$\Rightarrow x = \begin{cases} 8 \\ \cancel{-8} \end{cases} \quad x \geq 0$

FIND FIRST THE CORRESPONDING y CO-ORDINATES

$x=0, \quad y=2$

$x=8, \quad y = 8^2 - 6 \times 8^{\frac{4}{3}} + 2 = 64 - 6 \times 16 + 2 = 66 - 96 = -30$

$\therefore (0,2)$ & $(8,-30)$

DETERMINING THE NATURE OF THESE POINTS BY USING THE SECOND DERIVATIVE TEST

$\Rightarrow \frac{dy}{dx} = 2x - 8x^{\frac{1}{3}}$

$\Rightarrow \frac{d^2y}{dx^2} = 2 - \frac{8}{3}x^{-\frac{2}{3}}$

$\Rightarrow \frac{d^2y}{dx^2} = 2 - \frac{8}{3x^{\frac{2}{3}}}$

$\left.\frac{d^2y}{dx^2}\right|_{x=8} = 2 - \frac{8}{3 \times 8^{\frac{2}{3}}} = 2 - \frac{8}{12} = \frac{4}{3} > 0$

$\therefore (8,-30)$ IS A LOCAL MINIMUM

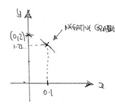
$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2 - \frac{8}{3 \times 0^{\frac{2}{3}}} = 2 - \frac{8}{0}$ ← WE CANNOT USE THIS TEST WITHOUT KNOWLEDGE OF LIMITING TECHNIQUES

CHECKING EITHER THE GRADIENT OR THE VALUE OF y TO THE "RIGHT" OF $x=0$ (AS $x \geq 0$)

- $\left.\frac{dy}{dx}\right|_{x=0.1} \approx -3.51\ldots$

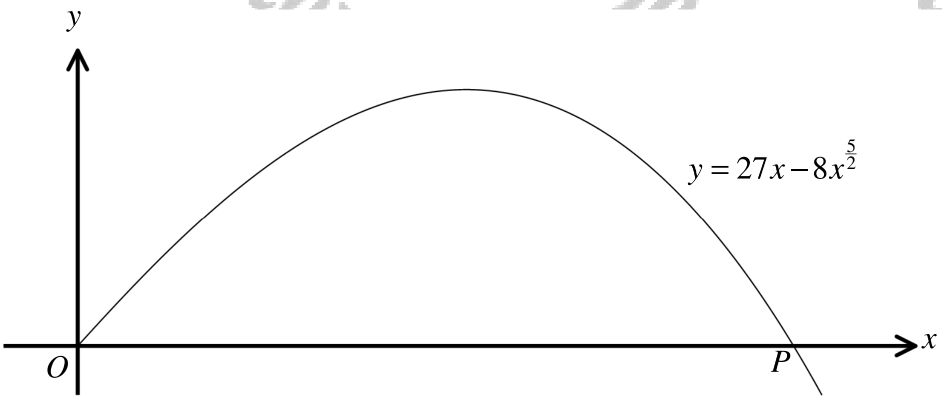
OR

- $\left. y \right|_{x=0.1} \simeq 1.73\ldots$



$\therefore (0,2)$ IS A LOCAL MAX

Question 223 (****)



The figure above shows the curve C with equation

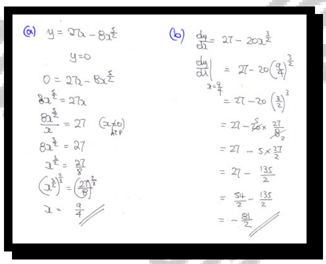
$$y = 27x - 8x^{\frac{5}{2}}, \; x \geq 0$$

The curve meets the x axis at the origin O and at the point P.

a) Find the exact x coordinate of P.

b) Calculate the gradient of C at the point P.

$\boxed{x = \frac{9}{4}}$, $\boxed{\frac{dy}{dx} = -\frac{81}{2}}$

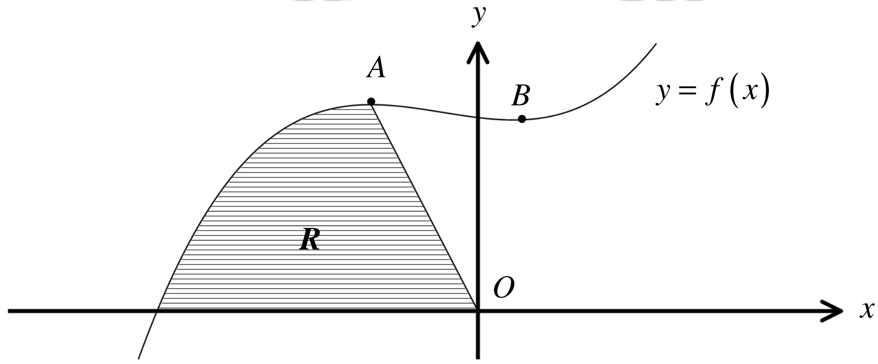


Question 224 (****)

$$f(x) = x^3 + x^2 - x + 15, \; x \in \mathbb{R}$$

a) Show that $(x+3)$ is a factor of $f(x)$.

The figure below shows the curve C with equation $y = f(x)$.



The points A and B are stationary points of C.

b) Find the exact coordinates of A and B.

The finite region R is bounded by the curve, the x axis and the straight line segment OA, where O is the origin.

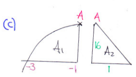
c) Determine the exact area R.

$\boxed{}$, $\boxed{A(-1,16), \; B\left(\frac{1}{3}, \frac{400}{27}\right)}$, $\boxed{\text{area} = \frac{92}{3}}$

(a) $f(x) = x^3 + x^2 - x + 15$
$f(-3) = -27 + 9 + 3 + 15 = 0$
$\therefore$ $x+3$ IS INDEED A FACTOR

(b) $f'(x) = 3x^2 + 2x - 1$
SOLVE FOR ZERO
$3x^2 + 2x - 1 = 0$
$(3x-1)(x+1) = 0$
$x = -1, \frac{1}{3}$ $\quad y = 16, \frac{400}{27}$
$\therefore A(-1,16) \quad B\left(\frac{1}{3}, \frac{400}{27}\right)$

(c)



$A_1 = \int_{-3}^{-1} x^3 + x^2 - x + 15 \, dx$
$A_1 = \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 15x\right]_{-3}^{-1}$
$A_1 = \left(\frac{1}{4} - \frac{1}{3} - \frac{1}{2} - 15\right) - \left(\frac{81}{4} - 9 - \frac{9}{2} - 45\right)$
$A_1 = \frac{68}{3}$
& $A_2 = 8$
$\therefore$ AREA $= \frac{68}{3} + 8 = \frac{92}{3}$

Question 225 (****)

The curve C has equation

$$y = \frac{1}{2x^2} + \frac{4}{3x^3}, \quad x > 0.$$

Show clearly that

$$x^2 \frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0.$$

$\boxed{}$, $\boxed{\text{proof}}$

$y = \frac{1}{2x^2} + \frac{4}{3x^3} = \frac{1}{2}x^{-2} + \frac{4}{3}x^{-3}$

$\frac{dy}{dx} = -x^{-3} - 4x^{-4} = -\frac{1}{x^3} - \frac{4}{x^4}$

$\frac{d^2y}{dx^2} = 3x^{-4} + 16x^{-5} = \frac{3}{x^4} + \frac{16}{x^5}$

HENCE

$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y$

$= x^2\left[\frac{3}{x^4} + \frac{16}{x^5}\right] + 6x\left[-\frac{1}{x^3} - \frac{4}{x^4}\right] + 6\left[\frac{1}{2x^2} + \frac{4}{3x^3}\right]$

$= \cancel{\frac{3}{x^2}} + \cancel{\frac{16}{x^3}} - \cancel{\frac{6}{x^2}} - \cancel{\frac{24}{x^3}} + \cancel{\frac{3}{x^2}} + \cancel{\frac{8}{x^3}}$

$= 0$

AS REQUIRED

Question 226 (****)

$$f(x) \equiv \sqrt{x} + \frac{8}{x^2}, \quad x > 0.$$

Find the range of values of x, for which $f(x)$ is increasing.

$\boxed{x > 4}$

$f(x) = \sqrt{x} + \frac{8}{x^2} = x^{\frac{1}{2}} + 8x^{-2}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 16x^{-3}$

"INCREASING" $\Longrightarrow f'(x) > 0$

$\Rightarrow \frac{1}{2}x^{-\frac{1}{2}} - 16x^{-3} > 0$

$\Rightarrow x^{-\frac{1}{2}} - 32x^{-3} > 0$

$\Rightarrow x^{-\frac{1}{2}} - \frac{32}{x^3} > 0$

MULTIPLY BY x^3 DOES NOT AFFECT INEQUALITY AS $x^3 > 0$

$\Rightarrow x^{-\frac{1}{2}}x^3 - 32 > 0$

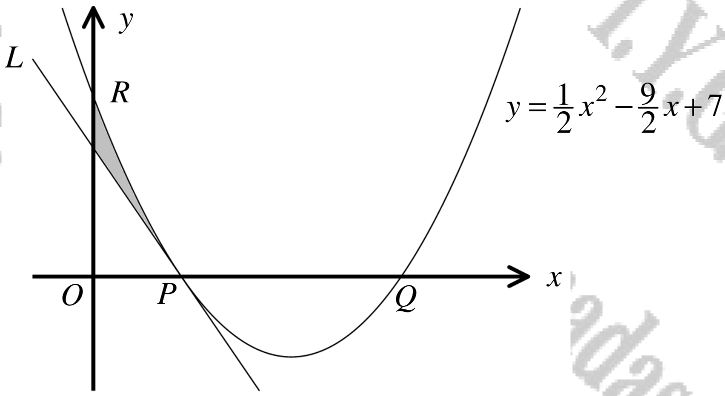
$\Rightarrow x^{\frac{5}{2}} - 32 > 0$

$\Rightarrow x^{\frac{5}{2}} > 32$

$\Rightarrow \left(x^{\frac{5}{2}}\right)^{\frac{2}{5}} > (32)^{\frac{2}{5}}$

$\Rightarrow x > 4$

Question 227 (****)



The diagram above shows the quadratic curve C with equation

$$y=\frac{1}{2}x^2-\frac{9}{2}x+7.$$

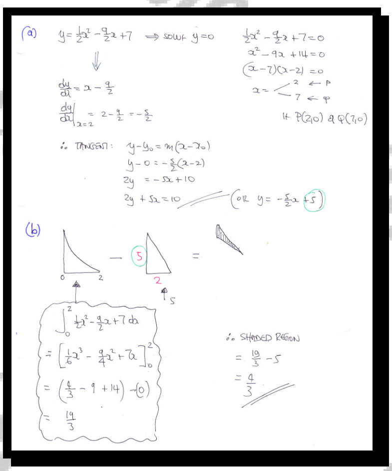
The curve crosses the x axis at the points P and Q, and the y axis at the point R.

The line L is the tangent to C at the point P.

a) Find an equation of L.

b) Find the exact area of the shaded region bounded by the tangent at P, the curve and the y axis.

$\boxed{2y+5x=10}$, $\boxed{\text{area}=\frac{4}{3}}$



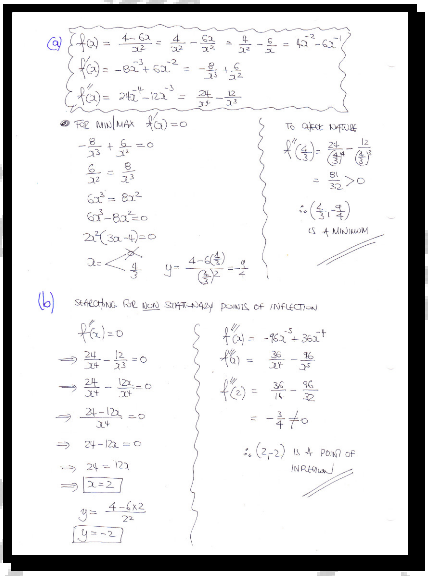
Question 228 (****)

$$f(x) = \frac{4-6x}{x^2}, \ x \neq 0.$$

a) Find the coordinates of the stationary point of $f(x)$ and determine its nature.

b) Show clearly that $f(x)$ has a point of inflection and determine its coordinates.

$\boxed{\min\left(\frac{4}{3}, -\frac{9}{4}\right)}$, $\boxed{(2,-2)}$



(a) $f(x) = \frac{4-6x}{x^2} = \frac{4}{x^2} - \frac{6x}{x^2} = \frac{4}{x^2} - \frac{6}{x} = 4x^{-2} - 6x^{-1}$

$f'(x) = -8x^{-3} + 6x^{-2} = -\frac{8}{x^3} + \frac{6}{x^2}$

$f''(x) = 24x^{-4} - 12x^{-3} = \frac{24}{x^4} - \frac{12}{x^3}$

- FOR MIN/MAX $f'(x) = 0$

$-\frac{8}{x^3} + \frac{6}{x^2} = 0$

$\frac{6}{x^2} = \frac{8}{x^3}$

$6x^3 = 8x^2$

$6x^3 - 8x^2 = 0$

$2x^2(3x-4) = 0$

$x = \cancel{0}$ or $\frac{4}{3}$ $\quad y = \frac{4-6\left(\frac{4}{3}\right)}{\left(\frac{4}{3}\right)^2} = -\frac{9}{4}$

TO CHECK NATURE

$f''\left(\frac{4}{3}\right) = \frac{24}{\left(\frac{4}{3}\right)^4} - \frac{12}{\left(\frac{4}{3}\right)^3}$

$= \frac{81}{32} > 0$

$\therefore \left(\frac{4}{3}, -\frac{9}{4}\right)$ IS A MINIMUM

(b) SEARCHING FOR NON STATIONARY POINTS OF INFLECTION

$f''(x) = 0$

$\Rightarrow \frac{24}{x^4} - \frac{12}{x^3} = 0$

$\Rightarrow \frac{24}{x^4} - \frac{12x}{x^4} = 0$

$\Rightarrow \frac{24-12x}{x^4} = 0$

$\Rightarrow 24 - 12x = 0$

$\Rightarrow 24 = 12x$

$\Rightarrow \boxed{x=2}$

$y = \frac{4-6\times2}{2^2}$

$\boxed{y=-2}$

$f'''(x) = -96x^{-5} + 36x^{-4}$

$f'''(x) = \frac{36}{x^4} - \frac{96}{x^5}$

$f'''(2) = \frac{36}{16} - \frac{96}{32}$

$= -\frac{3}{4} \neq 0$

$\therefore (2,-2)$ IS A POINT OF INFLECTION

Question 229 **(****)** **non calculator**

The population, P in hundreds, of a herd of zebra t years after a certain instant is given by the formula

$$P = 6t + \frac{96}{\sqrt{t}} - 60 \ , \ t > 0 .$$

a) Find the number of zebra in the herd after $2\frac{1}{4}$ years.

b) Find the time during which the herd's population is increasing.

$\boxed{P = 1750}$, $\boxed{t > 4}$

(a) $P = 6t + \frac{96}{\sqrt{t}} - 60$

when $t = 2\frac{1}{4} = \frac{9}{4}$

$\Rightarrow P = 6 \times \frac{9}{4} + \frac{96}{\sqrt{9/4}} - 60$

$\Rightarrow P = \frac{27}{2} + \frac{96}{3/2} - 60$

$\Rightarrow P = \frac{27}{2} + 64 - 60$

$\Rightarrow P = \frac{35}{2} = 17.5$

$\therefore$ 1750 zebras

(b) $P = 6t + 96t^{-\frac{1}{2}} - 60$

$\frac{dP}{dt} = 6 - 48t^{-\frac{3}{2}}$

INCREASING $\Rightarrow \frac{dP}{dt} > 0$

$\Rightarrow 6 - 48t^{-\frac{3}{2}} > 0$

$\Rightarrow -\frac{48}{t^{3/2}} > -6$

$\Rightarrow \frac{48}{t^{3/2}} < 6$

$\Rightarrow t^{3/2} > 8$

$\Rightarrow \left(t^{3/2}\right)^{2/3} > 8^{2/3}$

$\Rightarrow t > 4$

Question 230 (****)

$$f(x)=\frac{\left(x^{2}-3\right)^{2}}{x^{3}},\ x\neq 0.$$

a) Show clearly that

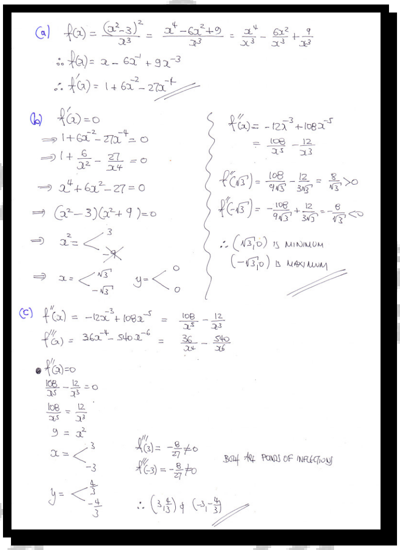
$$f'(x)=A+Bx^{-2}+Cx^{-4},$$

where A, B and C are constants to be found.

b) Find the exact coordinates of the stationary points of $f(x)$ and use $f''(x)$ to determine their nature.

c) Show that $f(x)$ has two points of inflection and determine their coordinates.

$\boxed{f'(x)=1+6x^{-2}-27x^{-4}}$, $\boxed{\min\left(\sqrt{3},0\right)}$, $\boxed{\max\left(-\sqrt{3},0\right)}$, $\boxed{\left(3,\frac{4}{3}\right),\left(-3,-\frac{4}{3}\right)}$



(a) $f(x)=\frac{(x^2-3)^2}{x^3}=\frac{x^4-6x^2+9}{x^3}=\frac{x^4}{x^3}-\frac{6x^2}{x^3}+\frac{9}{x^3}$

$\therefore f(x)=x-6x^{-1}+9x^{-3}$

$\therefore f'(x)=1+6x^{-2}-27x^{-4}$

(b) $f'(x)=0$

$\Rightarrow 1+6x^{-2}-27x^{-4}=0$

$\Rightarrow 1+\frac{6}{x^2}-\frac{27}{x^4}=0$

$\Rightarrow x^4+6x^2-27=0$

$\Rightarrow (x^2-3)(x^2+9)=0$

$\Rightarrow x^2=3$ or $\cancel{-9}$

$\Rightarrow x=\sqrt{3}$ or $-\sqrt{3}$ $\quad y=0$ or 0

$f''(x)=-12x^{-3}+108x^{-5}=\frac{108}{x^5}-\frac{12}{x^3}$

$f''(\sqrt{3})=\frac{108}{9\sqrt{3}}-\frac{12}{3\sqrt{3}}=\frac{8}{\sqrt{3}}>0$

$f''(-\sqrt{3})=-\frac{108}{9\sqrt{3}}+\frac{12}{3\sqrt{3}}=-\frac{8}{\sqrt{3}}<0$

$\therefore (\sqrt{3},0)$ IS MINIMUM

$(-\sqrt{3},0)$ IS MAXIMUM

(c) $f''(x)=-12x^{-3}+108x^{-5}=\frac{108}{x^5}-\frac{12}{x^3}$

$f'''(x)=36x^{-4}-540x^{-6}=\frac{36}{x^4}-\frac{540}{x^6}$

• $f''(x)=0$

$\frac{108}{x^5}-\frac{12}{x^3}=0$

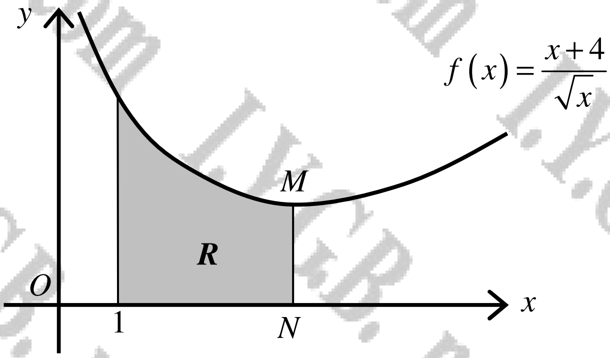
$\frac{108}{x^5}=\frac{12}{x^3}$

$9=x^2$

$x=3$ or -3 $\quad f'''(3)=-\frac{8}{27}\neq 0$, $f'''(-3)=-\frac{8}{27}\neq 0$ $\quad$ BOTH ARE POINTS OF INFLECTIONS

$y=\frac{4}{3}$ or $-\frac{4}{3}$ $\quad \therefore \left(3,\frac{4}{3}\right)$ & $\left(-3,-\frac{4}{3}\right)$

Question 231 (****)



The figure above shows the curve C with equation

$$f(x)=\frac{x+4}{\sqrt{x}},\ x>0.$$

a) Determine the coordinates of the minimum point of C, labelled as M.

The point N lies on the x axis so that MN is parallel to the y axis. The finite region R is bounded by C, the x axis, the straight line segment MN and the straight line with equation $x=1$.

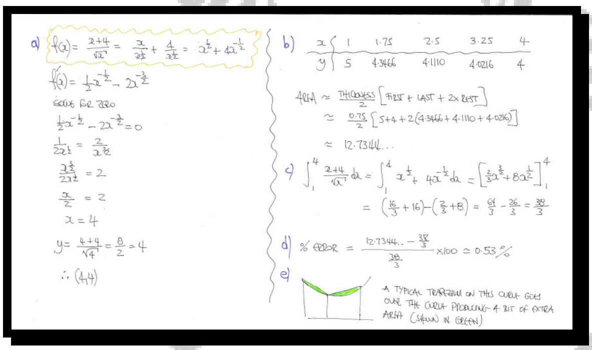
b) Use the trapezium rule with 4 strips of equal width to estimate the area of R.

c) Use integration to find the exact area of R.

d) Calculate the percentage error in using the trapezium rule to find the area of R.

e) Explain with the aid of a diagram why the trapezium rule overestimates the area of R.

, $\boxed{M(4,4)}$, $\boxed{\text{area}\approx 12.7344...}$, $\boxed{\text{area}=\dfrac{38}{3}}$, $\boxed{0.53\%}$



a) $f(x)=\dfrac{x+4}{\sqrt{x}}=\dfrac{x}{x^{\frac{1}{2}}}+\dfrac{4}{x^{\frac{1}{2}}}=x^{\frac{1}{2}}+4x^{-\frac{1}{2}}$

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}-2x^{-\frac{3}{2}}$

SOLVE FOR ZERO

$\frac{1}{2}x^{-\frac{1}{2}}-2x^{-\frac{3}{2}}=0$

$\dfrac{1}{2x^{\frac{1}{2}}}=\dfrac{2}{x^{\frac{3}{2}}}$

$\dfrac{x^{\frac{3}{2}}}{2x^{\frac{1}{2}}}=2$

$\dfrac{x}{2}=2$

$x=4$

$y=\dfrac{4+4}{\sqrt{4}}=\dfrac{8}{2}=4$

$\therefore (4,4)$

b)

x	1	1.75	2.5	3.25	4
y	5	4.3466	4.1110	4.0216	4

AREA $\simeq \dfrac{\text{THICKNESS}}{2}\left[\text{FIRST}+\text{LAST}+2\times\text{REST}\right]$

$\simeq \dfrac{0.75}{2}\left[5+4+2(4.3466+4.1110+4.0216)\right]$

$\simeq 12.7344...$

c) $\displaystyle\int_1^4 \frac{x+4}{\sqrt{x}}\,dx=\int_1^4 x^{\frac{1}{2}}+4x^{-\frac{1}{2}}\,dx=\left[\tfrac{2}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}\right]_1^4$

$=\left(\frac{16}{3}+16\right)-\left(\frac{2}{3}+8\right)=\frac{64}{3}-\frac{26}{3}=\frac{38}{3}$

d) % ERROR $=\dfrac{12.7344...-\frac{38}{3}}{\frac{38}{3}}\times100\simeq 0.53\%$

e) A TYPICAL TRAPEZIUM ON THIS CURVE GOES OVER THE CURVE PRODUCING A BIT OF EXTRA AREA (SHOWN IN GREEN)

Question 232 (****)

The curve C with equation $y = f(x)$ passes through the point $P(16,-5)$, and its gradient function $f'(x)$ is given by

$$f'(x) = \frac{x-6}{\sqrt{x}}, \quad x > 0.$$

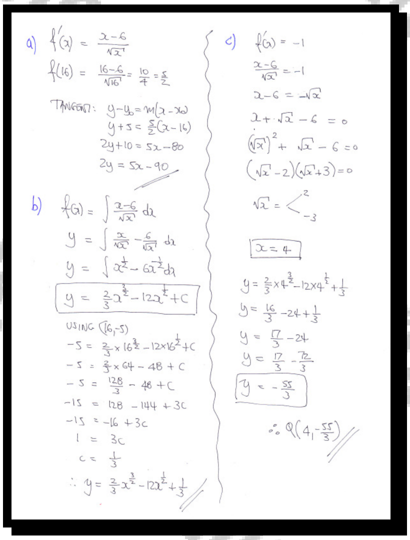
a) Find an equation of the tangent to C at P.

b) Determine an equation of C.

The point Q lies on C and the gradient of C at that point is -1.

c) Find the coordinates of Q.

$\boxed{}$, $\boxed{2y = 5x - 90}$, $\boxed{y = \frac{2}{3}x^{\frac{3}{2}} - 12x^{\frac{1}{2}} + \frac{1}{3}}$, $\boxed{Q\left(4, -\frac{55}{3}\right)}$



a) $f'(x) = \frac{x-6}{\sqrt{x}}$

$f(16) = \frac{16-6}{\sqrt{16}} = \frac{10}{4} = \frac{5}{2}$

TANGENT: $y - y_0 = m(x - x_0)$

$y + 5 = \frac{5}{2}(x - 16)$

$2y + 10 = 5x - 80$

$2y = 5x - 90$

b) $f(x) = \int \frac{x-6}{\sqrt{x}}\,dx$

$y = \int \frac{x}{\sqrt{x}} - \frac{6}{\sqrt{x}}\,dx$

$y = \int x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}\,dx$

$\boxed{y = \frac{2}{3}x^{\frac{3}{2}} - 12x^{\frac{1}{2}} + C}$

USING $(16,-5)$

$-5 = \frac{2}{3} \times 16^{\frac{3}{2}} - 12 \times 16^{\frac{1}{2}} + C$

$-5 = \frac{2}{3} \times 64 - 48 + C$

$-5 = \frac{128}{3} - 48 + C$

$-15 = 128 - 144 + 3C$

$-15 = -16 + 3C$

$1 = 3C$

$C = \frac{1}{3}$

$\therefore y = \frac{2}{3}x^{\frac{3}{2}} - 12x^{\frac{1}{2}} + \frac{1}{3}$

c) $f'(x) = -1$

$\frac{x-6}{\sqrt{x}} = -1$

$x - 6 = -\sqrt{x}$

$x + \sqrt{x} - 6 = 0$

$(\sqrt{x})^2 + \sqrt{x} - 6 = 0$

$(\sqrt{x} - 2)(\sqrt{x} + 3) = 0$

$\sqrt{x} = \begin{cases} 2 \\ -3 \end{cases}$

$\boxed{x = 4}$

$y = \frac{2}{3} \times 4^{\frac{3}{2}} - 12 \times 4^{\frac{1}{2}} + \frac{1}{3}$

$y = \frac{16}{3} - 24 + \frac{1}{3}$

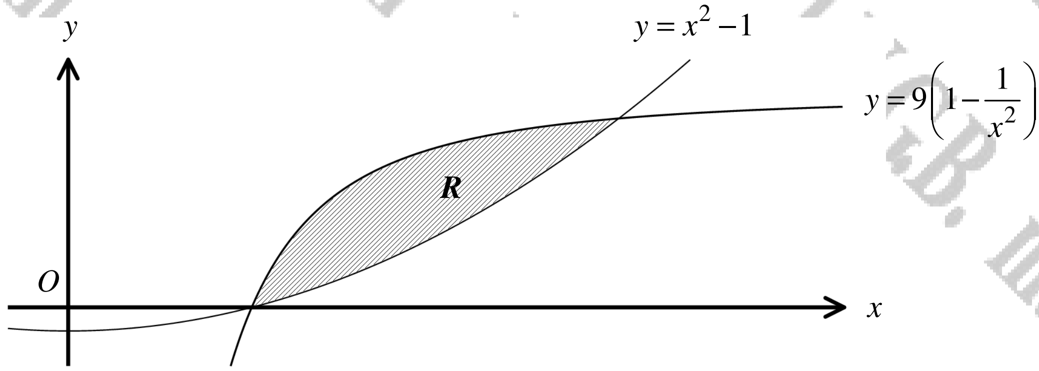
$y = \frac{17}{3} - 24$

$y = \frac{17}{3} - \frac{72}{3}$

$\boxed{y = -\frac{55}{3}}$

$\therefore Q\left(4, -\frac{55}{3}\right)$

Question 233 (****)



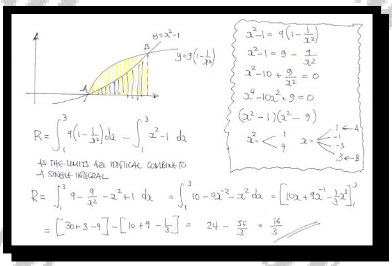
The figure above shows the graphs of the curves with equations

$$y = x^2 - 1 \quad \text{and} \quad y = 9\left(1 - \frac{1}{x^2}\right).$$

The finite region R is bounded by the two curves in the 1st quadrant, and is shown shaded in the figure above.

Determine the exact area of R.

$\boxed{}$, $\boxed{\frac{16}{3}}$



Question 234 (****)

The curve C has equation

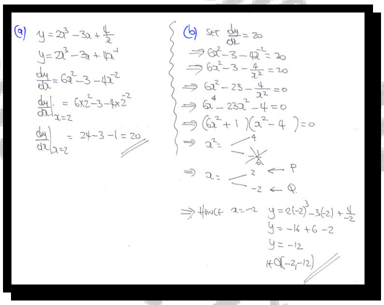
$$y = 2x^3 - 3x + \frac{4}{x}, \quad x \neq 0$$

The point $P(2,12)$ lies on C.

a) Find the gradient at P.

b) Determine the coordinates of another point Q that lies on C, so that the gradient at Q is the same gradient as P.

$\boxed{}$, $\boxed{\text{gradient} = 20}$, $\boxed{Q(-2,-12)}$



(a) $y = 2x^3 - 3x + \frac{4}{x}$

$y = 2x^3 - 3x + 4x^{-1}$

$\frac{dy}{dx} = 6x^2 - 3 - 4x^{-2}$

$\left.\frac{dy}{dx}\right|_{x=2} = 6\times2^2 - 3 - 4\times2^{-2}$

$\left.\frac{dy}{dx}\right|_{x=2} = 24 - 3 - 1 = 20$

(b) SET $\frac{dy}{dx} = 20$

$\Rightarrow 6x^2 - 3 - 4x^{-2} = 20$

$\Rightarrow 6x^2 - 3 - \frac{4}{x^2} = 20$

$\Rightarrow 6x^2 - 23 - \frac{4}{x^2} = 0$

$\Rightarrow 6x^4 - 23x^2 - 4 = 0$

$\Rightarrow (6x^2 + 1)(x^2 - 4) = 0$

$\Rightarrow x^2 = \begin{cases} 4 \\ \cancel{-\frac{1}{6}} \end{cases}$

$\Rightarrow x = \begin{cases} 2 & \leftarrow P \\ -2 & \leftarrow Q \end{cases}$

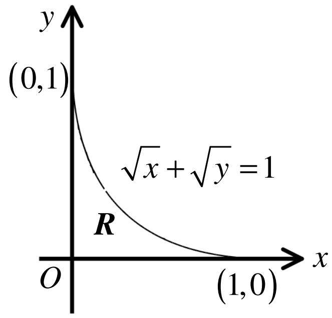
$\Rightarrow$ HENCE $x=-2$ $\quad y = 2(-2)^3 - 3(-2) + \frac{4}{-2}$

$y = -16 + 6 - 2$

$y = -12$

$\therefore Q(-2,-12)$

Question 235 **(****)**



The figure above shows the curve with equation

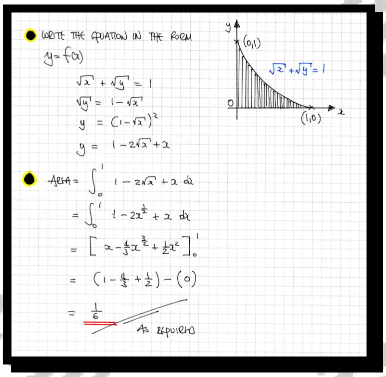
$$\sqrt{x}+\sqrt{y}=1\ ,\ x\in\mathbb{R},\ 0\le x\le 1.$$

The curve meets the coordinate axes at the points $(1,0)$ and $(0,1)$.

The finite region R is bounded by the curve and the coordinate axes.

Show that the area of R is $\frac{1}{6}$.

, $\boxed{\text{proof}}$


- WRITE THE EQUATION IN THE FORM $y=f(x)$

$\sqrt{x}+\sqrt{y}=1$

$\sqrt{y}=1-\sqrt{x}$

$y=(1-\sqrt{x})^2$

$y=1-2\sqrt{x}+x$

- AREA $=\int_0^1 1-2\sqrt{x}+x\,dx$

$=\int_0^1 1-2x^{\frac{1}{2}}+x\,dx$

$=\left[x-\frac{4}{3}x^{\frac{3}{2}}+\frac{1}{2}x^2\right]_0^1$

$=\left(1-\frac{4}{3}+\frac{1}{2}\right)-(0)$

$=\frac{1}{6}$ AS REQUIRED

Question 236 (****) non calculator

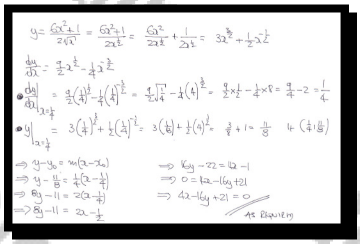
The curve C has equation

$$y = \frac{6x^2 + 1}{2\sqrt{x}}, \; x > 0$$

Show that an equation of the tangent to C at the point where $x = \frac{1}{4}$ is

$$4x - 16y + 21 = 0.$$

$\boxed{\text{proof}}$



$y = \frac{6x^2+1}{2\sqrt{x}} = \frac{6x^2+1}{2x^{\frac{1}{2}}} = \frac{6x^2}{2x^{\frac{1}{2}}} + \frac{1}{2x^{\frac{1}{2}}} = 3x^{\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{9}{2}x^{\frac{1}{2}} - \frac{1}{4}x^{-\frac{3}{2}}$

• $\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}} = \frac{9}{2}\left(\frac{1}{4}\right)^{\frac{1}{2}} - \frac{1}{4}\left(\frac{1}{4}\right)^{-\frac{3}{2}} = \frac{9}{2}\sqrt{\frac{1}{4}} - \frac{1}{4}(4)^{\frac{3}{2}} = \frac{9}{2}\times\frac{1}{2} - \frac{1}{4}\times 8 = \frac{9}{4} - 2 = \frac{1}{4}$

• $\left.y\right|_{x=\frac{1}{4}} = 3\left(\frac{1}{4}\right)^{\frac{3}{2}} + \frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{1}{2}} = 3\left(\frac{1}{8}\right) + \frac{1}{2}(4)^{\frac{1}{2}} = \frac{3}{8} + 1 = \frac{11}{8}$ $\quad \left(\frac{1}{4}, \frac{11}{8}\right)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - \frac{11}{8} = \frac{1}{4}\left(x - \frac{1}{4}\right)$

$\Rightarrow 8y - 11 = 2\left(x - \frac{1}{4}\right)$

$\Rightarrow 8y - 11 = 2x - \frac{1}{2}$

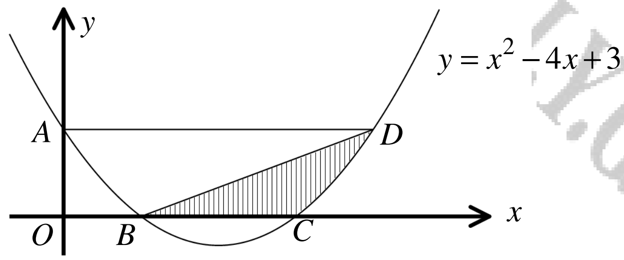
$\Rightarrow 16y - 22 = 4x - 1$

$\Rightarrow 0 = 4x - 16y + 21$

$\Rightarrow 4x - 16y + 21 = 0$

AS REQUIRED

Question 237 (****)



The figure above shows a quadratic curve with equation

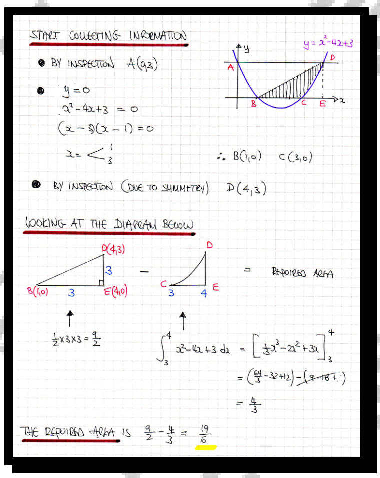
$$y = x^2 - 4x + 3.$$

The points A, B and C are the points where the curve meets the coordinate axes.

The point D lies on the curve so that AD is parallel to the x axis.

Calculate the exact area of the shaded region, bounded by the curve, the x axis and the straight line segment BD.

$\boxed{}$, $\boxed{\text{area} = \frac{19}{6}}$



Question 238 (****) **non calculator**

The curve C has equation

$$y = \frac{12x^2 + 1}{8\sqrt{x}}, \quad x > 0$$

Find the exact coordinates of the turning point of C.

$$\boxed{\left(\frac{1}{6}, \frac{\sqrt{6}}{6}\right)}$$

$$y = \frac{12x^2+1}{8\sqrt{x}} = \frac{12x^2+1}{8x^{\frac{1}{2}}} = \frac{12x^2}{8x^{\frac{1}{2}}} + \frac{1}{8x^{\frac{1}{2}}} = \frac{3}{2}x^{\frac{3}{2}} + \frac{1}{8}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{9}{4}x^{\frac{1}{2}} - \frac{1}{16}x^{-\frac{3}{2}}$$

• SOLVING FOR ZERO

$$\Rightarrow \frac{9}{4}x^{\frac{1}{2}} - \frac{1}{16}x^{-\frac{3}{2}} = 0$$

$$\Rightarrow \frac{9}{4}x^{\frac{1}{2}} = \frac{1}{16}x^{-\frac{3}{2}}$$

$$\Rightarrow 36x^{\frac{1}{2}} = x^{-\frac{3}{2}}$$

$$\Rightarrow 36x^{\frac{1}{2}} = \frac{1}{x^{\frac{3}{2}}}$$

$$\Rightarrow 36x^{\frac{1}{2}}x^{\frac{3}{2}} = 1$$

$$\Rightarrow 36x^2 = 1$$

$$\Rightarrow x^2 = \frac{1}{36}$$

$$\Rightarrow x = +\frac{1}{6} \quad (x>0)$$

$$y = \frac{12x^2+1}{8\sqrt{x}}$$

$$y = \frac{12 \times \frac{1}{36} + 1}{8 \times \sqrt{\frac{1}{6}}}$$

$$y = \frac{\frac{1}{3}+1}{\frac{8}{\sqrt{6}}}$$

$$y = \frac{\frac{4}{3}}{\frac{8}{\sqrt{6}}}$$

$$y = \frac{4\sqrt{6}}{24}$$

$$y = \frac{\sqrt{6}}{6}$$

$$\therefore \left(\frac{1}{6}, \frac{\sqrt{6}}{6}\right)$$

Question 239 **(****)**

A curve has equation

$$y = 6\sqrt[3]{x^5} - 15\sqrt[3]{x^4} - 80x + 16, \qquad x \in \mathbb{R}, \qquad x \geq 0.$$

Find the coordinates of the stationary point of the curve and determine whether it is a local maximum, a local minimum or a point of inflexion.

$\boxed{\text{local minimum at } (16, -2800)}$

WRITE THE EQUATION IN INDICIAL FORM AND DIFFERENTIATE

$$\Rightarrow y = 6\sqrt[3]{x^5} - 15\sqrt[3]{x^4} - 80x + 16$$
$$\Rightarrow y = 6x^{\frac{5}{3}} - 15x^{\frac{4}{3}} - 80x + 16$$
$$\Rightarrow \frac{dy}{dx} = 10x^{\frac{2}{3}} - 20x^{\frac{1}{3}} - 80$$

FOR STATIONARY POINTS, SOLVING $\frac{dy}{dx} = 0$

$$\Rightarrow 0 = 10x^{\frac{2}{3}} - 20x^{\frac{1}{3}} - 80$$
$$\Rightarrow x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 8 = 0$$
$$\Rightarrow \left(x^{\frac{1}{3}}\right)^2 - 2\left(x^{\frac{1}{3}}\right) - 8 = 0$$
$$\Rightarrow a^2 - 2a - 8 = 0, \quad \text{WHERE } a = x^{\frac{1}{3}}$$
$$\Rightarrow (a+2)(a-4) = 0$$
$$\Rightarrow a = \begin{cases} -2 \\ 4 \end{cases}$$
$$\Rightarrow x^{\frac{1}{3}} = \begin{cases} -2 \\ 4 \end{cases}$$
$$\Rightarrow x = \begin{cases} \cancel{-8} \quad (x \geqslant 0) \\ 64 \end{cases}$$
$$\Rightarrow y = 6 \times 64^{\frac{5}{3}} - 15 \times 64^{\frac{4}{3}} - 80 \times 64 + 16$$
$$= 6 \times 1024 - 15 \times 256 - 80 \times 64 + 16$$
$$= 6144 - 3840 - 5120 + 16$$
$$= -2800$$

DETERMINING THE NATURE BY THE SECOND DERIVATIVE TEST

$$\frac{dy}{dx} = 10x^{\frac{2}{3}} - 20x^{\frac{1}{3}} - 80$$
$$\frac{d^2y}{dx^2} = \frac{20}{3}x^{-\frac{1}{3}} - \frac{20}{3}x^{-\frac{2}{3}}$$
$$\frac{d^2y}{dx^2} = \frac{20}{3}\left[\frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt[3]{x^2}}\right]$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=64} = \frac{20}{3}\left[\frac{1}{\sqrt[3]{64}} - \frac{1}{\sqrt[3]{64^2}}\right] = \frac{5}{4} > 0$$

$\therefore$ (16, −2800) IS A LOCAL MINIMUM

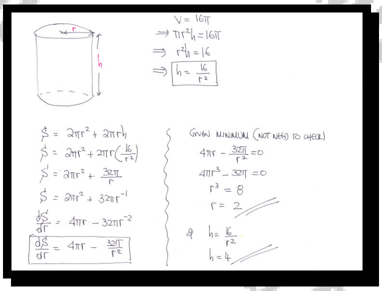
Question 240 (****)

A tank is in the shape of a closed right circular cylinder of radius r m and height h m.

The tank has a volume of 16π m^3 and is made of thin sheet metal.

Given the surface area of the tank is a minimum, determine the value of r and the value of h.

$\boxed{}$, $\boxed{r=2}$, $\boxed{h=4}$



Question 241 (****+)

The curve C has equation

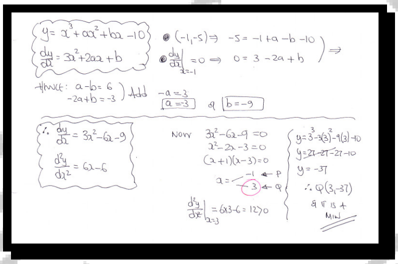
$$y = x^3 + ax^2 + bx - 10,$$

where a and b are constants.

The curve has two stationary points P and Q.

Given the coordinates of P are $(-1,-5)$, find the coordinates of Q and use $\frac{d^2y}{dx^2}$ to determine its nature.

$\boxed{}$, $\boxed{Q(3,-37),\text{min}}$



Question 242 (****+)

It is given that

$$\int_1^2 kx^2 + a \ dx = 11 \qquad \text{and} \qquad \int_1^k \frac{6}{x^2} \ dx = a,$$

where a and k are constants.

Determine the possible values of k.

MP1-U, $\boxed{k = 3 \cup k = -\frac{6}{7}}$

LOOKING AT THE FIRST EQUATION

$$\int_1^2 kx^2 + a \ dx = 11 \implies \left[\tfrac{1}{3}kx^3 + ax\right]_1^2 = 11$$

$$\implies \left(\tfrac{8}{3}k + 2a\right) - \left(\tfrac{1}{3}k + a\right) = 11$$

$$\implies \tfrac{7}{3}k + a = 11$$

LOOKING AT THE SECOND EQUATION

$$\int_1^k \frac{6}{x^2} \ dx = a \implies \left[-\frac{6}{x}\right]_1^k = a$$

$$\implies -\frac{6}{k} + 6 = a$$

SOLVING SIMULTANEOUSLY BY SUBSTITUTION

$$\implies \tfrac{7}{3}k + \left(-\tfrac{6}{k} + 6\right) = 11$$

$$\implies \tfrac{7}{3}k - \tfrac{6}{k} + 6 = 11 \quad \times 3k$$

$$\implies 7k^2 - 18 + 18k = 33k$$

$$\implies 7k^2 - 15k - 18 = 0$$

$$\implies (7k+6)(k-3) = 0$$

$$k = \begin{cases} 3 \\ -\frac{6}{7} \end{cases}$$

Question 243 **(****+)**

The profit of a small business, £ P is modelled by the equation

$$P = \frac{(54x + 6y - xy - 324)^2}{3x},$$

where x and y are positive variables associated with the running of the company.

It is further known that x and y constrained by the relation

$$3x + y = 54.$$

a) Show clearly that

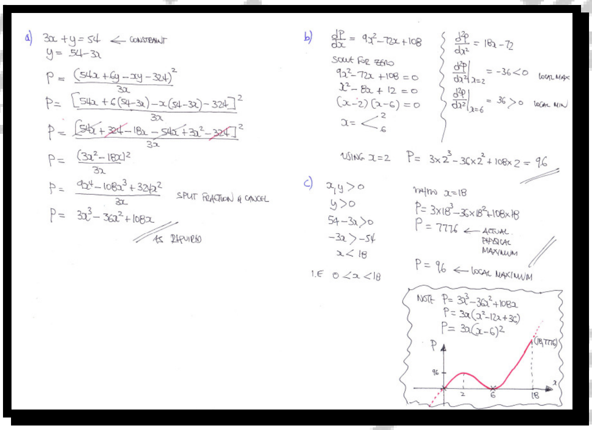
$$P = 108x - 36x^2 + 3x^3.$$

b) Hence show that the stationary value of P produces a maximum value of £96.

The owner is very concerned about the very small profit and shows the calculations to a mathematician. The mathematician agrees that the calculations are correct but he asserts that the profit is substantially higher.

c) Explain, by calculations, the mathematician's reasoning.

$\boxed{}$, $\boxed{\text{proof}}$



a) $3x + y = 54 \leftarrow$ CONSTRAINT

$y = 54 - 3x$

$P = \frac{(54x + 6y - xy - 324)^2}{3x}$

$P = \frac{[54x + 6(54 - 3x) - x(54 - 3x) - 324]^2}{3x}$

$P = \frac{[\cancel{54x} + \cancel{324} - 18x - \cancel{54x} + 3x^2 - \cancel{324}]^2}{3x}$

$P = \frac{(3x^2 - 18x)^2}{3x}$

$P = \frac{9x^4 - 108x^3 + 324x^2}{3x}$ SPLIT FRACTION & CANCEL

$P = 3x^3 - 36x^2 + 108x$

AS REQUIRED

b) $\frac{dP}{dx} = 9x^2 - 72x + 108$

SOLVE FOR ZERO

$9x^2 - 72x + 108 = 0$

$x^2 - 8x + 12 = 0$

$(x-2)(x-6) = 0$

$x = \begin{cases} 2 \\ 6 \end{cases}$

$\frac{d^2P}{dx^2} = 18x - 72$

$\left.\frac{d^2P}{dx^2}\right|_{x=2} = -36 < 0$ LOCAL MAX

$\left.\frac{d^2P}{dx^2}\right|_{x=6} = 36 > 0$ LOCAL MIN

USING $x = 2$ $P = 3 \times 2^3 - 36 \times 2^2 + 108 \times 2 = 96$

c) $x, y > 0$

$y > 0$

$54 - 3x > 0$

$-3x > -54$

$x < 18$

I.E $0 < x < 18$

TAKING $x = 18$

$P = 3 \times 18^3 - 36 \times 18^2 + 108 \times 18$

$P = 7776 \leftarrow$ ACTUAL PHYSICAL MAXIMUM

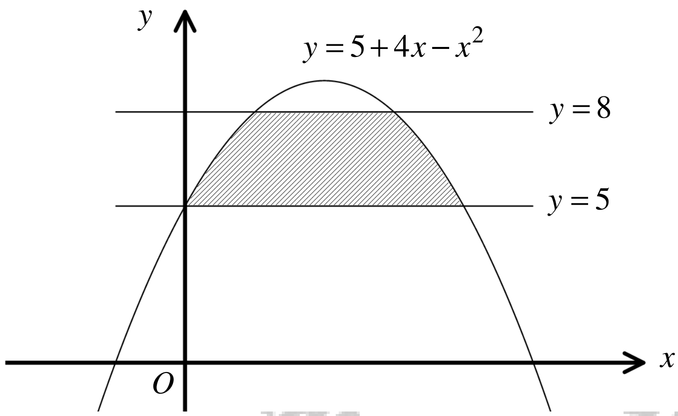
$P = 96 \leftarrow$ LOCAL MAXIMUM

NOTE $P = 3x^3 - 36x^2 + 108x$

$P = 3x(x^2 - 12x + 36)$

$P = 3x(x-6)^2$

Question 244 (****+)



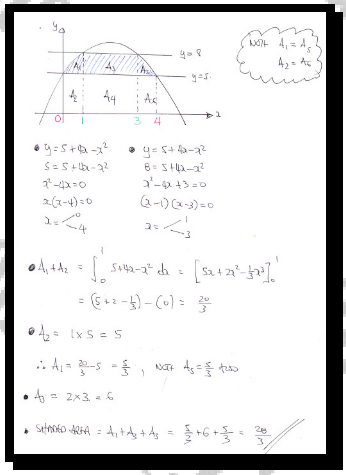
The figure above shows the curve C with equation

$$y = 5 + 4x - x^2,$$

intersected by the horizontal straight lines with equations $y = 5$ and $y = 8$.

Calculate the exact area of the shaded region, bounded by C and the two straight lines.

$\boxed{}$, $\boxed{\text{area} = \frac{28}{3}}$



Question 245 **(****+)** **non calculator**

$$y = \frac{1}{\sqrt{x}} + 5\sqrt{x}\,,\ \ x > 0\,.$$

Show clearly that

$$\int_1^4 y^3 \ dx = 1931\,.$$

proof

$$y = \frac{1}{\sqrt{x}} + 5\sqrt{x} = x^{-\frac{1}{2}} + 5x^{\frac{1}{2}}$$

$$y^3 = \left(x^{-\frac{1}{2}} + 5x^{\frac{1}{2}}\right)^3$$

$$y^3 = \left(x^{-\frac{1}{2}} + 5x^{\frac{1}{2}}\right)\left(x^{-\frac{1}{2}} + 5x^{\frac{1}{2}}\right)^2$$

$$y^3 = \left(x^{-\frac{1}{2}} + 5x^{\frac{1}{2}}\right)\left(x^{-1} + 10\cancel{x^{-\frac{1}{2}}x^{\frac{1}{2}}} + 25x\right)$$

$$y^3 = x^{-\frac{3}{2}} + 10x^{-\frac{1}{2}} + 25x^{\frac{1}{2}} + 5x^{-\frac{1}{2}} + 50x^{\frac{1}{2}} + 125x^{\frac{3}{2}}$$

$$y^3 = x^{-\frac{3}{2}} + 15x^{-\frac{1}{2}} + 75x^{\frac{1}{2}} + 125x^{\frac{3}{2}}$$

HENCE $$\int_1^4 y^3\,dx = \int_1^4 x^{-\frac{3}{2}} + 15x^{-\frac{1}{2}} + 75x^{\frac{1}{2}} + 125x^{\frac{3}{2}}\,dx$$

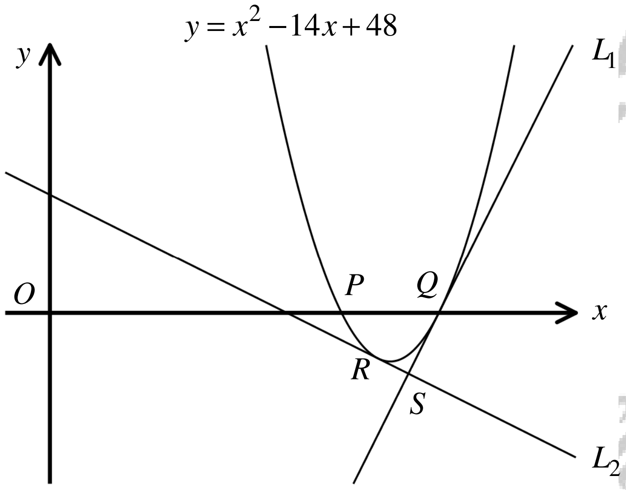
$$= \left[-2x^{-\frac{1}{2}} + 30x^{\frac{1}{2}} + 50x^{\frac{3}{2}} + 50x^{\frac{5}{2}}\right]_1^4$$

$$= (-1 + 60 + 400 + 1600) - (-2 + 30 + 50 + 50)$$

$$= 2059 - 128$$

$$= 1931$$

Question 246 (****+)



The figure above shows the curve with equation

$$y = x^2 - 14x + 48 .$$

The curve crosses the x axis at the points $P(x_1, 0)$ and $Q(x_2, 0)$, where $x_2 > x_1$.

The tangent to the curve at Q is the straight line L_1.

The tangent to the curve at some point R is denoted by L_2.

Given further that L_1 meets L_2 at right angles, find the coordinates of the point of intersection between L_1 meets L_2, denoted in the figure by S.

$\boxed{}$, $\boxed{S\left(\frac{59}{8}, -\frac{5}{4}\right)}$

START BY FINDING THE COORDINATES OF Q

$y = x^2 - 14x + 48$
$y = (x-6)(x-8)$
$\therefore$ $P(6,0)$ $Q(8,0)$

FIND THE GRADIENT AT Q

$\frac{dy}{dx} = 2x - 14$
$\left.\frac{dy}{dx}\right|_{x=8} = 2 \times 8 - 14 = 2$

EQUATION OF TANGENT (L_1)

$y - y_0 = m(x - x_0)$
$y - 0 = 2(x - 8)$
$y = 2x - 16$

AS L_2 IS PERPENDICULAR TO L_2 ITS GRADIENT IS $-\frac{1}{2}$

$\frac{dy}{dx} = -\frac{1}{2} \Rightarrow 2x - 14 = -\frac{1}{2}$
$\Rightarrow 4x - 28 = -1$
$\Rightarrow 4x = 27$
$\Rightarrow x = \frac{27}{4}$ ← POINT R

$y = \left(\frac{27}{4}\right)^2 - 14 \times \frac{27}{4} + 48$
$y = -\frac{15}{16}$

$\therefore R\left(\frac{27}{4}, -\frac{15}{16}\right)$

FIND THE EQUATION OF L_2

$y - y_0 = m(x - x_0)$
$y + \frac{15}{16} = -\frac{1}{2}\left(x - \frac{27}{4}\right)$

SOLVING SIMULTANEOUSLY

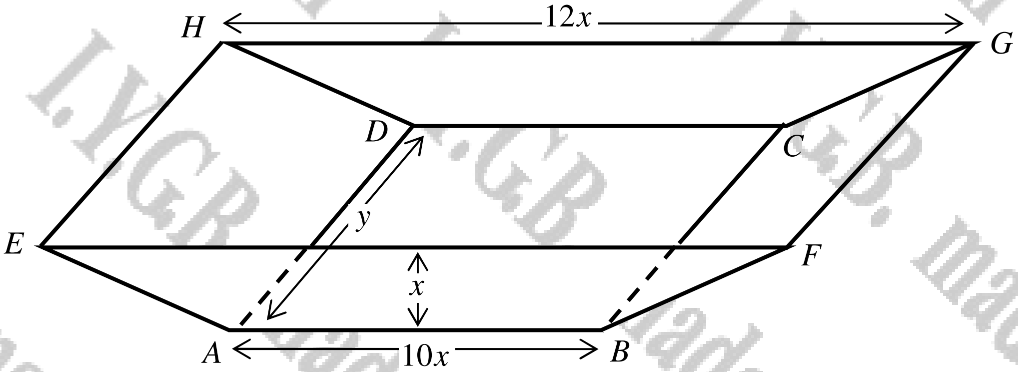
$y + \frac{15}{16} = -\frac{1}{2}\left(x - \frac{27}{4}\right)$ & $y = 2x - 16$

$(2x - 16) + \frac{15}{16} = -\frac{1}{2}\left(x - \frac{27}{4}\right)$ $\times 16$
$32x - 256 + 15 = -8\left(x - \frac{27}{4}\right)$
$32x - 241 = -8x + 54$
$40x = 295$
$x = \frac{59}{8}$

& $y = 2\left(\frac{59}{8}\right) - 16$
$y = -\frac{5}{4}$

$\therefore S\left(\frac{59}{8}, -\frac{5}{4}\right)$

Question 247 **(****+)**



The figure above shows the design of a baking tray with a horizontal rectangular base $ABCD$, measuring $10x$ cm by y cm.

The faces $ABFE$ and $DCGH$ are isosceles trapeziums, parallel to each other.

The lengths of the edges EF and HG are $12x$ cm.

The faces $ADHE$ and $BCGF$ are identical rectangles.

The height of the tray is x cm.

The capacity of the tray is 1980 cm^3.

a) Show that the surface area, A cm^2, of the tray is given by

$$A = 22x^2 + \frac{360}{x}\left(5+\sqrt{2}\right).$$

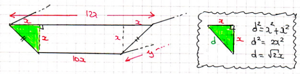
b) Determine the value of x for which A is stationary, showing that this value of x minimizes the value for A.

c) Calculate the minimum surface area of the tray.

$\boxed{}$, $\boxed{x \approx 3.744}$, $\boxed{A_{\text{min}} \approx 925}$

[solution overleaf]

a)



THE TOTAL SURFACE AREA

$$\Rightarrow A = \underbrace{10xy}_{\text{FLOOR}} + \underbrace{2\left[\frac{12x+10x}{2}x\right]}_{\text{TRAPEZOIDAL SIDES}} + \underbrace{2dy}_{\text{SLOPING SIDES}}$$

$$\Rightarrow A = 10xy + 22x^2 + 2y(\sqrt{2}x)$$

$$\Rightarrow A = 22x^2 + 10xy + 2\sqrt{2}xy$$

$$\Rightarrow A = 22x^2 + (10 + 2\sqrt{2})xy$$

$$\Rightarrow A = 22x^2 + (10 + 2\sqrt{2})\left(\frac{180}{x}\right)$$

$$\Rightarrow A = 22x^2 + \frac{360}{x}(5 + \sqrt{2})$$

AS REQUIRED

CONSTRAINT ON CAPACITY

$$V = 1980$$

$$1980 = \underbrace{\left(\frac{12x+10x}{2}\right)x}_{\text{AREA OF TRAPEZIUM ABFE}} \times y$$

$$11x^2y = 1980$$

$$x^2y = 180$$

$$x(xy) = 180$$

$$xy = \frac{180}{x}$$

b) DIFFERENTIATE THE ABOVE EQUATION & SOLVING FOR ZERO

$$\Rightarrow A = 22x^2 + 360(5+\sqrt{2})x^{-1}$$

$$\Rightarrow \frac{dA}{dx} = 44x - 360(5+\sqrt{2})x^{-2}$$

$$\Rightarrow \frac{dA}{dx} = 44x - \frac{360(5+\sqrt{2})}{x^2}$$

SOLVING FOR ZERO

$$\Rightarrow 0 = 44x - \frac{360(5+\sqrt{2})}{x^2}$$

$$\Rightarrow 0 = 44x^3 - 360(5+\sqrt{2})$$

$$\Rightarrow x^3 = \frac{90(5+\sqrt{2})}{11}$$

$$\Rightarrow x \approx 3.744$$

USING THE SECOND DERIVATIVE

$$\Rightarrow \frac{d^2A}{dx^2} = 44 + 720(5+\sqrt{2})x^{-3}$$

$$\Rightarrow \frac{d^2A}{dx^2} = 132 > 0$$

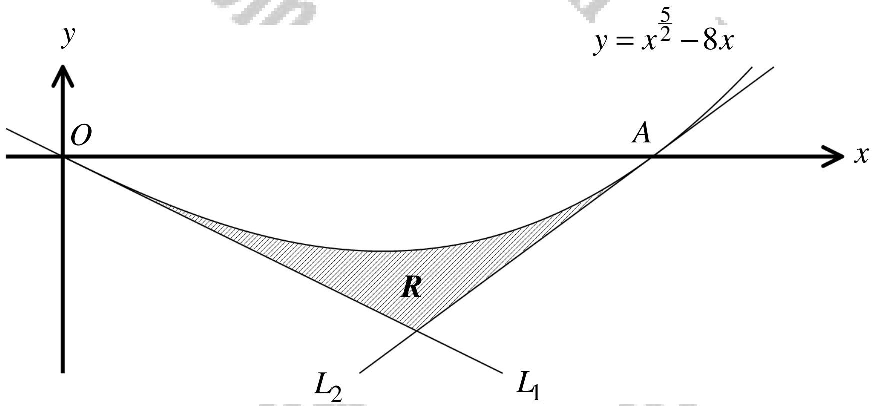
INDEED $x \approx 3.744$ MINIMIZES A

c) $A = 22x^2 + \frac{360}{x}(5+\sqrt{2})$

$$\Rightarrow A_{MIN} = 22(3.744)^2 + \frac{360}{3.744}(5+\sqrt{2})$$

$$\Rightarrow A_{MIN} \simeq 925 \text{ cm}^3$$

Question 248 (****+)



The figure above shows the graph of the curve C with equation

$$y = x^{\frac{5}{2}} - 8x, \ x \geq 0.$$

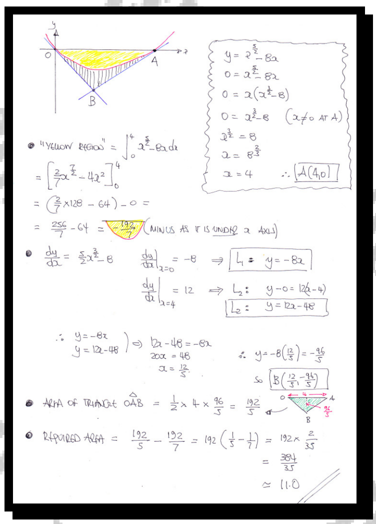
The curve meets the x axis at the origin O and at the point A.

The tangent to C at O is denoted by L_1 and the tangent to C at A is denoted by L_2.

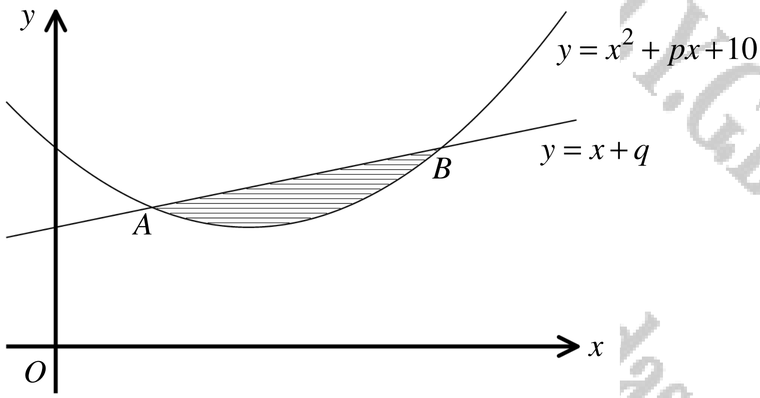
The finite region R, shown shaded in the figure above, is bounded by C, L_1 and L_2.

Determine the area of R.

$$\boxed{\frac{384}{35} \approx 11.0}$$



Question 249 (****+)



The figure above shows the graphs of a quadratic curve and a straight line with respective equations

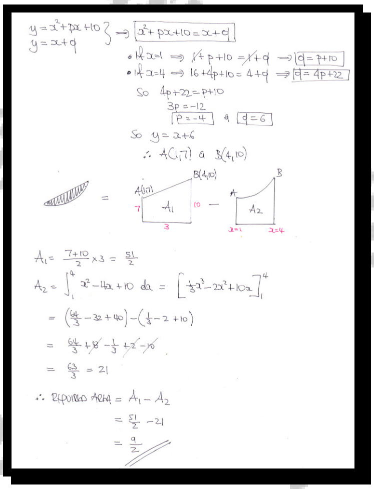
$$y = x^2 + px + 10 \quad \text{and} \quad y = x + q,$$

where p and q are constants.

The curve and the straight line intersect at the points A and B, whose x coordinates are 1 and 4, respectively.

Determine the area of the finite region bounded by the curve and the straight line, shown shaded in the figure.

$\boxed{\text{area} = \frac{9}{2}}$



$\left.\begin{array}{l} y = x^2 + px + 10 \\ y = x + q \end{array}\right\} \Rightarrow \boxed{x^2 + px + 10 = x + q}$

- If $x = 1 \Rightarrow \cancel{1} + p + 10 = \cancel{1} + q \Rightarrow \boxed{q = p + 10}$
- If $x = 4 \Rightarrow 16 + 4p + 10 = 4 + q \Rightarrow \boxed{q = 4p + 22}$

So $4p + 22 = p + 10$

$3p = -12$

$\boxed{p = -4}$ & $\boxed{q = 6}$

So $y = x + 6$

$\therefore A(1,7)$ & $B(4,10)$

= A(1,7) B(4,10) 7 A_1 10 3 − A B A_2 $x=1$ $x=4$

$A_1 = \frac{7+10}{2} \times 3 = \frac{51}{2}$

$A_2 = \int_1^4 x^2 - 4x + 10 \, dx = \left[\frac{1}{3}x^3 - 2x^2 + 10x\right]_1^4$

$= \left(\frac{64}{3} - 32 + 40\right) - \left(\frac{1}{3} - 2 + 10\right)$

$= \frac{64}{3} + \cancel{8} - \frac{1}{3} + \cancel{2} - \cancel{10}$

$= \frac{63}{3} = 21$

$\therefore$ REQUIRED AREA $= A_1 - A_2$

$= \frac{51}{2} - 21$

$= \frac{9}{2}$

Question 250 **(****+)**

The volume, V cm^3, of a soap bubble is modelled by the formula

$$V = (p - qt)^2, \ t \geq 0,$$

where p and q are positive constants, and t is the time in seconds, measured after a certain instant.

When $t = 1$ the volume of a soap bubble is 9 cm^3 and at that instant its volume is decreasing at the rate of 6 cm^3 per second.

Determine the value of p and the value of q.

$\boxed{}$, $\boxed{p = 4}$, $\boxed{q = 1}$

$V = (p-qt)^2 \implies V = p^2 - 2pqt + q^2t^2$

$\frac{dV}{dt} = -2pq + 2q^2t$

$t=1, V=9$

$9 = (p - q\times 1)^2$

$9 = (p-q)^2$

$\boxed{p - q = 3 \text{ or } -3}$

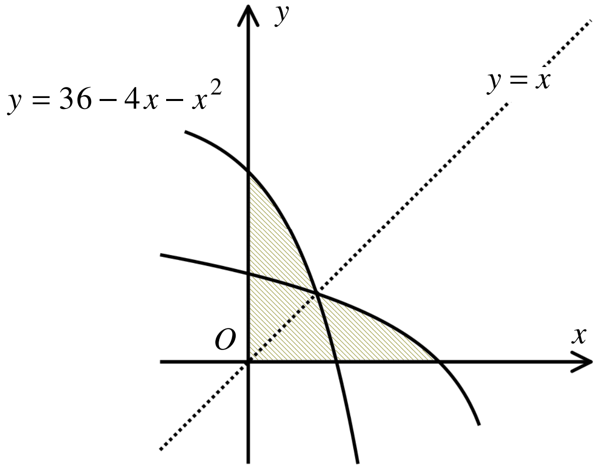
$-6 = -2pq + 2q^2 \times 1$

$\boxed{2pq - 6 = 2q^2}$

- $p - q = 3$
  $p = q + 3$
  $\downarrow$
  $2q(q+3) - 6 = 2q^2$
  $\cancel{2q^2} + 6q - 6 = \cancel{2q^2}$
  $6q = 6$
  $q = 1$
  $p = 4$

- $p - q = -3$
  $p = q - 3$
  $\downarrow$
  $2q(q-3) - 6 = 2q^2$
  $\cancel{2q^2} - 6q - 6 = \cancel{2q^2}$
  $-6q = 6$
  $\cancel{q = -1}$ $(q > 0)$

Question 251 (****+)



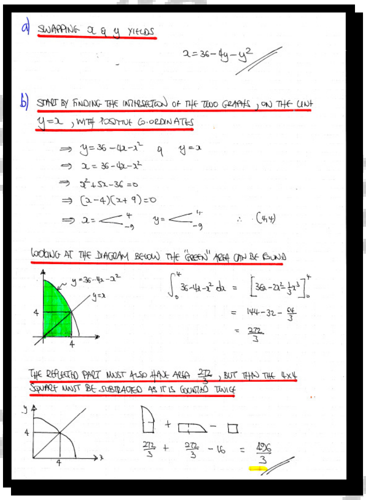
The figure above shows the graph of the curve with equation $y = 36 - 4x - x^2$ and its reflection about the straight line with equation $y = x$.

a) Write down the equation of the curve which is the reflection of the curve with equation $y = 36 - 4x - x^2$ about the straight line with equation $y = x$.

b) Determine an exact value for the area of the finite region bounded by the curve with equation $y = 36 - 4x - x^2$, its reflection about the straight line with equation $y = x$, and the positive x and y axes.

This region is shown shaded in the figure above.

$\boxed{x = 36 - 4y - y^2}$, $\boxed{\text{AREA} = \frac{496}{3} = 165\frac{1}{3}}$



a) SWAPPING x & y YIELDS

$x = 36 - 4y - y^2$

b) START BY FINDING THE INTERSECTION OF THE TWO GRAPHS, ON THE LINE $y = x$, WITH POSITIVE CO-ORDINATES

$\Rightarrow y = 36 - 4x - x^2$ & $y = x$

$\Rightarrow x = 36 - 4x - x^2$

$\Rightarrow x^2 + 5x - 36 = 0$

$\Rightarrow (x-4)(x+9) = 0$

$\Rightarrow x = \begin{cases} 4 \\ -9 \end{cases} \quad y = \begin{cases} 4 \\ -9 \end{cases} \quad \therefore (4,4)$

LOOKING AT THE DIAGRAM BELOW THE "GREEN" AREA CAN BE FOUND

$\int_0^4 36 - 4x - x^2 \, dx = \left[36x - 2x^2 - \frac{1}{3}x^3\right]_0^4$

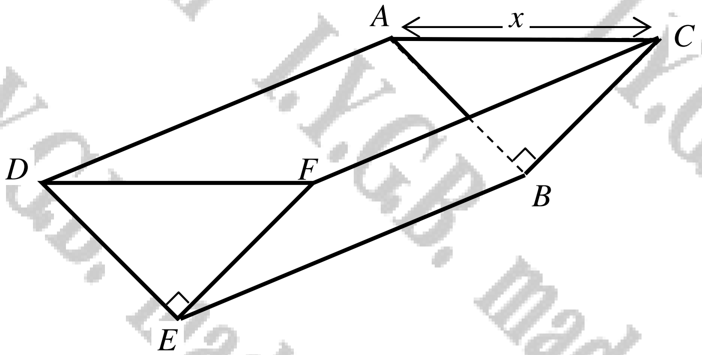
$= 144 - 32 - \frac{64}{3}$

$= \frac{272}{3}$

THE REFLECTED PART MUST ALSO HAVE AREA $\frac{272}{3}$, BUT THEN THE 4x4 SQUARE MUST BE SUBTRACTED AS IT IS COUNTED TWICE

$\frac{272}{3} + \frac{272}{3} - 16 = \frac{496}{3}$

Question 252 (****+)



The figure above shows the design of a horse feeder which in the shape of a hollow, open topped triangular prism.

The triangular faces at the two ends of the feeder are isosceles and right angled, so that $AB = BC = DE = EF$ and $\widehat{ABC} = \widehat{DEF} = 90°$.

The triangular faces are vertical, and the edges AD, BE and CF are horizontal.

The capacity of the feeder is 4 m^3.

a) Show that the surface area, $A \text{ m}^2$, of the feeder is given by

$$A = \frac{1}{2}x^2 + \frac{16\sqrt{2}}{x},$$

where x is the length of AC.

b) Determine by differentiation the value of x for which A is stationary, giving the answer in the form $k\sqrt{2}$, where k is an integer.

c) Show that the value of x found in part (**b**) gives the minimum value for A.

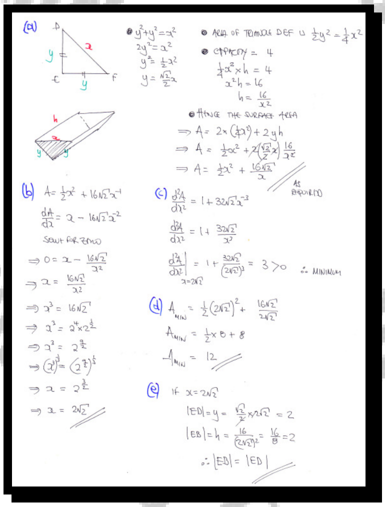
[continues overleaf]

[continued from overleaf]

d) Show, by exact calculations, that the minimum surface area of the feeder is 12 m^2.

e) Show further that the length ED is equal to the length EB.

$\boxed{}$, $\boxed{x = 2\sqrt{2} \approx 2.82}$, $\boxed{ED = EB = 2}$



(a)

- $y^2 + y^2 = x^2$
  $2y^2 = x^2$
  $y^2 = \frac{1}{2}x^2$
  $y = \frac{\sqrt{2}}{2}x$

- AREA OF TRIANGLE DEF IS $\frac{1}{2}y^2 = \frac{1}{4}x^2$
- CAPACITY = 4
  $\frac{1}{4}x^2 \times h = 4$
  $x^2 h = 16$
  $h = \frac{16}{x^2}$
- HENCE THE SURFACE AREA
  $\Rightarrow A = 2 \times \left(\frac{1}{4}x^2\right) + 2yh$
  $\Rightarrow A = \frac{1}{2}x^2 + \cancel{2}\left(\frac{\sqrt{2}}{\cancel{2}}x\right)\frac{16}{x^2}$
  $\Rightarrow A = \frac{1}{2}x^2 + \frac{16\sqrt{2}}{x}$ AS REQUIRED

(b) $A = \frac{1}{2}x^2 + 16\sqrt{2}x^{-1}$

$\frac{dA}{dx} = x - 16\sqrt{2}x^{-2}$

SOLVE FOR ZERO

$\Rightarrow 0 = x - \frac{16\sqrt{2}}{x^2}$
$\Rightarrow x = \frac{16\sqrt{2}}{x^2}$
$\Rightarrow x^3 = 16\sqrt{2}$
$\Rightarrow x^3 = 2^4 \times 2^{\frac{1}{2}}$
$\Rightarrow x^3 = 2^{\frac{9}{2}}$
$\Rightarrow (x^3)^{\frac{1}{3}} = \left(2^{\frac{9}{2}}\right)^{\frac{1}{3}}$
$\Rightarrow x = 2^{\frac{3}{2}}$
$\Rightarrow x = 2\sqrt{2}$

(c) $\frac{d^2A}{dx^2} = 1 + 32\sqrt{2}x^{-3}$

$\frac{d^2A}{dx^2} = 1 + \frac{32\sqrt{2}}{x^3}$

$\left.\frac{d^2A}{dx^2}\right|_{x=2\sqrt{2}} = 1 + \frac{32\sqrt{2}}{(2\sqrt{2})^3} = 3 > 0$ $\therefore$ MINIMUM

(d) $A_{MIN} = \frac{1}{2}(2\sqrt{2})^2 + \frac{16\sqrt{2}}{2\sqrt{2}}$

$A_{MIN} = \frac{1}{2} \times 8 + 8$

$A_{MIN} = 12$

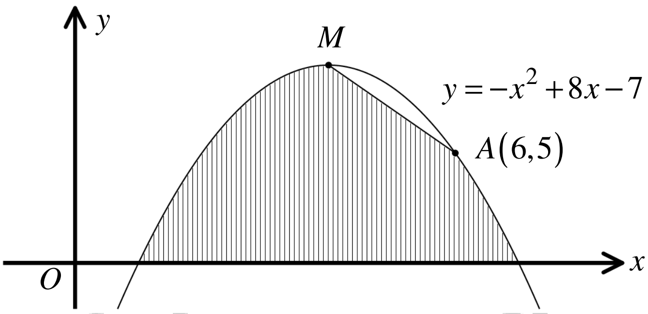
(e) IF $x = 2\sqrt{2}$

$|ED| = y = \frac{\sqrt{2}}{\cancel{2}} \times \cancel{2}\sqrt{2} = 2$

$|EB| = h = \frac{16}{(2\sqrt{2})^2} = \frac{16}{8} = 2$

$\therefore |EB| = |ED|$

Question 253 (****+)



The figure above shows the quadratic curve with equation

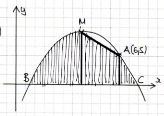
$$y = -x^2 + 8x - 7.$$

The point M is the maximum point of the curve and A is another point on the curve whose coordinates are $(6,5)$.

Find the exact area of the shaded region, bounded by the curve, the x axis and the straight line segment from A to M.

$\boxed{}$, $\boxed{\text{area} = \frac{104}{3}}$

FIND THE CO-ORDS OF M, BY COMPLETING THE SQUARE (OR CALCULUS)



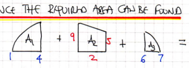
$\Rightarrow y = -x^2 + 8x - 7$
$\Rightarrow -y = x^2 - 8x + 7$
$\Rightarrow -y = (x-4)^2 - 16 + 7$
$\Rightarrow -y = (x-4)^2 - 9$
$\Rightarrow y = 9 - (x-4)^2$

$\therefore M(4,9)$

ALSO THE CO-ORDINATES OF B & C ARE NEEDED

$\Rightarrow y = 0$
$\Rightarrow -x^2 + 8x - 7 = 0$
$\Rightarrow x^2 - 8x + 7 = 0$
$\Rightarrow (x-1)(x-7) = 0$
$\Rightarrow x = \begin{cases} 1 \\ 7 \end{cases}$ $\quad \therefore B(1,0) \quad C(7,0)$

HENCE THE REQUIRED AREA CAN BE FOUND



$A_1 = \int_1^4 -x^2+8x-7 \, dx \quad A_2 = \frac{9+5}{2} \times 2 \quad A_3 = \int_6^7 -x^2+8x-7 \, dx$
$A_2 = 14$

$A_1 = \int_1^4 -x^2 + 8x - 7 \, dx = \left[-\frac{1}{3}x^3 + 4x^2 - 7x \right]_1^4$
$= \left(-\frac{64}{3} + 64 - 28\right) - \left(-\frac{1}{3} + 4 - 7\right)$
$= \frac{44}{3} - \left(-\frac{10}{3}\right)$
$= 18$

$A_3 = \int_6^7 -x^2 + 8x - 7 \, dx = \left[-\frac{1}{3}x^3 + 4x^2 - 7x \right]_6^7$
$= \left(-\frac{343}{3} + 196 - 49\right) - (-72 + 144 - 42)$
$= \frac{98}{3} - 30$
$= \frac{8}{3}$

THE AREA OF THE SHADED REGION IS $A_1 + A_2 + A_3$
$18 + 14 + \frac{8}{3} = \frac{104}{3}$

Question 254 (****+)

A curve C has equation

$$y = 2x^3 - 5x^2 + a, \; x \in \mathbb{R},$$

where a is a constant.

The tangent to C at the point where $x = 2$ and the normal to C at the point where $x = 1$, meet at the point Q.

Given that Q lies on the x axis, determine in any order …

a) … the value of a.

b) … the coordinates of Q.

$\boxed{}$, $\boxed{a = \frac{8}{3}}$, $\boxed{Q\left(\frac{7}{3}, 0\right)}$

$y = 2x^3 - 5x^2 + a$

- $\frac{dy}{dx} = 6x^2 - 10x$

$\left.\frac{dy}{dx}\right|_{x=2} = 6\times2^2 - 10\times2 = 4$

$\left.\frac{dy}{dx}\right|_{x=1} = 6\times1^2 - 10\times1 = -4$

$\therefore$ NORMAL GRADIENT $\frac{1}{4}$

- WHEN $x = 2$

$y = 2\times2^3 - 5\times2^2 + a$

$y = 16 - 20 + a$

$y = a - 4$ i.e. $(2, a-4)$

WHEN $x = 1$

$y = 2 - 5 + a$

$y = a - 3$ i.e. $(1, a-3)$

- EQUATION OF TANGENT

$y - (a-4) = 4(x-2)$

- EQUATION OF NORMAL

$y - (a-3) = \frac{1}{4}(x-1)$

$\Rightarrow$ NEXT FIND THE x INTERCEPT FOR EACH LINE, i.e. $y = 0$

TANGENT: $-a + 4 = 4x - 8$

NORMAL: $-a + 3 = \frac{1}{4}x - \frac{1}{4}$

$\Rightarrow \begin{cases} 4x = 12 - a \\ \frac{1}{4}x = 3 + \frac{1}{4} - a \end{cases}$

$\Rightarrow \begin{cases} x = \frac{12-a}{4} \\ x = 13 - 4a \end{cases}$

BUT THESE x INTERCEPTS MUST BE THE SAME (LINES INTERSECT)

$\frac{12-a}{4} = 13 - 4a$

$12 - a = 52 - 16a$

$15a = 40$

$a = \frac{8}{3}$

& $x = 13 - 4 \times \frac{8}{3}$

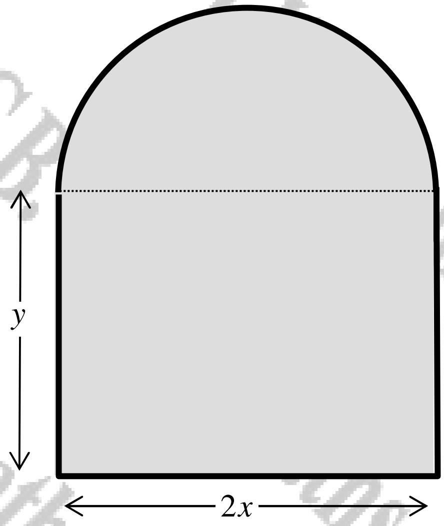
$x = 13 - \frac{32}{3}$

$x = \frac{39-32}{3}$

$x = \frac{7}{3}$ i.e. $Q\left(\frac{7}{3}, 0\right)$

Question 255 (****+)

The figure below shows the design of a window which is the shape of a semicircle attached to rectangle.



The diameter of the semicircle is $2x$ metres and is attached to one side of the rectangle also measuring $2x$ meters. The other side of the rectangle is y metres.

The total area of the window is 2 m^2.

a) Show that perimeter, P m, is given by

$$P = \frac{1}{2}(4+\pi)x + \frac{2}{x}.$$

b) Determine by differentiation an exact value of x for which P is stationary.

[continues overleaf]

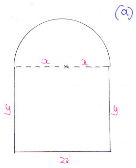
[continued from overleaf]

c) Show that the value of x found in part (b) gives the minimum value for P.

d) Show that when P takes a minimum value $x = y$.

$$\boxed{x = \frac{2}{\sqrt{\pi+4}} \approx 0.748}$$

(a)



- $A = 2$

$\Rightarrow 2xy + \frac{1}{2} \times \pi x^2 = 2$

$\Rightarrow 4xy + \pi x^2 = 4$

$\Rightarrow \boxed{y = \frac{4 - \pi x^2}{4x}}$

- $P = 2y + 2x + \frac{1}{2}(2\pi x)$

$\Rightarrow P = 2y + 2x + \pi x$

$\Rightarrow P = 2 \times \left(\frac{4 - \pi x^2}{4x}\right) + 2x + \pi x$

$\Rightarrow P = \frac{4 - \pi x^2}{2x} + 2x + \pi x$

$\Rightarrow P = \frac{2}{x} - \frac{\pi x}{2} + 2x + \pi x$

$\Rightarrow P = \frac{\pi x}{2} + 2x + \frac{2}{x}$

$\Rightarrow P = \frac{1}{2}x(\pi + 4) + \frac{2}{x}$

AS REQUIRED

(b) $P = \frac{1}{2}(\pi+4)x + 2x^{-1}$

$\boxed{\frac{dP}{dx} = \frac{1}{2}(\pi+4) - 2x^{-2}}$

SOLVE FOR ZERO

$\frac{1}{2}(\pi+4) - \frac{2}{x^2} = 0$

$\frac{1}{2}(\pi+4) = \frac{2}{x^2}$

$x^2 = \frac{2}{\frac{1}{2}(\pi+4)}$

$x^2 = \frac{4}{\pi+4}$

$x = \frac{2}{\sqrt{\pi+4}}$ $(x>0)$

(c) $\frac{d^2P}{dx^2} = 4x^{-3} = \frac{4}{x^3}$

$\left.\frac{d^2P}{dx^2}\right|_{x=\frac{2}{\sqrt{\pi+4}}} = \frac{4}{\left(\frac{2}{\sqrt{\pi+4}}\right)^3} > 0$

$\therefore$ INDEED A MINIMUM

(d) $y = \frac{4 - \pi x^2}{4x}$

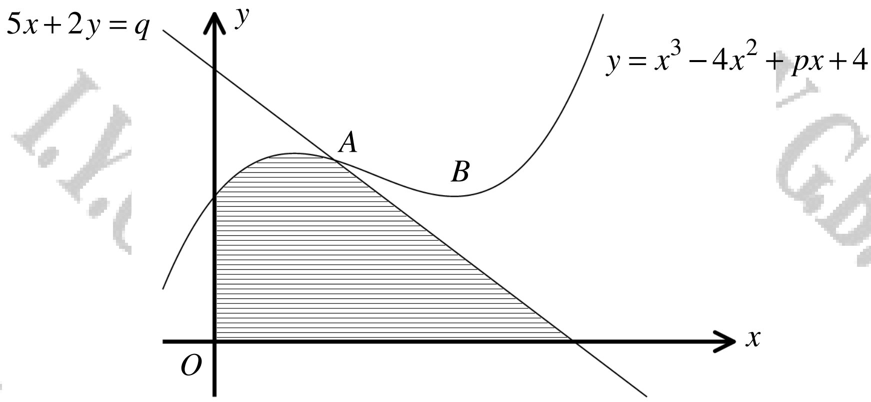
$y = \frac{4 - \pi\left(\frac{4}{\pi+4}\right)}{4 \times \left(\frac{2}{\sqrt{\pi+4}}\right)} = \frac{4 - \frac{4\pi}{\pi+4}}{\frac{8}{(\pi+4)^{\frac{1}{2}}}}$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY $\pi+4$

$y = \frac{4(\pi+4) - 4\pi}{8(\pi+4)^{\frac{1}{2}}} = \frac{\cancel{4\pi} + 16 - \cancel{4\pi}}{8(\pi+4)^{\frac{1}{2}}} = \frac{2}{(\pi+4)^{\frac{1}{2}}} = x$

AS REQUIRED

Question 256 (****+)



The figure above shows a curve C and a straight line L, with respective equations

$$y = x^3 - 4x^2 + px + 4 \qquad \text{and} \qquad 5x + 2y = q,$$

where p and q are constants.

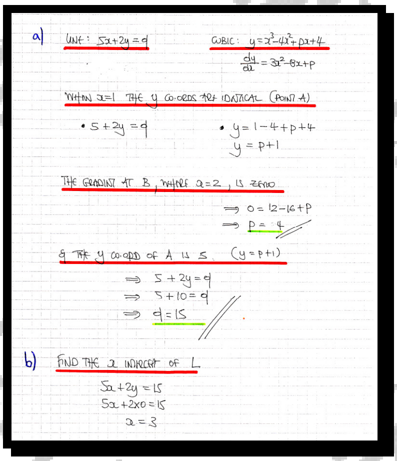
C and L intersect at the point A. The point B is a turning point C.

Given that the respective x coordinates of A and B are 1 and 2, determine …

a) … the value of p and the value of q

b) … the area of the shaded region bounded by C, L and the coordinate axes.

$\boxed{}$, $\boxed{p = 4,\ q = 15}$, $\boxed{\text{area} = \frac{119}{12}}$



a) LINE: $5x+2y=q$ CUBIC: $y=x^3-4x^2+px+4$

$\frac{dy}{dx}=3x^2-8x+p$

WHEN $x=1$ THE y CO-ORDS ARE IDENTICAL (POINT A)

• $5+2y=q$ • $y=1-4+p+4$, $y=p+1$

THE GRADIENT AT B, WHERE $x=2$, IS ZERO

$\Rightarrow 0=12-16+p$

$\Rightarrow p=4$

& THE y CO-ORD OF A IS 5 $(y=p+1)$

$\Rightarrow 5+2y=q$

$\Rightarrow 5+10=q$

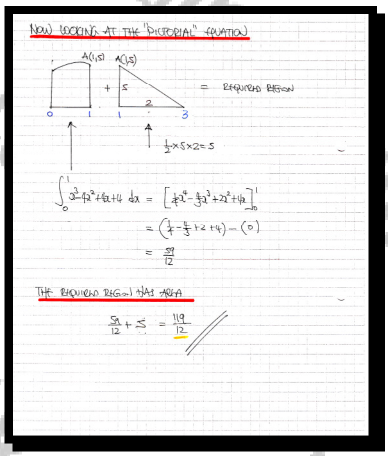
$\Rightarrow q=15$

b) FIND THE x INTERCEPT OF L

$5x+2y=15$

$5x+2\times0=15$

$x=3$



NOW LOOKING AT THE "PICTORIAL" EQUATION

A(1,5) A(1,5) + 5 2 = REQUIRED REGION

0 1 1 3

$\frac{1}{2}\times5\times2=5$

$\int_0^1 x^3-4x^2+4x+4\,dx = \left[\frac{1}{4}x^4-\frac{4}{3}x^3+2x^2+4x\right]_0^1$

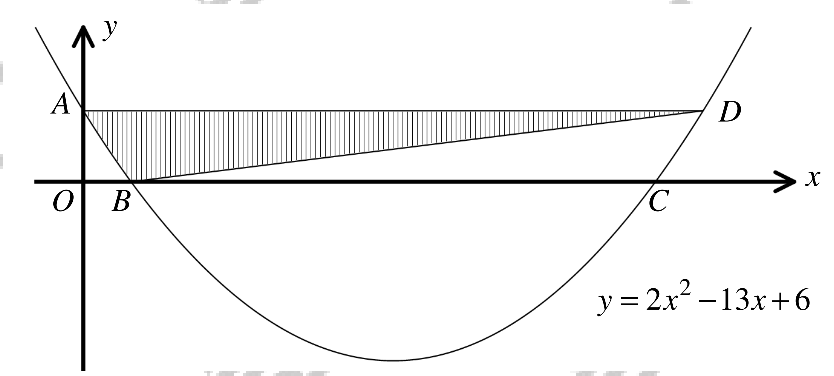
$=\left(\frac{1}{4}-\frac{4}{3}+2+4\right)-(0)$

$=\frac{59}{12}$

THE REQUIRED REGION HAS AREA

$\frac{59}{12}+5=\frac{119}{12}$

Question 257 (****+)



The figure above shows the curve with equation

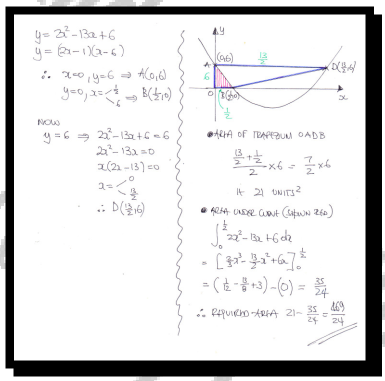
$$y = 2x^2 - 13x + 6 .$$

The points A, B and C are the points where the curve meets the coordinate axes.

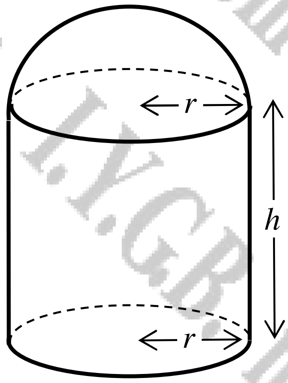
The point D is such so that the straight line segment AD is parallel to the x axis.

Find the exact area of the shaded region, bounded by the curve and the straight line segments BD and AD.

$\boxed{}$, $\boxed{\text{area} = \frac{469}{24}}$



Question 258 (****+)



The figure above shows a hollow container consisting of a right circular cylinder of radius r cm and of height h cm joined to a hemisphere of radius r cm.

The cylinder is open on one of the circular ends and the hemisphere is also open on its circular base. The cylinder is joined to the hemisphere at their open ends so that the resulting object is completely sealed.

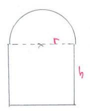
a) Given that volume of the container is exactly $2880\pi\ \text{cm}^3$, show clearly that the total surface area of the container, $S\ \text{cm}^2$, is given by

$$S = \frac{5\pi}{3r}\left(r^3 + 3456\right).$$

b) Show further than when S is minimum, $r = h$.

proof

a)



$V = 2880\pi$

$2880\pi = \left(\frac{1}{2}\times\frac{4}{3}\pi r^3\right) + \left(\pi r^2 h\right)$

$2880\pi = \frac{2}{3}\pi r^3 + \pi r^2 h$

$2880 = \frac{2}{3}r^3 + r^2 h$

$r^2 h = 2880 - \frac{2}{3}r^3$

$\boxed{rh = \frac{2880}{r} - \frac{2}{3}r^2}$

$S = \pi r^2 + 2\pi r h + \frac{1}{2}(4\pi r^2)$

$S = \pi r^2 + 2\pi r h + 2\pi r^2$

$S = 3\pi r^2 + 2\pi (rh)$

$S = 3\pi r^2 + 2\pi\left(\frac{2880}{r} - \frac{2}{3}r^2\right)$

$S = 3\pi r^2 + \frac{5760\pi}{r} - \frac{4\pi}{3}r^2$

$S = \frac{5}{3}\pi r^2 + \frac{5760\pi}{r}$

$S = \frac{5\pi}{3r}\left[r^3 + 3456\right]$ // AS REQUIRED

b) $S = \frac{5}{3}\pi r^2 + 5760\pi r^{-1}$

$\frac{dS}{dr} = \frac{10}{3}\pi r - 5760\pi r^{-2}$

$\frac{d^2S}{dr^2} = \frac{10}{3}\pi + 11520\pi r^{-3}$

- FOR MIN/MAX

$\frac{10}{3}\cancel{\pi} r - \frac{5760\cancel{\pi}}{r^2} = 0$

$\frac{10r}{3} = \frac{5760}{r^2}$

$10r^3 = 17280$

$r^3 = 1728$

$r = 12$ //

- $\left.\frac{d^2S}{dr^2}\right|_{r=12} = \frac{10}{3}\pi + \frac{11520\pi}{12^3}$

$\frac{d^2S}{dr^2} = 10\pi > 0$

$\therefore$ A MIN VALUE FOR S //

FINALLY $rh = \frac{2880}{r} - \frac{2}{3}r$

$12h = \frac{2880}{12} - \frac{2}{3}\times 12^2$

$12h = 240 - 96$

$12h = 144$

$h = 12$

$\therefore h = r$ INDEED //

Question 259 (****+)

A quadratic curve C passes through the points $P(a,b)$ and $Q(2a,2b)$, where a and b are constants.

The gradient at any given point on C is given by

$$\frac{dy}{dx} = 2x - 6.$$

Find an equation for C, in terms of a.

C1, $\boxed{y = x^2 - 6x + 2a^2}$

$\frac{dy}{dx} = 2x - 6$

$y = \int 2x - 6 \, dx$

$\boxed{y = x^2 - 6x + c}$

$P(a,b) \Rightarrow b = a^2 - 6a + c$
$Q(2a,2b) \Rightarrow 2b = 4a^2 - 12a + c$ $\Rightarrow$

$2b = 2a^2 - 12a + 2c$
$2b = 4a^2 - 12a + c$ $\Rightarrow$

$\Rightarrow 2a^2 - \cancel{12a} + 2c = 4a^2 - \cancel{12a} + c$

$\Rightarrow c = 2a^2$

$\therefore y = x^2 - 6x + 2a^2$

Question 260 (****+)

A curve has equation

$$y = 2x^3 + \frac{k}{x} - 19, \qquad x > 0,$$

where k is a positive constant.

If the y coordinate of the stationary point of this curve is 45, find the value of k.

$\boxed{}$, $\boxed{k = 96}$

FIND THE x COORDINATE OF THE STATIONARY POINT, IN TERMS OF k

$$y = 2x^3 + \frac{k}{x} - 19$$

$$\frac{dy}{dx} = 6x^2 - \frac{k}{x^2}$$

SOLVING $\frac{dy}{dx} = 0$

$$6x^2 - \frac{k}{x^2} = 0$$

$$6x^2 = \frac{k}{x^2}$$

$$x^4 = \frac{k}{6}$$

$$x = \frac{k^{1/4}}{6^{1/4}}$$

NOW $f\left(\frac{k^{1/4}}{6^{1/4}}\right) = 45$

$$45 = 2\left(\frac{k^{1/4}}{6^{1/4}}\right)^3 + \frac{k}{\frac{k^{1/4}}{6^{1/4}}} - 19$$

$$64 = 2\left(\frac{k^{3/4}}{6^{3/4}}\right) + \frac{6^{1/4} \times k}{k^{1/4}}$$

$$64 = \frac{2}{6^{3/4}}k^{\frac{3}{4}} + 6^{\frac{1}{4}}k^{\frac{3}{4}} \quad \Big\} \times 6^{\frac{3}{4}}$$

$$64 \times 6^{\frac{3}{4}} = 2k^{\frac{3}{4}} + 6k^{\frac{3}{4}}$$

$$64 \times 6^{\frac{3}{4}} = 8 \times k^{\frac{3}{4}}$$

$$8 \times 6^{\frac{3}{4}} = k^{\frac{3}{4}}$$

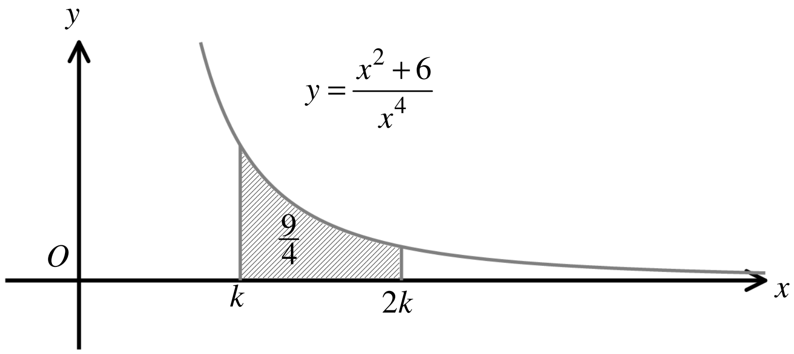
$$\left(k^{\frac{3}{4}}\right)^{\frac{4}{3}} = \left(8 \times 6^{\frac{3}{4}}\right)^{\frac{4}{3}}$$

$$k = 8^{\frac{4}{3}} \times 6$$

$$k = 16 \times 6$$

$\therefore k = 96$

Question 261 (****+)



The figure above shows the graph of the curve with equation

$$y = \frac{x^2+6}{x^4}, \quad x > 0.$$

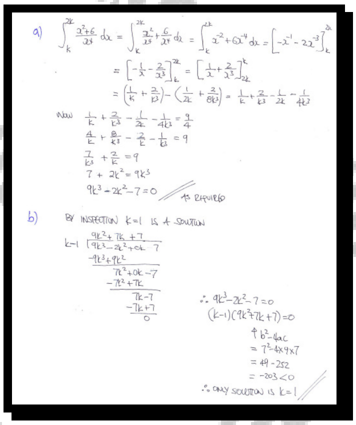
The area of the region between the curve and the x axis for $k \le x \le 2k$, where k is a positive constant, is exactly $\frac{9}{4}$.

a) Show clearly that

$$9k^3 - 2k^2 - 7 = 0.$$

b) Hence find the value of k, showing further that there is no other value of k which satisfies the equation of part **(b)**.

$\boxed{}$, $\boxed{k=1}$



a) $\int_k^{2k} \frac{x^2+6}{x^4}\,dx = \int_k^{2k} \frac{x^2}{x^4} + \frac{6}{x^4}\,dx = \int_k^{2k} x^{-2} + 6x^{-4}\,dx = \left[-x^{-1} - 2x^{-3}\right]_k^{2k}$

$= \left[-\frac{1}{x} - \frac{2}{x^3}\right]_k^{2k} = \left[\frac{1}{x} + \frac{2}{x^3}\right]_{2k}^{k}$

$= \left(\frac{1}{k} + \frac{2}{k^3}\right) - \left(\frac{1}{2k} + \frac{2}{8k^3}\right) = \frac{1}{k} + \frac{2}{k^3} - \frac{1}{2k} - \frac{1}{4k^3}$

Now $\frac{1}{k} + \frac{2}{k^3} - \frac{1}{2k} - \frac{1}{4k^3} = \frac{9}{4}$

$\frac{4}{k} + \frac{8}{k^3} - \frac{2}{k} - \frac{1}{k^3} = 9$

$\frac{7}{k^3} + \frac{2}{k} = 9$

$7 + 2k^2 = 9k^3$

$9k^3 - 2k^2 - 7 = 0$ // AS REQUIRED

b) BY INSPECTION $k=1$ IS A SOLUTION

$k-1 \,\overline{)\,9k^3 - 2k^2 + 0k - 7}$ giving $9k^2 + 7k + 7$

$-9k^3 + 9k^2$

$7k^2 + 0k - 7$

$-7k^2 + 7k$

$7k - 7$

$-7k + 7$

0

$\therefore 9k^3 - 2k^2 - 7 = 0$

$(k-1)(9k^2+7k+7) = 0$

$b^2 - 4ac$

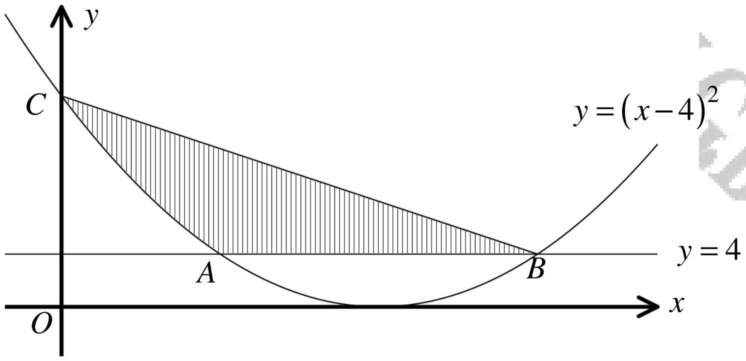
$= 7^2 - 4 \times 9 \times 7$

$= 49 - 252$

$= -203 < 0$

$\therefore$ ONLY SOLUTION IS $k=1$ //

Question 262 (****+)



The diagram above shows the curve with equation

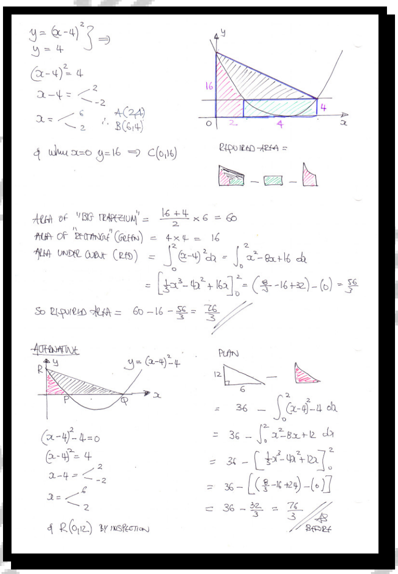
$$y=(x-4)^2,\ x\in\mathbb{R},$$

intersected by the straight line with equation $y=4$, at the points A and B.

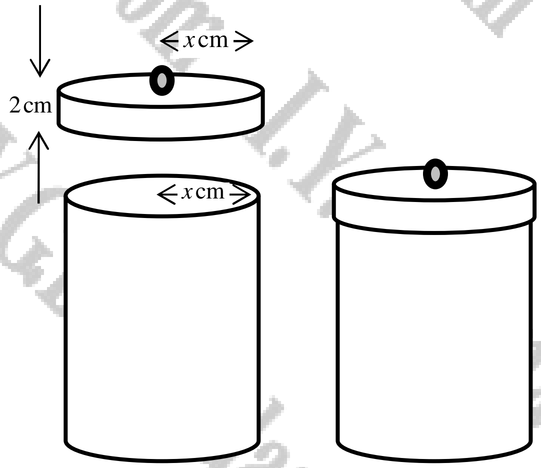
The curve meets the y axis at the point C.

Calculate the exact area of the shaded region, bounded by the curve and the straight line segments AB and BC.

$\boxed{}$, $\boxed{\text{area} = \frac{76}{3}}$



Question 263 (****+)



The figure above shows the design of coffee jar with a "push on" lid.

The jar is in the shape of a right circular cylinder of radius x cm. It is fitted with a lid of width 2 cm, which fits tightly on the top of the jar, so it may be assumed that it has the same radius as the jar.

The jar and its lid is made of sheet metal and there is no wastage.

The total metal used to make the jar and its lid is 190π cm^2.

(*This figure does not include the handle of the lid which is made of different material.*)

a) Show that volume of the jar, V cm^3, is given by

$$V = \pi\left(95x - 2x^2 - x^3\right).$$

b) Determine by differentiation the value of x for which V is stationary.

[continues overleaf]

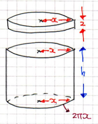
[continued from overleaf]

c) Show that the value of x found in part **(b)** gives the maximum value for V.

d) Hence determine the maximum volume of the jar.

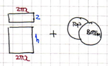
$\boxed{}$, $\boxed{x = 5 \text{ cm}}$, $\boxed{V_{\max} = 300\pi \approx 942 \text{ cm}^3}$

a)



CONSTRAINT ON SURFACE AREA

$\Rightarrow A = 190\pi$

$\Rightarrow$ 

$= 190\pi$

$\Rightarrow \cancel{2\pi} x(h+2) + 2(\cancel{\pi} x^2) = 190\cancel{\pi}$

$\Rightarrow 2x(h+2) + 2x^2 = 190$

$\Rightarrow x(h+2) + x^2 = 95$

$\Rightarrow xh + 2x + x^2 = 95$

$\Rightarrow \boxed{xh = 95 - 2x - x^2}$

VOLUME OF THE JAR

$\Rightarrow V = \pi r^2 h$

$\Rightarrow V = \pi x^2 h$

$\Rightarrow V = \pi x(xh)$

$\Rightarrow V = \pi x(95 - 2x - x^2)$

$\Rightarrow V = \pi(95x - 2x^2 - x^3)$

AS REQUIRED

b) DIFFERENTIATE & SOLVE FOR ZERO

$\Rightarrow V = \pi(95x - 2x^2 - x^3)$

$\Rightarrow \frac{dV}{dx} = \pi(95 - 4x - 3x^2)$

$\Rightarrow 0 = \cancel{\pi}(95 - 4x - 3x^2)$

$\Rightarrow 3x^2 - 4x - 95 = 0$

$\Rightarrow (3x + 19)(x - 5) = 0$

$\Rightarrow x = \begin{cases} 5 \\ \cancel{-\frac{19}{3}} \end{cases}$

c) USING THE 2ND DERIVATIVE TEST

$\frac{dV}{dx} = \pi(95 - 4x - 3x^2)$

$\frac{d^2V}{dx^2} = \pi(-4 - 6x)$

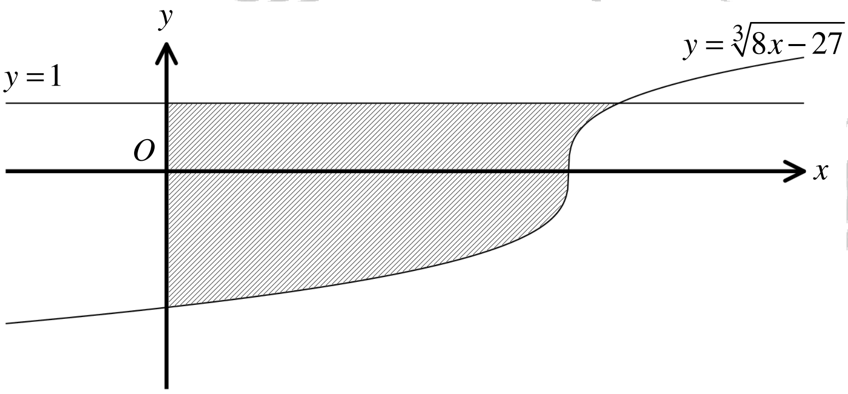
$\left.\frac{d^2V}{dx^2}\right|_{x=5} = -34\pi < 0$ INDEED A MAXIMUM

d) $V = \pi(95x - 2x^2 - x^3)$

$V_{MAX} = \pi(95 \times 5 - 2 \times 5^2 - 5^3)$

$V_{MAX} = 300\pi \approx 943 \text{ cm}^3$

Question 264 (****+)



The figure above shows the curve C with equation

$$y=\sqrt[3]{8x-27}\,,\ x\in\mathbb{R}\,.$$

The finite region R, shown shaded in the figure, is bounded by C, the y axis and the straight line with equation $y=1$.

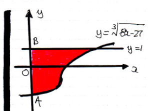
When the lengths are measured in m, R models the design of a yacht rudder.

Show that the area of the yacht rudder is $11\ \text{m}^2$.

proof

LOOKING AT THE DIAGRAM IT BECOMES OBVIOUS THAT THE INTEGRATION NEEDED IS WITH RESPECT TO y

- $x=0 \Rightarrow y=\sqrt[3]{-27}$
  $\Rightarrow y=-3$
  $\Rightarrow A(0,-3)$



$y=\sqrt[3]{8x-27}$
$y^3=8x-27$
$y^3+27=8x$
$x=\frac{1}{8}(y^3+27)$
$x=\frac{1}{8}y^3+\frac{27}{8}$

- $\text{AREA}=\int_{y_1}^{y_2} x(y)\ dy$

$\text{AREA}=\int_{-3}^{1}\frac{1}{8}y^3+\frac{27}{8}\ dy$

$\text{AREA}=\left[\frac{1}{32}y^4+\frac{27}{8}y\right]_{-3}^{1}$

$\text{AREA}=\left(\frac{1}{32}+\frac{27}{8}\right)-\left(\frac{81}{32}-\frac{81}{8}\right)$

$\text{AREA}=\frac{109}{32}-\left(-\frac{243}{32}\right)$

$\text{AREA}=11$

AS REQUIRED

Question 265 (****+)

The curve C has equation

$$y = \frac{x^3\left(5x\sqrt{x}-128\right)}{\sqrt{x}}, \quad x \in \mathbb{R}, \quad x > 0.$$

a) Determine expressions for $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^3y}{dx^3}$.

b) Show that the y coordinate of the stationary point of C is $-k\sqrt[3]{4}$, where k is a positive integer.

c) Evaluate $\dfrac{d^2y}{dx^2}$ at the stationary point of C.

Give the answer in terms of $\sqrt[3]{2}$.

d) Find the value of $\dfrac{d^3y}{dx^3}$ at the point on C, where $\dfrac{d^2y}{dx^2}=0$.

$\boxed{}$, $\boxed{\dfrac{dy}{dx} = 20x^3 - 320x^{\frac{3}{2}}}$, $\boxed{\dfrac{d^2y}{dx^2} = 60x^2 - 480x^{\frac{1}{2}}}$, $\boxed{\dfrac{d^3y}{dx^3} = 120x - 240x^{-\frac{1}{2}}}$,

$\boxed{k = 3072}$, $\boxed{960\sqrt[3]{2}}$, $\boxed{360}$

a) START BY REWRITING THE EQUATION IN INDICIAL FORM, THEN DIFFERENTIATE

$$y = \frac{x^3(5x\sqrt{x}-128)}{\sqrt{x}} = \frac{5x^4\sqrt{x}-128x^3}{\sqrt{x}} = \frac{5x^4\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} - \frac{128x^3}{\sqrt{x}}$$

- $y = 5x^4 - 128x^{\frac{5}{2}}$
- $\dfrac{dy}{dx} = 20x^3 - 320x^{\frac{3}{2}}$
- $\dfrac{d^2y}{dx^2} = 60x^2 - 480x^{\frac{1}{2}}$
- $\dfrac{d^3y}{dx^3} = 120x - 240x^{-\frac{1}{2}}$

b) FOR STATIONARY POINTS $\dfrac{dy}{dx}=0$

$\Rightarrow 20x^3 - 320x^{\frac{3}{2}} = 0$

$\Rightarrow x^3 - 16x^{\frac{3}{2}} = 0$

$\Rightarrow x^3 = 16x^{\frac{3}{2}}$ $(x \neq 0)$

$\Rightarrow \dfrac{x^3}{x^{3/2}} = 16$

$\Rightarrow x^{\frac{3}{2}} = 16$

$\Rightarrow \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = 16^{\frac{2}{3}}$

$\Rightarrow x^1 = \left(2^4\right)^{\frac{2}{3}}$

$\Rightarrow x = 2^{\frac{8}{3}}$

SUBSTITUTE INTO y & TIDY

$\Rightarrow y = 5x^4 - 128x^{\frac{5}{2}}$

$\Rightarrow y = x^{\frac{5}{2}}\left[5x^{\frac{3}{2}} - 128\right]$

$\Rightarrow y = \left(2^{\frac{8}{3}}\right)^{\frac{5}{2}}\left[5 \times 16 - 128\right]$

$\Rightarrow y = 2^{\frac{20}{3}}\left[80 - 128\right]$

$\Rightarrow y = 2^{6\frac{2}{3}}(-48)$

$\Rightarrow y = 2^6 \times 2^{\frac{2}{3}} \times (-48)$

$\Rightarrow y = -48 \times 64 \times \left(2^2\right)^{\frac{1}{3}}$

$\Rightarrow y = -3072 \times \sqrt[3]{4}$

c) $\dfrac{d^2y}{dx^2} = 60x^2 - 480x^{\frac{1}{2}}$

$\Rightarrow \dfrac{d^2y}{dx^2} = 60x^{\frac{1}{2}}\left(x^{\frac{3}{2}} - 8\right)$

$\Rightarrow \left.\dfrac{d^2y}{dx^2}\right|_{x^{\frac{3}{2}}=16} = 60\left(2^{\frac{8}{3}}\right)^{\frac{1}{2}}(16-8) = 60 \times 2^{\frac{4}{3}} \times 8 = 60 \times 2^1 \times 2^{\frac{1}{3}} \times 8$

$\Rightarrow \left.\dfrac{d^2y}{dx^2}\right|_{x^{\frac{3}{2}}=16} = 960\sqrt[3]{2}$

d) FIRSTLY $\dfrac{d^2y}{dx^2} = 0$

$\Rightarrow 60x^2 - 480x^{\frac{1}{2}} = 0$

$\Rightarrow x^2 - 8x^{\frac{1}{2}} = 0$

$\Rightarrow x^2 = 8x^{\frac{1}{2}}$ $x \neq 0$

$\Rightarrow \dfrac{x^2}{x^{\frac{1}{2}}} = 8$

$\Rightarrow x^{\frac{3}{2}} = 8$

$\Rightarrow \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = 8^{\frac{2}{3}}$

$\Rightarrow x^1 = \left(\sqrt[3]{8}\right)^2$

$\Rightarrow x = 4$

FINALLY

$\Rightarrow \dfrac{d^3y}{dx^3} = 120x - 240x^{-\frac{1}{2}}$

$\Rightarrow \left.\dfrac{d^3y}{dx^3}\right|_{x=4} = 120 \times 4 - 240 \times 4^{-\frac{1}{2}}$

$= 480 - 240 \times \dfrac{1}{2}$

$= 480 - 120 = 360$

Question 266 (****+)

Use differentiation to establish the number of real solutions of the equation

$$3x^4 - 4x^3 - 12x^2 + 15 = 0.$$

You are not expected to solve the equation.

$\boxed{}$, $\boxed{\text{2 solutions}}$

- FIRSTLY LET US NOTE THAT ALL POLYNOMIALS ARE CONTINUOUS, IE THE GRAPHS HAVE NO ASYMPTOTES OR OTHER DISCONTINUITIES – WRITE THE EQUATION AS A FUNCTION & LOOK FOR INTERSECTIONS WITH THE x AXIS

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 15$$
$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$f'(x) = 12x(x^2 - x - 2)$$
$$f'(x) = 12x(x+1)(x-2)$$

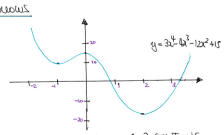
- STATIONARY VALUES $x = 0, -1, 2$

$f(0) = 15$ IE $(0, 15)$

$f(-1) = 3 + 4 - 12 + 15 = 10$ I.E. $(-1, 10)$

$f(2) = 48 - 32 - 48 + 15 = -17$ IE. $(2, -17)$

- A QUICK SKETCH FOLLOWS



$\therefore$ 2 SOLUTIONS

- NOTE – CHANGE OF SIGN WILL YIELD THE INTERVAL

Question 267 (****+)

The gradient function of a curve satisfies the following relationship

$$(x+1)\frac{dy}{dx}+16=4(2x+y).$$

The normal to the curve at the point P has equation $x+3y=6$.

Determine the coordinates of P.

$\boxed{}$, $\boxed{P(3,1)}$

START BY FINDING THE GRADIENT OF THE NORMAL

$x+3y=6$

$3y=-x+6$

$y=-\frac{1}{3}x+2$

$\therefore$ "TANGENT GRADIENT IS $+3$

SUB THIS GRADIENT INTO THE GRADIENT EXPRESSION

$\Rightarrow (x+1)\times 3+16=4(2x+y)$

$\Rightarrow 3x+3+16=8x+4y$

$\Rightarrow 19=5x+4y$

SOLVING SIMULTANEOUSLY WITH THE NORMAL $y=-\frac{1}{3}x+2$

$\Rightarrow 19=5x+4\left(-\frac{1}{3}x+2\right)$

$\Rightarrow 19=5x-\frac{4}{3}x+8$

$\Rightarrow 11=5x-\frac{4}{3}x$

$\Rightarrow 33=15x-4x$

$\Rightarrow 11x=33$

$\Rightarrow x=3$

FINALLY $y=-\frac{1}{3}x+2$

$y=-\frac{1}{3}\times 3+2$

$y=1$

$\therefore P(3,1)$

Question 268 (****+)

A quartic curve C has equation

$$y = x^3(x+2), \quad x \in \mathbb{R}.$$

a) Sketch the graph of C.

The sketch must contain the coordinates of any stationary points and the coordinates of the points of intersection with the coordinate axes.

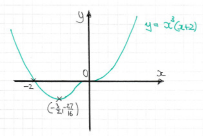
b) Show that there is only one point on C where the gradient is 10.

$\boxed{}$, $\boxed{\text{proof}}$

a) $\boxed{y = x^3(x+2), \quad x \in \mathbb{R}}$

EXPAND AND START COLLECTING INFORMATION FOR THE SKETCH

- $y = x^4 + 2x^3 = x^3(x+2)$
- $(0,0)$, IT STATIONARY POINT OF INFLEXION
- $(-2,0)$, STANDARD x INTERCEPT
- AS $x \to +\infty$, $y \sim x^4$, IT GETS LARGE & POSITIVE
- AS $x \to -\infty$, $y \sim x^4$, IT IT GETS LARGE & POSITIVE



- $\frac{dy}{dx} = 4x^3 + 6x^2 = 2x^2(2x+3)$

SOLVING FOR ZERO YIELDS $x = 0$ (POINT OF INFLECTION), $x = -\frac{3}{2}$ (MINIMUM AS SEEN ON THE GRAPH)

$$y\Big|_{x=-\frac{3}{2}} = \left(-\frac{3}{2}\right)^3\left(-\frac{3}{2}+2\right) = -\frac{27}{8} \times \frac{1}{2} = -\frac{27}{16}$$

b) SET THE GRADIENT FUNCTION EQUAL TO 10

$$\Rightarrow 4x^3 + 6x^2 = 10$$
$$\Rightarrow 2x^3 + 3x^2 = 5$$
$$\Rightarrow 2x^3 + 3x^2 - 5 = 0$$

BY INSPECTION $x=1$ IS A SOLUTION – PROCEED BY LONG DIVISION OR MANIPULATION

$$\Rightarrow 2x^2(x-1) + 5x(x-1) + 5(x-1) = 0$$
$$\Rightarrow (2x^2 + 5x + 5)(x-1) = 0$$

CHECK THE DISCRIMINANT OF THE QUADRATIC

$$\Rightarrow b^2 - 4ac = 5^2 - 4 \times 2 \times 5 = 25 - 40 = -15 < 0$$

$\therefore$ NO MORE SOLUTIONS & ONLY POINT ON THE CURVE IS THE POINT $P(1,3)$

Question 269 (****+)

The function $y = f(x)$ satisfies the following relationships

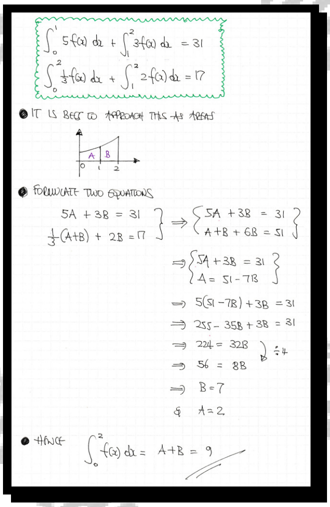
$$\int_0^1 5f(x)\,dx + \int_1^2 3f(x)\,dx = 31$$

$$\int_0^2 \frac{1}{3}f(x)\,dx + \int_1^2 2f(x)\,dx = 17.$$

Determine the value of

$$\int_0^2 f(x)\,dx.$$

$\boxed{\text{SP-I}}$, $\boxed{9}$



$$\int_0^1 5f(x)\,dx + \int_1^2 3f(x)\,dx = 31$$
$$\int_0^2 \tfrac{1}{3}f(x)\,dx + \int_1^2 2f(x)\,dx = 17$$

- IT IS BEST TO APPROACH THIS AS AREAS

- FORMULATE TWO EQUATIONS

$$\left.\begin{array}{l} 5A + 3B = 31 \\ \frac{1}{3}(A+B) + 2B = 17 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 5A + 3B = 31 \\ A + B + 6B = 51 \end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{l} 5A + 3B = 31 \\ A = 51 - 7B \end{array}\right\}$$

$$\Rightarrow 5(51 - 7B) + 3B = 31$$
$$\Rightarrow 255 - 35B + 3B = 31$$
$$\Rightarrow 224 = 32B \quad \div 4$$
$$\Rightarrow 56 = 8B$$
$$\Rightarrow B = 7$$
$$\& \quad A = 2$$

- HENCE $\int_0^2 f(x)\,dx = A + B = 9$

Question 270 (****+)

A curve C and a straight L have respective equations

$$C: \ y = 4x^2 - 6x + 3 \quad \text{and} \quad L: \ 2x - 4y + 3 = 0 .$$

Show that L is a normal to C, at some point.

$\boxed{}$, $\boxed{\text{proof}}$

$C: \ y = 4x^2 - 6x + 3 \quad \bullet \quad L: \ 2x - 4y + 3 = 0$

- START BY SOLVING THE EQUATIONS SIMULTANEOUSLY

$\Rightarrow 2x - 4[4x^2 - 6x + 3] + 3 = 0$

$\Rightarrow 2x - 16x^2 + 24x - 12 + 3 = 0$

$\Rightarrow 0 = 16x^2 + 26x + 9$

$\Rightarrow 0 = (2x - 1)(8x - 9)$

$\Rightarrow x = \begin{cases} \frac{1}{2} \\ \frac{9}{8} \end{cases}$

- OBTAIN THE GRADIENT OF THE LINE

$\Rightarrow 2x - 4y + 3 = 0$

$\Rightarrow 2x + 3 = 4y$

$\Rightarrow y = \underbrace{\tfrac{1}{2}}_{m = \frac{1}{2}}x + \frac{3}{4}$

- EVALUATE THE GRADIENT AT THE INTERSECTION POINTS OF THE CURVE AND THE LINE

$\frac{dy}{dx} = 8x - 6$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = 8\left(\frac{1}{2}\right) - 6 = -2 \leftarrow$ GRADIENT OF TANGENT AT $x = \frac{1}{2}$

$\therefore$ L IS A NORMAL TO C AT $x = \frac{1}{2}$

Question 271 (****+)

A curve C and a straight line L have respective equations

$$C: \; y = ax^{\frac{3}{2}} - bx^{\frac{1}{2}} \qquad \text{and} \qquad L: \; y = 8x - 32,$$

where a and b are non zero constants.

Given that L is a tangent to C at the point where $x = 4$ is, determine the value of a and the value of b.

$\boxed{}$, $\boxed{a = 4, \;\; b = 16}$

$C: \; y = ax^{\frac{3}{2}} + bx^{\frac{1}{2}}, x > 0$ • $L: \; y = 8x - 32$

- "L IS A TANGENT TO THE CURVE AT $x=4$"

  - $x = 4$
    $y = 8 \times 4 - 32$
    $y = 0$
  - GRADIENT OF L IS 8

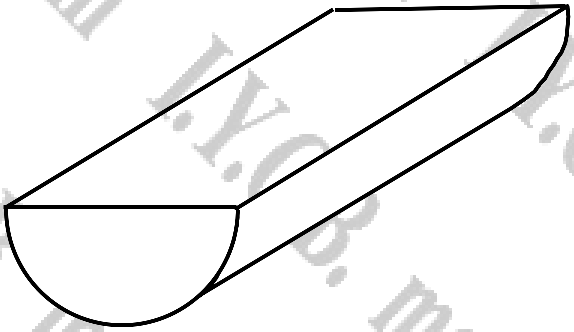
- $y = ax^{\frac{3}{2}} - bx^{\frac{1}{2}}$
  $\frac{dy}{dx} = \frac{3}{2}ax^{\frac{1}{2}} - \frac{1}{2}bx^{-\frac{1}{2}}$

- WHEN $x = 4$, $\frac{dy}{dx} = 8$
  $\Rightarrow 8 = \frac{3}{2}a \times 4^{\frac{1}{2}} - \frac{1}{2}b \times 4^{-\frac{1}{2}}$
  $\Rightarrow 8 = 3a - \frac{1}{4}b$
  $32 = 12a - b$

- WHEN $x = 0$, $y = 0$
  $\Rightarrow 0 = a \times 4^{\frac{3}{2}} - b \times 4^{\frac{1}{2}}$
  $\Rightarrow 0 = 8a - 2b$
  $\Rightarrow 2b = 8a$
  $\Rightarrow b = 4a$

- SOLVING THE EQUATIONS YIELDS
  $\Rightarrow 32 = 12a - (4a)$
  $\Rightarrow 32 = 8a$
  $\Rightarrow a = 4$
  &
  $b = 16$

Question 272 (****+)

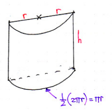


The figure above shows a solid prism, which is in the shape a right semi-circular cylinder.

The total surface area of the 4 faces of the prism is $\sqrt[3]{27\pi}$.

Given that the measurements of the prism are such so that its volume is maximized, find in exact simplified form the volume of the prism.

$\boxed{}$, $\boxed{V_{\max} = \dfrac{\pi}{\pi + 2}}$



- LET THE RADIUS BE r & THE VERTICAL HEIGHT h
- CONSTRAINT ON SURFACE AREA

$\Rightarrow \pi r^2 + \pi r h + 2rh = \sqrt[3]{27\pi}$

$\Rightarrow \pi r h + 2rh = \sqrt[3]{27\pi} - \pi r^2$

$\Rightarrow rh(\pi + 2) = 3\pi^{\frac{1}{3}} - \pi r^2$

$\Rightarrow \boxed{rh = \dfrac{3\pi^{\frac{1}{3}} - \pi r^2}{\pi + 2}}$

- NOW LOOKING AT THE VOLUME

$\Rightarrow V = \frac{1}{2}(\pi r^2 h) = \frac{1}{2}\pi r (rh) = \frac{1}{2}\pi r \left(\dfrac{3\pi^{\frac{1}{3}} - \pi r^2}{\pi + 2}\right)$

$\Rightarrow V = \dfrac{\pi r (3\pi^{\frac{1}{3}} - \pi r^2)}{2(\pi + 2)}$

$\Rightarrow \boxed{V = \dfrac{\pi}{2(\pi+2)}\left[3\pi^{\frac{1}{3}} r - \pi r^3\right]}$

- DIFFERENTIATE & SOLVE FOR ZERO

$\Rightarrow \dfrac{dV}{dr} = \dfrac{\pi}{2(\pi+2)}\left[3\pi^{\frac{1}{3}} - 3\pi r^2\right]$

$\Rightarrow 0 = \cancel{\dfrac{\pi}{2(\pi+2)}}\left[3\pi^{\frac{1}{3}} - 3\pi r^2\right]$

$\Rightarrow \cancel{3}\pi^{\frac{1}{3}} - \cancel{3}\pi r^2 = 0$

$\Rightarrow \pi^{\frac{1}{3}} = \pi r^2$

$\Rightarrow r^2 = \pi^{-\frac{2}{3}}$

$\Rightarrow r^2 = \dfrac{1}{\pi^{2/3}}$

$\Rightarrow r = +\dfrac{1}{\pi^{1/3}}$

FINALLY TO OBTAIN THE MAXIMUM VOLUME

$\Rightarrow V = \dfrac{\pi}{2(\pi+2)}\left[3\pi^{\frac{1}{3}} r - \pi r^3\right]$

$\Rightarrow V = \dfrac{\pi r}{2(\pi+2)}\left[3\pi^{\frac{1}{3}} - \pi r^2\right]$

$\Rightarrow V_{MAX} = \dfrac{\pi}{2(\pi+2)}\left(\dfrac{1}{\pi^{1/3}}\right)\left[3\pi^{\frac{1}{3}} - \pi \times \dfrac{1}{\pi^{2/3}}\right]$

$\Rightarrow V_{MAX} = \dfrac{\pi^{\frac{2}{3}}}{2(\pi+2)}\left[3\pi^{\frac{1}{3}} - \pi^{\frac{1}{3}}\right]$

$\Rightarrow V_{MAX} = \dfrac{\pi^{\frac{2}{3}}}{2(\pi+2)} \times 2\pi^{\frac{1}{3}}$

$\Rightarrow V_{MAX} = \dfrac{2\pi}{2(\pi+2)}$

$\Rightarrow V_{MAX} = \dfrac{\pi}{\pi+2}$

Question 273 (*****)

A curve C and a straight line L have respective equations

$$C: \ y = 4x\sqrt{x} - \frac{25x^2}{16} \quad \text{and} \quad L: \ x + 2y = 18.$$

a) Show that L is a tangent to C, at some point to be found.

b) Verify the answer of part **(a)** by an alternative method.

c) Show further that L does not meet C again.

$\boxed{}$, $\boxed{\text{proof}}$

a) SOLVE THE EQUATIONS SIMULTANEOUSLY OR BY VERIFICATION

$x + 2y = 18$ AND $y = 4x\sqrt{x} - \frac{25}{16}x^2$

- $2y = -x + 18$
- $y = -\frac{1}{2}x + 9$

- $y = 4x^{\frac{3}{2}} - \frac{25}{16}x^2$
- $\frac{dy}{dx} = 6x^{\frac{1}{2}} - \frac{25}{8}x$

SET THE GRADIENT EQUAL TO $-\frac{1}{2}$

$\Rightarrow 6x^{\frac{1}{2}} - \frac{25}{8}x = -\frac{1}{2}$

$\Rightarrow 48x^{\frac{1}{2}} - 25x = -4$

$\Rightarrow 0 = 25x - 48x^{\frac{1}{2}} - 4$

$\Rightarrow 25(x^{\frac{1}{2}})^2 - 48(x^{\frac{1}{2}}) - 4 = 0$

$\Rightarrow (25x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2) = 0$

$\Rightarrow x^{\frac{1}{2}} = +\sqrt{x} = \begin{cases} 2 \\ \cancel{-\frac{2}{25}} \end{cases}$

$\Rightarrow x = 4$

USE THE CURVE TO FIND THE y CO-ORDINATE

$\Rightarrow y = 4 \times 4 \times \sqrt{4} - \frac{25}{16} \times 4^2 = 32 - 25 = 7$

$\Rightarrow y = 7$

CHECK THAT $(4,7)$ SATISFIES THE EQUATION OF THE LINE

$4 + 2 \times 7 = 18$

INDEED $x + 2y = 18$ IS A TANGENT TO $y = 4x\sqrt{x} - \frac{25}{16}x^2$ AT $(4,7)$

b) AS VERIFICATION WAS USED IN THE FIRST PART, PROCEED BY SOLVING THE EQUATIONS SIMULTANEOUSLY

$x + 2y = 18$ $\qquad$ $y = 4x\sqrt{x} - \frac{25}{16}x^2$

$x + 2\left[4x\sqrt{x} - \frac{25}{16}x^2\right] = 18$

$x + 8x^{\frac{3}{2}} - \frac{25}{8}x^2 - 18 = 0$

WE ARE LOOKING FOR A REPEATED ROOT FOR THIS EQUATION

DIFFERENTIATE – THE REPEATED SOLUTION MUST ALSO BE A SOLUTION OF THE DERIVATIVE

$\Rightarrow 1 + 12x^{\frac{1}{2}} - \frac{25}{4}x = 0$

$\Rightarrow 4 + 48x^{\frac{1}{2}} - 25x = 0$

$\Rightarrow 25x - 48x^{\frac{1}{2}} - 4 = 0$

$\Rightarrow 25(x^{\frac{1}{2}})^2 - 48x^{\frac{1}{2}} - 4 = 0$

$\Rightarrow (25x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2) = 0$

$\Rightarrow x^{\frac{1}{2}} = +\sqrt{x} = \begin{cases} 2 \\ \cancel{-\frac{2}{25}} \end{cases}$

$\Rightarrow x = 4$ & $y = 7$

INDEED A TANGENT

c) STARTING WITH A "REPEATED" ROOT

$(x^{\frac{1}{2}} - 2)^2 = x - 4x^{\frac{1}{2}} + 4$

LONG DIVIDE WITH THE EQUATION

$\Rightarrow x + 8x^{\frac{3}{2}} - \frac{25}{8}x^2 - 18 = 0$

$\Rightarrow 8x + 64x^{\frac{3}{2}} - 25x^2 - 144 = 0$

$\Rightarrow 25x^2 - 64x^{\frac{3}{2}} - 8x + 144 = 0$

THUS WE HAVE

$$\begin{array}{r|l} & 25x + 36x^{\frac{1}{2}} + 36 \\ \hline x - 4x^{\frac{1}{2}} + 4 & 25x^2 - 64x^{\frac{3}{2}} - 8x + 0x^{\frac{1}{2}} + 144 \\ & -25x^2 + 100x^{\frac{3}{2}} - 100x \\ \hline & 36x^{\frac{3}{2}} - 108x + 0x^{\frac{1}{2}} + 144 \\ & -36x^{\frac{3}{2}} + 144x - 144x^{\frac{1}{2}} \\ \hline & 36x - 144x^{\frac{1}{2}} + 144 \\ & -36x + 144x^{\frac{1}{2}} - 144 \\ \hline & 0 \quad 0 \quad 0 \end{array}$$

FACTORIZING FULLY

$(x^{\frac{1}{2}} - 2)^2(25x + 36x^{\frac{1}{2}} + 36) = 0$

$b^2 - 4ac = 36^2 - 4 \times 25 \times 36$

$= 36^2 - 100 \times 36$

$= 36 \times (-64) < 0$

NO MORE SOLUTIONS – THE TANGENT DOESN'T MEET AGAIN

Question 274 (*****)

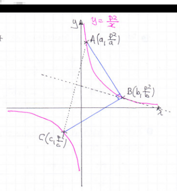
The distinct points A, B and C lie on the curve with equation

$$xy = p^2,$$

where p is a positive constant.

Given that ABC is a right angle, show that the tangent to the curve at B, is perpendicular to AC.

$\boxed{}$, $\boxed{\text{proof}}$



- LET THE CO-ORDINATES OF A, B & C BE

$A\left(a, \frac{p^2}{a}\right)$, $B\left(b, \frac{p^2}{b}\right)$, $C\left(c, \frac{p^2}{c}\right)$

WHERE a, b, c ARE DISTINCT

- GRADIENT OF AB

$$\frac{\frac{p^2}{b} - \frac{p^2}{a}}{b-a} = \frac{\frac{ap^2 - bp^2}{ab}}{b-a} = \frac{ap^2 - bp^2}{ab(b-a)} = \frac{p^2(a-b)}{ab(b-a)} = -\frac{p^2\cancel{(a-b)}}{ab\cancel{(a-b)}} = -\frac{p^2}{ab}$$

$a \neq b$

- BY ANALOGY AS THERE IS NOTHING SPECIAL ABOUT a, b, c

GRADIENT OF $BC = -\frac{p^2}{bc}$

GRADIENT OF $AC = -\frac{p^2}{ac}$

- NEXT WE NEED THE GRADIENT OF THE TANGENT AT B

$$y = \frac{p^2}{x}$$

$$\frac{dy}{dx} = -\frac{p^2}{x^2}$$

$$\left.\frac{dy}{dx}\right|_{x=b} = -\frac{p^2}{b^2} \leftarrow \text{TANGENT GRADIENT AT } B$$

- BUT WE HAVE A CONSTRAINT AS $AB \perp BC$

$$\Rightarrow -\frac{p^2}{ab} \times \frac{-p^2}{bc} = -1$$

$$\Rightarrow \underline{\underline{p^4 = -ab^2c}}$$

- FINALLY CHECKING THE GRADIENT OF THE TANGENT AT B, AND THE GRADIENT OF AC

$$\underbrace{-\frac{p^2}{b^2}}_{\text{TANGENT GRADIENT AT } B} \times \underbrace{\frac{-p^2}{ac}}_{\text{GRADIENT } AC} = \frac{p^4}{ab^2c} = \frac{-ab^2c}{ab^2c} = -1$$

INDEED PERPENDICULAR //

Question 275 (*****)

A quadratic curve has equation

$$f(x) \equiv x^2 + 6x + 20 + k\left(x^2 - 3x - 12\right),$$

where k is a constant.

Given that the point $P(-2, p)$ is the minimum point of the curve, determine the value of each of the constants p and k.

$\boxed{p = \frac{80}{7}, k = \frac{2}{7}}$

USING CALCULUS, NOTING k IS A CONSTANT

$$f(x) = x^2 + 6x + 20 + k(x^2 - 3x - 12)$$
$$f'(x) = 2x + 6 + k(2x - 3)$$

NOW $f'(-2) = 0$, SO WE HAVE

$$0 = -4 + 6 + k(-4 - 3)$$
$$-2 = -7k$$
$$k = \frac{2}{7}$$

FINALLY $f(-2) = p$, SO WE HAVE

$$p = 4 - 12 + 20 - \frac{2}{7}(4 + 6 - 12)$$
$$p = 12 + \frac{2}{7}(-2)$$
$$p = \frac{80}{7}$$

ALTERNATIVE BY COMPLETING THE SQUARE AS $f(x)$ IS A QUADRATIC

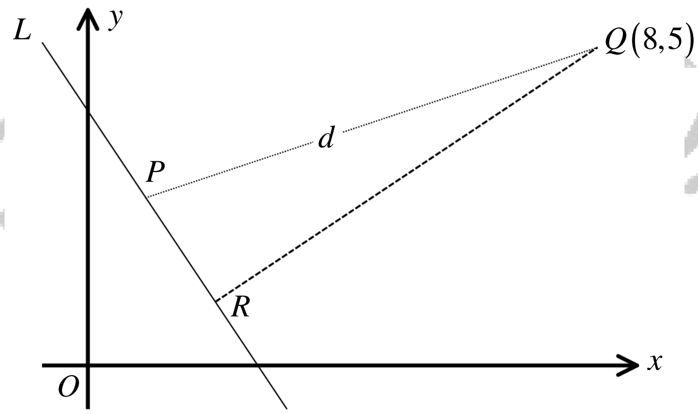
$$f(x) = (k+1)x^2 + (6 - 3k)x + (20 - 12k)$$
$$f(x) = (k+1)\left[x^2 + \frac{6-3k}{k+1}x + \frac{20-12k}{k+1}\right]$$
$$f(x) = (k+1)\left[\left[x + \frac{6-3k}{2(k+1)}\right]^2 - g(k)\right]$$

NOW WE HAVE

$$\frac{6-3k}{2(k+1)} = 2$$
$$\Rightarrow 6 - 3k = 2 \times 2(k+1)$$
$$\Rightarrow 6 - 3k = 4k + 4$$
$$\Rightarrow 2 = 7k$$
$$\Rightarrow k = \frac{2}{7}$$

AND THE VALUE OF p FOLLOWS

Question 276 (*****)



The straight line L has equation $3x+2y=8$.

The point $P(x,y)$ lies on L and the point $Q(8,5)$ lies outside L. The point R lies on L so that QR is perpendicular to L. The length PQ is denoted by d.

a) Show clearly that

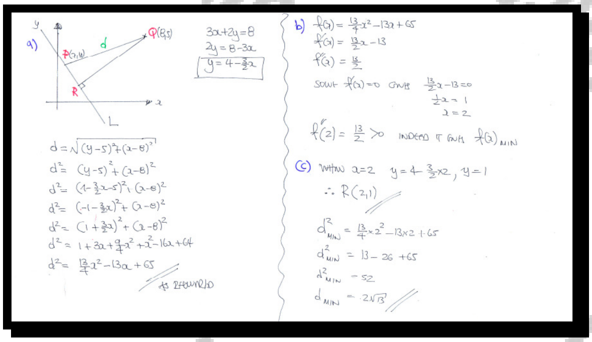
$$d^2 = 65 - 13x + \frac{13}{4}x^2.$$

Let $f(x) = 65 - 13x + \frac{13}{4}x^2$.

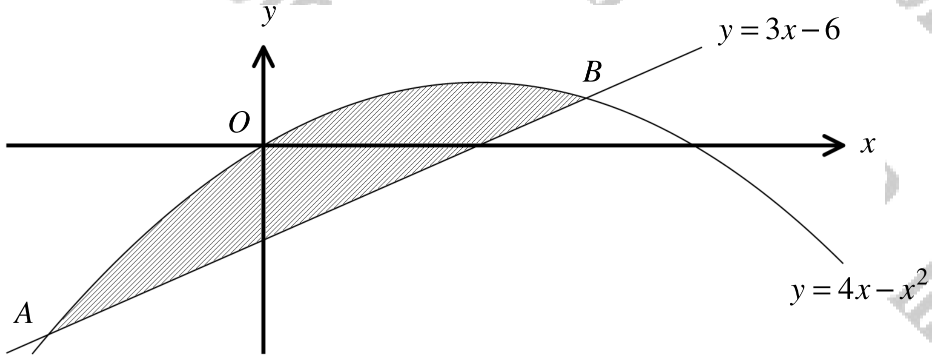
b) Use **differentiation** to find the stationary value of $f(x)$, fully justifying that this value of x minimizes the value of $f(x)$.

c) State the coordinates of R and find, as an exact surd, the shortest distance of the point Q from L

$\boxed{x=2}$, $\boxed{R(2,1)}$, $\boxed{\sqrt{52}}$



Question 277 (*****)



The figure above shows the graph of the curve C with equation

$$y = 4x - x^2,$$

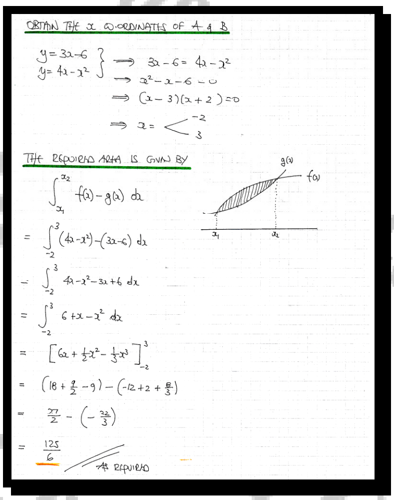
intersected by the straight line L with equation

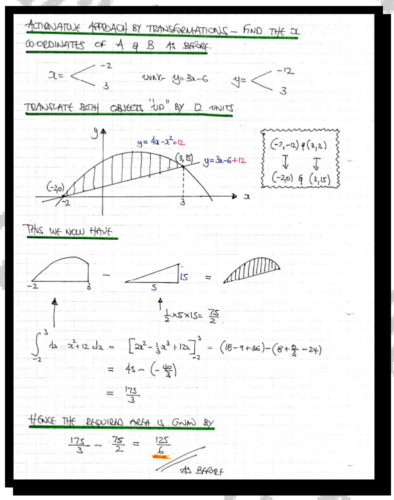
$$y = 3x - 6.$$

The finite region R is bounded by C and L.

Show that the area of R, shown shaded in the above figure, is $\frac{125}{6}$.

$\boxed{}$, $\boxed{\text{proof}}$





Question 278 (*****)

An **open** box is to be made of thin sheet metal, in the shape of a cuboid with a square base of length x and height h.

The box is to have a **fixed** volume.

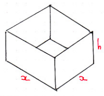
Determine the value of x, in terms of h, when the surface area of the box is minimum.

$\boxed{}$, $\boxed{\text{proof}}$

THE BOX HAS TO HAVE A FIXED VOLUME

$x^2h = \text{constant volume}$

$x^2h = V$



THE SURFACE AREA OF THE BOX IS GIVEN BY

$A = x^2 + 4xh$

$A = x^2 + \frac{4x^2h}{x}$

$A = x^2 + \frac{4V}{x}$

$A = x^2 + 4Vx^{-1}$

DIFFERENTIATING & SETTING TO ZERO

$\frac{dA}{dx} = 2x - 4Vx^{-2}$

$0 = 2x - \frac{4V}{x^2}$

$\frac{4V}{x^2} = 2x$

$2x^3 = 4V$

$x^3 = 2V$

$x = \sqrt[3]{2V}$

CHECK WHETHER THIS VALUE OF x PRODUCES A MINIMUM OR MAXIMUM

$\frac{d^2A}{dx^2} = 2 + 8Vx^{-3} = 2 + \frac{8V}{x^3}$

$\left.\frac{d^2A}{dx^2}\right|_{x=\sqrt[3]{2V}} = 2 + \frac{8V}{2V} = 6 > 0$ INDEED THIS VALUE OF x MINIMIZES

USING $x^2h = V$

$\Rightarrow h = \frac{V}{x^2}$

$\Rightarrow h = \frac{V}{(\sqrt[3]{2V})^2} = \frac{V}{[(2V)^{\frac{1}{3}}]^2} = \frac{V}{(2V)^{\frac{2}{3}}}$

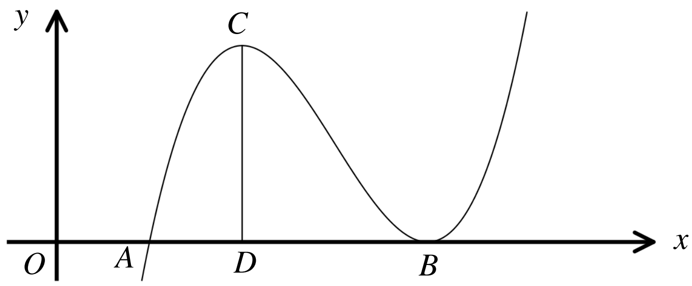
$\Rightarrow h = \frac{V}{2^{\frac{2}{3}}V^{\frac{2}{3}}} = \frac{V^{\frac{1}{3}}}{2^{\frac{2}{3}}} = \frac{V^{\frac{1}{3}} \times 2^{\frac{1}{3}}}{2^{\frac{2}{3}} \times 2^{\frac{1}{3}}}$

$\Rightarrow h = \frac{2^{\frac{1}{3}}V^{\frac{1}{3}}}{2} = \frac{(2V)^{\frac{1}{3}}}{2}$

HENCE $h = \frac{x}{2}$

$\therefore x = 2h$ WHEN SURFACE IS MINIMUM

Question 279 (*****)



The figure above shows a cubic curve whose coefficient of x^3 is 1.

The curve crosses the x axis at $A(a,0)$ and touches the x axis at $B(b,0)$, where a and b are positive constants.

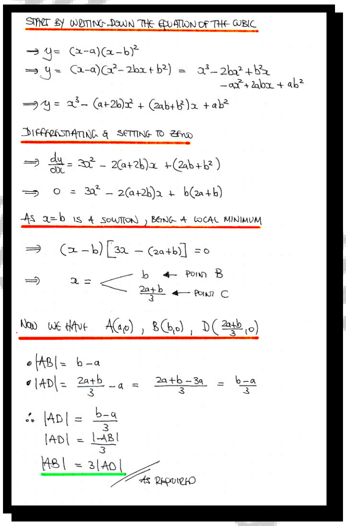
The point C is a local maximum of the curve.

The point D lies on the x axis so that CD is parallel to the y axis.

Show, with a detailed method, that

$$|AB| = 3|AD|.$$

$\boxed{}$, $\boxed{\text{proof}}$



START BY WRITING DOWN THE EQUATION OF THE CUBIC

$\Rightarrow y = (x-a)(x-b)^2$

$\Rightarrow y = (x-a)(x^2-2bx+b^2) = x^3-2bx^2+b^2x-ax^2+2abx+ab^2$

$\Rightarrow y = x^3-(a+2b)x^2+(2ab+b^2)x+ab^2$

DIFFERENTIATING & SETTING TO ZERO

$\Rightarrow \frac{dy}{dx} = 3x^2-2(a+2b)x+(2ab+b^2)$

$\Rightarrow 0 = 3x^2-2(a+2b)x+b(2a+b)$

AS $x=b$ IS A SOLUTION, BEING A LOCAL MINIMUM

$\Rightarrow (x-b)[3x-(2a+b)]=0$

$\Rightarrow x = \begin{cases} b & \leftarrow \text{POINT B} \\ \frac{2a+b}{3} & \leftarrow \text{POINT C} \end{cases}$

NOW WE HAVE $A(a,0)$, $B(b,0)$, $D\left(\frac{2a+b}{3},0\right)$

- $|AB| = b-a$
- $|AD| = \frac{2a+b}{3}-a = \frac{2a+b-3a}{3} = \frac{b-a}{3}$

$\therefore |AD| = \frac{b-a}{3}$

$|AD| = \frac{|AB|}{3}$

$|AB| = 3|AD|$

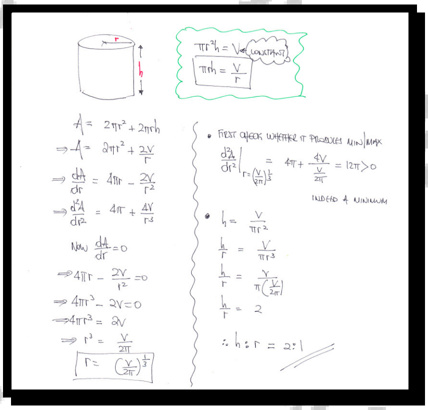
AS REQUIRED

Question 280 (*****)

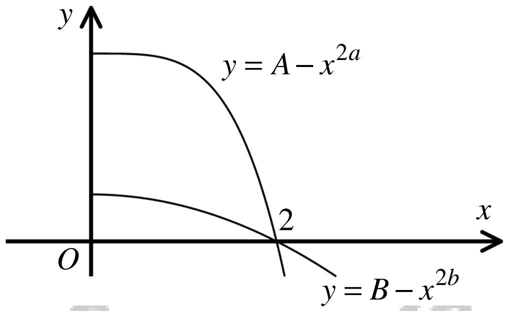
A solid right circular cylinder of fixed volume has radius r and height h.

Show clearly that when the surface area of the cylinder is minimum $h:r = 2:1$.

proof



Question 281 (*****)



The figure above shows the curves with equations

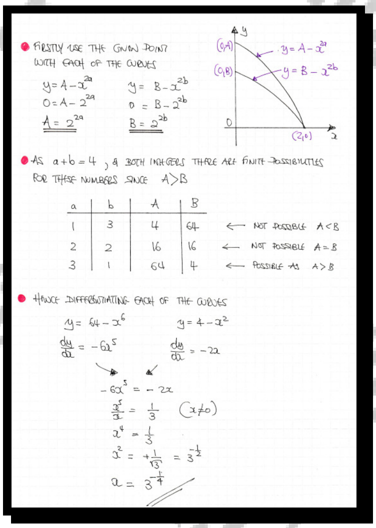
$$y = A - x^{2a} \quad \text{and} \quad y = B - x^{2b} \ , \quad x \geq 0 ,$$

where A and B are positive constants with $A > B$, and a and b are positive integers with $a + b = 4$.

Both curves meet the x axis at the point $(2,0)$.

Find the exact value of x for which both curves have the same gradient.

$\boxed{}$, $\boxed{x = 3^{-\frac{1}{4}}}$



• FIRSTLY USE THE GIVEN POINT WITH EACH OF THE CURVES

$y = A - x^{2a}$ $\qquad$ $y = B - x^{2b}$

$0 = A - 2^{2a}$ $\qquad$ $0 = B - 2^{2b}$

$\underline{\underline{A = 2^{2a}}}$ $\qquad$ $\underline{\underline{B = 2^{2b}}}$

$(0,A)$ $\quad$ $y = A - x^{2a}$ $\quad$ $(0,B)$ $\quad$ $y = B - x^{2b}$ $\quad$ O $\quad$ $(2,0)$

• AS $a+b=4$, & BOTH INTEGERS THERE ARE FINITE POSSIBILITIES FOR THESE NUMBERS SINCE $A > B$

a	b	A	B	
1	3	4	64	← NOT POSSIBLE $A<B$
2	2	16	16	← NOT POSSIBLE $A=B$
3	1	64	4	← POSSIBLE AS $A>B$

• HENCE DIFFERENTIATING EACH OF THE CURVES

$y = 64 - x^6$ $\qquad$ $y = 4 - x^2$

$\frac{dy}{dx} = -6x^5$ $\qquad$ $\frac{dy}{dx} = -2x$

$-6x^5 = -2x$

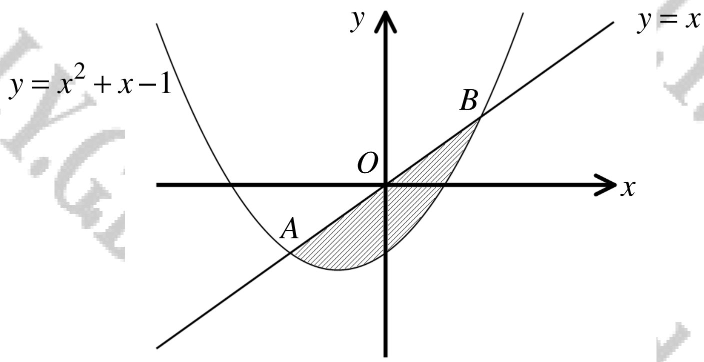
$\frac{x^5}{x} = \frac{1}{3} \quad (x \neq 0)$

$x^4 = \frac{1}{3}$

$x^2 = +\frac{1}{\sqrt{3}} = 3^{-\frac{1}{2}}$

$x = 3^{-\frac{1}{4}}$

Question 282 **(*****)**



The figure above shows the graph of the curve C with equation

$$y = x^2 + x - 1,$$

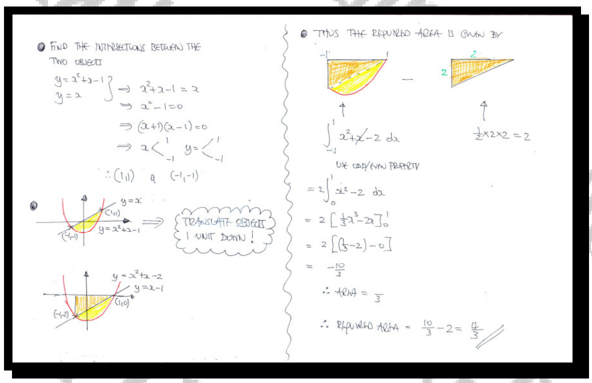
intersected by the straight line L with equation

$$y = x.$$

The points A and B, are the points of intersection between C and L, as shown in the above figure. The finite region R is bounded by C and L.

Show that the area of R, shown shaded in the above figure, is $\frac{4}{3}$.

$\boxed{\text{proof}}$



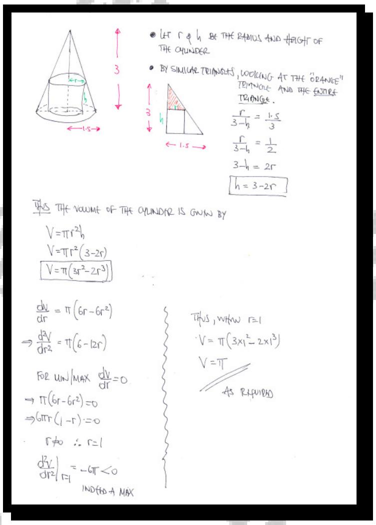
Question 283 (*****)

A solid right circular cylinder is to be cut out of a solid right circular cone, whose radius is 1.5 m and its height is 3 m.

The axis of symmetry of the cone coincides with the axis of symmetry of the cylinder which passes though its circular ends. The circumference of one end of the cylinder is in contact with the curved surface of the cone and the other end of the cylinder lies on the base of the cone.

Show that the maximum volume of the cylinder to be cut out is π m^3.

$\boxed{}$, $\boxed{\text{proof}}$


- LET r & h BE THE RADIUS AND HEIGHT OF THE CYLINDER
- BY SIMILAR TRIANGLES, LOOKING AT THE "ORANGE" TRIANGLE AND THE ENTIRE TRIANGLE.

$$\frac{r}{3-h} = \frac{1.5}{3}$$
$$\frac{r}{3-h} = \frac{1}{2}$$
$$3-h = 2r$$
$$\boxed{h = 3-2r}$$

THUS THE VOLUME OF THE CYLINDER IS GIVEN BY

$$V = \pi r^2 h$$
$$V = \pi r^2 (3-2r)$$
$$\boxed{V = \pi(3r^2 - 2r^3)}$$

$$\frac{dV}{dr} = \pi(6r - 6r^2)$$
$$\Rightarrow \frac{d^2V}{dr^2} = \pi(6 - 12r)$$

FOR MIN/MAX $\frac{dV}{dr} = 0$

$$\Rightarrow \pi(6r - 6r^2) = 0$$
$$\Rightarrow 6\pi r(1-r) = 0$$
$$r \neq 0 \quad \therefore r = 1$$
$$\left.\frac{d^2V}{dr^2}\right|_{r=1} = -6\pi < 0$$

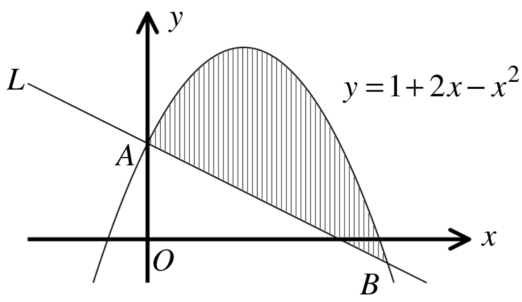
INDEED A MAX

THUS, WHEN $r=1$

$$V = \pi(3\times1^2 - 2\times1^3)$$
$$V = \pi$$

AS REQUIRED

Question 284 (*****)



The diagram above shows part of the curve C, with equation

$$y = 1 + 2x - x^2 .$$

The curve crosses the y axis at the point A.

The straight line L is the normal to C at A.

The point B is a point of intersection between C and A.

Find the exact area of the finite region, bounded by C and L.

$\boxed{}$, $\boxed{\text{area} = \frac{125}{48}}$

$y = 1 + 2x - x^2$ & $A(0,1)$ BY INSPECTION

$\frac{dy}{dx} = 2 - 2x$

$\left.\frac{dy}{dx}\right|_{x=0} = 2$

∴ EQUATION OF NORMAL

$y - y_0 = m(x - x_0)$

$y - 1 = -\frac{1}{2}(x - 0)$

$\boxed{y = 1 - \frac{1}{2}x}$

FIND INTERSECTION BETWEEN L & C

$\left.\begin{array}{l} y = 1 - \frac{1}{2}x \\ y = 1 + 2x - x^2 \end{array}\right\} \Rightarrow \cancel{1} - \frac{1}{2}x = \cancel{1} + 2x - x^2$

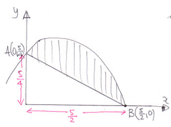
$\Rightarrow x^2 - \frac{5}{2}x = 0$

$\Rightarrow 2x^2 - 5x = 0$

$\Rightarrow x(2x - 5) = 0$

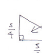
$\Rightarrow x = \begin{cases} 0 \leftarrow A \\ \frac{5}{2} \leftarrow B \end{cases}$

∴ $y = 1 - \frac{1}{2}\left(\frac{5}{2}\right) = 1 - \frac{5}{4} = -\frac{1}{4}$ ∴ $B\left(\frac{5}{2}, -\frac{1}{4}\right)$



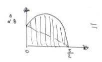
• TRANSLATE ENTIRE PICTURE UPWARDS BY $\frac{1}{4}$

∴ $y = \frac{5}{4} + 2x - x^2$



AREA $= \frac{1}{2} \times \frac{5}{4} \times \frac{5}{2} = \frac{25}{16}$

∴



$= \int_0^{\frac{5}{2}} \frac{5}{4} + 2x - x^2 \, dx = \left[\frac{5}{4}x + x^2 - \frac{1}{3}x^3\right]_0^{\frac{5}{2}}$

$\left(\frac{25}{8} + \frac{25}{4} - \frac{1}{3} \times \frac{125}{8}\right) - (0) = \frac{25}{6}$

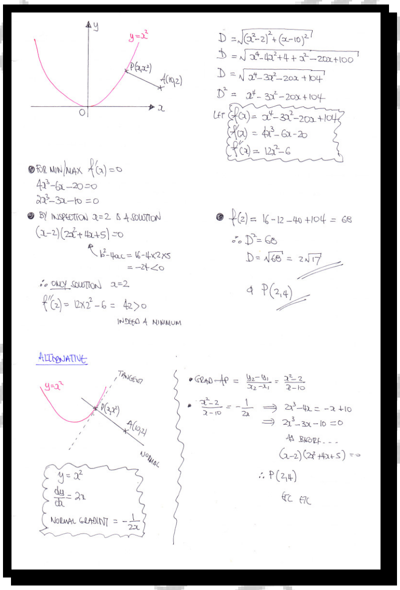
∴ REQUIRED AREA $= \frac{25}{6} - \frac{25}{16} = \frac{125}{48}$

Question 285 (*****)

The point P lies on the curve with equation $y = x^2$, so that its distance from the point $A(10,2)$ is **least**.

Determine the coordinates of P and the distance AP.

$\boxed{P(2,4)}$, $\boxed{d_{\text{min}} = 2\sqrt{17}}$



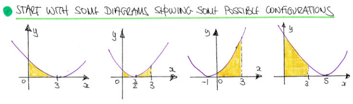
Question 286 (*****)

The curve with equation $y = x^2$, $x \in \mathbb{R}$, is translated parallel to the x axis, so that R denotes the area of the finite region bounded by the coordinated axes, the curve and the straight line with equation $x = 3$.

Determine the least value of R.

$\boxed{}$, $\boxed{\frac{9}{4}}$

- START WITH SOME DIAGRAMS SHOWING SOME POSSIBLE CONFIGURATIONS



- GET AN EXPRESSION FOR THE AREA IN TERMS OF t, FOR $y = (x-t)^2$

$$\text{AREA} = \int_0^3 (x-t)^2 \, dx = \int_0^3 x^2 - 2tx + t^2 \, dx$$

$$= \left[\tfrac{1}{3}x^3 - tx^2 + t^2 x\right]_0^3 = (9 - 9t + 3t^2) - (0)$$

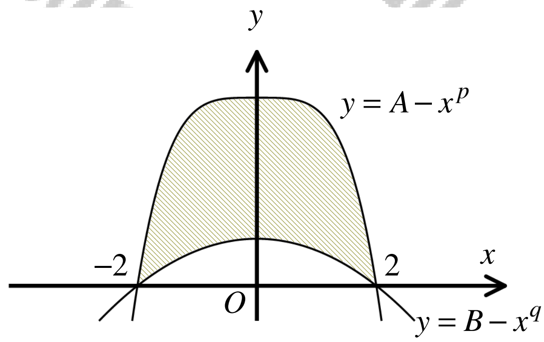
- LET $f(t) = 3t^2 - 9t + 9$
- BY COMPLETING THE SQUARE (OR CALCULUS)

$$\text{AREA} = f(t) = 3\left[t^2 - 3t + 3\right]$$
$$= 3\left[\left(t - \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + 3\right]$$
$$= 3\left[\left(t - \tfrac{3}{2}\right)^2 + \tfrac{3}{4}\right]$$
$$= 3\left(t - \tfrac{3}{2}\right)^2 + \tfrac{9}{4}$$

$\therefore$ MINIMUM AREA IS $\frac{9}{4}$

(IT OCCURS WHEN $t = \frac{3}{2}$, IE DIAGRAM 2)

Question 287 (*****)



The figure above shows the curves with equations

$$y = A - x^p \quad \text{and} \quad y = B - x^q ,$$

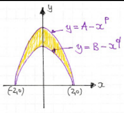
where A and B are positive constants with $A > B$, and p and q are positive integers with $p + q = 8$.

Both curves meet the x axis at the points $(-2,0)$ and $(2,0)$.

Find the exact value of the area bounded between the two curves.

$\boxed{}$, $\boxed{\text{area} = \frac{4384}{21}}$

- START BY USING THE POINTS $(\pm 2, 0)$ WITH EACH OF THE EQUATIONS



$(2,0)$: $0 = A - 2^p$ $\quad$ $0 = B - 2^q$

$\underline{\underline{2^p = A}}$ $\quad$ $\underline{\underline{B = 2^q}}$

$(-2,0)$: $0 = A - (-2)^p$ $\quad$ $0 = B - (-2)^q$

$\underline{\underline{(-2)^p = A}}$ $\quad$ $\underline{\underline{B = (-2)^q}}$

(FROM THESE TWO EQUATIONS WE KNOW THAT p & q ARE EVEN)

- NEXT WE ARE GIVEN THAT $p + q = 8$ — CONSTRUCT A TABLE

p	q	A	B	
2	6	$2^2 = 4$	$2^6 = 64$	← NOT POSSIBLE
4	4	$2^4 = 16$	$2^4 = 16$	← NOT POSSIBLE
6	2	$2^6 = 64$	$2^2 = 4$	← ONLY POSSIBLE

- NOW WE CAN FIND THE AREA

$$\Rightarrow \text{AREA} = \int_{-2}^{2} 4 - x^p \, dx - \int_{-2}^{2} B - x^q \, dx$$

$$\Rightarrow \text{AREA} = \int_{-2}^{2} 64 - x^6 \, dx - \int_{-2}^{2} 4 - x^2 \, dx$$

$$\Rightarrow \text{AREA} = \int_{-2}^{2} 60 - x^6 + x^2 \, dx$$

AS THE INTEGRAND IS EVEN WE HAVE

$$\text{AREA} = 2\int_{0}^{2} 60 - x^6 + x^2 \, dx$$

$$= 2\left[60x - \tfrac{1}{7}x^7 + \tfrac{1}{3}x^3\right]_0^2$$

$$= 2\left[\left(120 - \frac{128}{7} + \frac{8}{3}\right) - 0\right]$$

$$= 2\left[\frac{120\times21 - 128\times3 + 8\times7}{21}\right]$$

$$= \frac{2}{21}\left[2400 + 120 - 384 + 56\right]$$

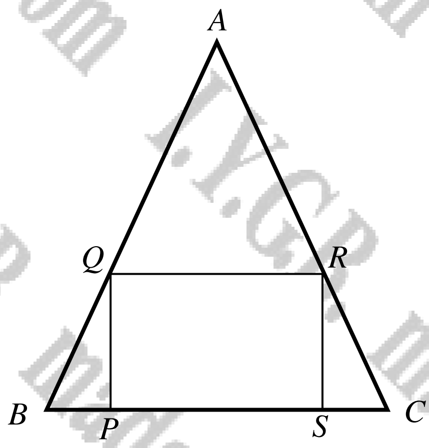
$$= \frac{2}{21}\left[2576 - 384\right]$$

$$= \frac{2}{21}\left[2000 + 16 + 176\right]$$

$$= \frac{2}{21} \times 2192$$

$$= \frac{4384}{21}$$

Question 288 (*****)

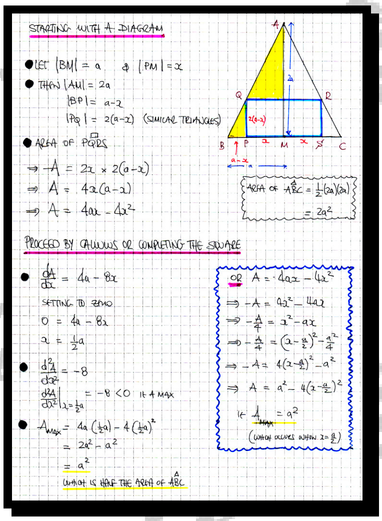


The figure above shows an isosceles triangle ABC, where $|AB| = |AC|$ and a rectangle $PQRS$ drawn inside the triangle.

The points P and S lie on BC, the point Q lies on AB and the point R lies on AC.

Given that the base of the triangle BC is equal in length to its height, show clearly that the largest area that the rectangle $PQRS$ can achieve is $\frac{1}{2}$ the area of the triangle ABC.

, $\boxed{\text{proof}}$



Question 289 (*****)

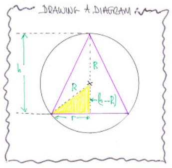
A right circular cone of radius r and height h is to be cut out of a sphere of radius R.

It is a requirement that the circumference of the base of the cone and its vertex lie on the surface of the sphere.

Determine, in exact form in terms of R, and with full justification, the maximum volume of the cone that can be cut out of this sphere.

$\boxed{}$, $\boxed{V_{\text{max}} = \dfrac{32\pi R^3}{81}}$

DRAWING A DIAGRAM



- LOOKING AT THE YELLOW TRIANGLE

$\Rightarrow r^2 + (h-R)^2 = R^2$

$\Rightarrow r^2 + h^2 - 2hR + \cancel{R^2} = \cancel{R^2}$

$\Rightarrow r^2 = 2hR - h^2$

- VOLUME OF THE CONE

$\Rightarrow V = \frac{1}{3}\pi r^2 h$

$\Rightarrow V = \frac{1}{3}\pi(2hR - h^2)h$

$\Rightarrow V = \frac{1}{3}\pi(2Rh^2 - h^3)$

- BY DIFFERENTIATION

$\boxed{\frac{dV}{dh} = \frac{1}{3}\pi(4Rh - 3h^2)}$

- SOLVING FOR ZERO

$\Rightarrow 4Rh - 3h^2 = 0$

$\Rightarrow h(4R - 3h) = 0$

$\Rightarrow h = \begin{cases} 0 \\ \frac{4}{3}R \end{cases}$

- $\boxed{\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4R - 6h)}$

$\left.\frac{d^2V}{dh^2}\right|_{h=0} = \frac{4}{3}\pi R > 0$ (LOCAL MINIMUM)

$\left.\frac{d^2V}{dh^2}\right|_{h=\frac{4}{3}R} = \frac{1}{3}\pi\left(4R - 6\times\frac{4}{3}h\right) = -\frac{4}{3}\pi R < 0$

LOCAL MAXIMUM

- $V_{\text{MAX}} = \frac{1}{3}\pi\left[2R \times \frac{16}{9}R^2 - \frac{64}{27}R^3\right] = \frac{1}{3}\pi\left[\frac{32}{9}R^3 - \frac{64}{27}R^3\right]$

$= \frac{1}{3}\pi\left[\frac{96}{27}R^3 - \frac{64}{27}R^3\right] = \frac{1}{3}\pi \times \frac{32}{27}R^3 = \frac{32}{81}\pi R^3$

Question 290 (*****)

A mobile phone wholesaler buys a certain brand of phone for £35 a unit and sells it to shops for £100 a unit.

In a typical week the wholesaler expects to sell 500 of these phones.

Research however showed that on a typical week for every £1 reduced of the selling price of this phone, an extra 20 sales can be achieved.

Determine the **selling** price for this phone if the weekly profit is to be maximized, and find this maximum weekly profit.

$\boxed{}$, $\boxed{\text{£80, maximum profit £40500}}$

LET THE PROFIT BE P

$P = 500 \times (100 - 35)$

$P = 520 \times (99 - 35)$

$P = 540 \times (98 - 35)$

SO IN GENERAL ...

$P = [500 + 20x][(100 - x) - 35]$

$P = (500 + 20x)(65 - x)$

$P = 20(x + 25)(65 - x)$

$P = -20(x + 25)(x - 65)$

$\boxed{P = -20(x^2 - 40x - 1625)}$

- $\frac{dP}{dx} = -20(2x - 40)$
- $\frac{d^2P}{dx^2} = -40 < 0$ (INDEPENDENT OF x)

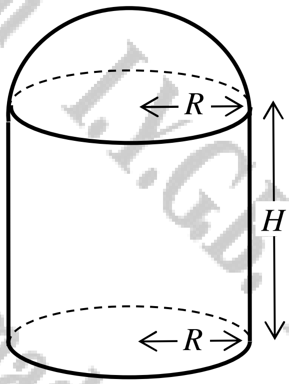
THUS $\frac{dP}{dx} = 0$ GIVES $x = 20$

$P_{MAX} = -20[20^2 - 40 \times 20 - 1625]$

$P_{MAX} = 40500$

$\therefore$ MAX PROFIT OF £40500 WHEN SELLING AT £80 EACH

Question 291 (*****)



The figure above shows a hollow container consisting of a right circular cylinder of radius R and of height H joined to a hemisphere of radius R.

The cylinder is open on one of the circular ends and the hemisphere is also open on its circular base. The cylinder is joined to the hemisphere at their open ends so that the resulting object is completely sealed.

Given that volume of the container is V, show the surface area of the container is minimised when $R = H$, and hence show further that this minimum surface area is

$$\sqrt[3]{45\pi V^2}.$$

$\boxed{}$, $\boxed{\text{proof}}$

START WITH AN EXPRESSION FOR THE VOLUME OF THE COMPOSITE

$V = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right) + \pi R^2 H$

$V = \frac{2}{3}\pi R^3 + \pi R^2 H$

NEXT AN EXPRESSION FOR THE VARIABLE SURFACE AREA

$A = \pi R^2 + 2\pi RH + \frac{1}{2}(4\pi R^2)$

$A = \pi R^2 + 2\pi RH + 2\pi R^2$

$A = 3\pi R^2 + 2\pi RH$

MANIPULATE THE VOLUME EXPRESSION AS FOLLOWS

$V = \frac{2}{3}\pi R^3 + \pi R^2 H$) $\times 2$

$2V = \frac{4}{3}\pi R^3 + 2\pi R^2 H$) $\div R$

$\frac{2V}{R} = \frac{4}{3}\pi R^2 + 2\pi RH$

$2\pi RH = \frac{2V}{R} - \frac{4}{3}\pi R^2$

FINALLY SUBSTITUTE THE ABOVE INTO THE SURFACE AREA EXPRESSION

$A = 3\pi R^2 + 2\pi RH$

$A = 3\pi R^2 + \frac{2V}{R} - \frac{4}{3}\pi R^2$

$A = \frac{5}{3}\pi R^2 + \frac{2V}{R}$ (NOTE THAT R MAY VARY BUT V IS A CONSTANT)

NEXT DIFFERENTIATE A W.R.T R & SOLVE FOR ZERO

$\frac{dA}{dR} = \frac{10}{3}\pi R - \frac{2V}{R^2}$

$0 = \frac{10}{3}\pi R - \frac{2V}{R^2}$

$\frac{10}{3}\pi R = \frac{2V}{R^2}$

$10\pi R^3 = 6V$

$R^3 = \frac{3V}{5\pi}$ IE $R = \left(\frac{3V}{5\pi}\right)^{\frac{1}{3}}$

JUSTIFY THE NATURE

$\frac{d^2A}{dR^2} = \frac{10}{3}\pi + \frac{4V}{R^3}$

$\left.\frac{d^2A}{dR^2}\right|_{R^3 = \frac{3V}{5\pi}} = \frac{10}{3}\pi + \frac{4V}{\frac{3V}{5\pi}} = \frac{10}{3}\pi + \frac{20}{3}\pi = 10\pi > 0$

INDEED MINIMIZES

NOW RETURNING TO THE CONSTRAINT

$2\pi RH = \frac{2V}{R} - \frac{4}{3}\pi R^2$ WITH $R^3 = \frac{3V}{5\pi}$

$2\pi R^2 H = 2V - \frac{4}{3}\pi R^3$

$6\pi R^2 H = 6V - 4\pi R^3$

$3\pi R^2 H = 3V - 2\pi R^3$ NOW $V = \frac{5\pi R^3}{3}$

$3\pi R^2 H = 3\left(\frac{5\pi R^3}{3}\right) - 2\pi R^3$

$3\pi R^2 H = 5\pi R^3 - 2\pi R^3$

$3\pi R^2 H = 3\pi R^3$

$H = R$ // AS REQUIRED

FINALLY TO FIND THE MINIMUM SURFACE AREA

$A = \frac{5}{3}\pi R^2 + \frac{2V}{R} = \frac{1}{R}\left[\frac{5}{3}\pi R^3 + 2V\right]$

$A_{MIN} = \frac{1}{R}\left[\frac{5}{3}\pi\left(\frac{3V}{5\pi}\right) + 2V\right]$

$A_{MIN} = \frac{1}{R}\left[V + 2V\right]$

$A_{MIN} = \frac{3V}{R}$

$A_{MIN} = 3V \times R^{-1}$

$A_{MIN} = 3V \times \left(\frac{5\pi}{3V}\right)^{\frac{1}{3}}$

$A_{MIN} = \left(27V^3\right)^{\frac{1}{3}} \times \left(\frac{5\pi}{3V}\right)^{\frac{1}{3}}$

$A_{MIN} = \left(27V^3 \times \frac{5\pi}{3V}\right)^{\frac{1}{3}}$

$A_{MIN} = \left(45\pi V^2\right)^{\frac{1}{3}}$

$A_{MIN} = \sqrt[3]{45\pi V^2}$ // AS REQUIRED

Question 292 (*****)

A curve C has equation

$$y = \frac{2x+3}{\sqrt{2x-1}}, \; x \in \mathbb{R}, \; x > \frac{1}{2}.$$

Find the coordinates of the stationary point of C, further determining the nature of this point.

You may not use the product rule, the quotient rule or logarithmic differentiation in this question.

$\boxed{\text{SYN}}$, $\boxed{\max\left(\frac{5}{2},4\right)}$

THE QUOTIENT RULE COULD BE USED HERE, BUT A QUICK SUBSTITUTION WHICH ALLOWS TO SPLIT THE FRACTION WOULD BE QUICKER HERE

$$\Rightarrow y = \frac{2x+3}{\sqrt{2x-1}}$$

- LET $t = 2x-1$ — THIS TRANSLATES THE GRAPH 1 UNIT TO THE "RIGHT", THEN IT HALVES THE x CO-ORDS

$$\Rightarrow y = \frac{(t+1)+3}{\sqrt{t}} = \frac{t+4}{t^{\frac{1}{2}}} = t^{\frac{1}{2}} + 4t^{-\frac{1}{2}}$$

- DIFFERENTIATE W.R.T t

$$\Rightarrow \frac{dy}{dt} = \frac{1}{2}t^{-\frac{1}{2}} - 2t^{-\frac{3}{2}}$$

- SOLVING FOR ZERO YIELDS.

$$\Rightarrow 0 = \frac{1}{2}t^{-\frac{1}{2}} - 2t^{-\frac{3}{2}}$$
$$\Rightarrow 2t^{-\frac{3}{2}} = \frac{1}{2}t^{-\frac{1}{2}}$$
$$\Rightarrow \frac{2}{t^{3/2}} = \frac{1}{2t^{1/2}}$$
$$\Rightarrow 4t^{\frac{1}{2}} = t^{3/2} \quad (t \neq 0)$$
$$\Rightarrow \frac{t^{3/2}}{t^{1/2}} = 4$$
$$\Rightarrow t = 4$$

- DIFFERENTIATE AGAIN TO CHECK THE NATURE, WHICH IS NOT AFFECTED BY THESE TRANSFORMATIONS

$$\frac{dy}{dt} = \frac{1}{2}t^{-\frac{1}{2}} - 2t^{-\frac{3}{2}}$$
$$\frac{d^2y}{dt^2} = -\frac{1}{4}t^{-\frac{3}{2}} + \frac{3}{4}t^{-\frac{5}{2}} = \frac{1}{4}t^{-\frac{5}{2}}\left[3-t\right]$$
$$\left.\frac{dy}{dt^2}\right|_{t=4} = \frac{1}{4} \times 4^{-\frac{5}{2}} \times (3-4) = -\frac{1}{4} \times \frac{1}{32} = -\frac{1}{128} < 0$$

- USING $t=4$ TO FIND THE VALUE OF y (NOT AFFECTED)

$$y = \frac{t+4}{\sqrt{t}}$$
$$\left. y \right|_{t=4} = \frac{4+4}{\sqrt{4}} = 4$$

- REVERSING THE TRANSFORMATION IN x

$$t = 2x-1$$
$$4 = 2x-1$$
$$5 = 2x$$
$$x = \frac{5}{2}$$

HENCE THERE IS A MAXIMUM AT $\left(\frac{5}{2}, 4\right)$ //

Question 293 (*****)

The point P, whose y coordinate is 2, lies on the curve with equation

$$y = \frac{k + 8x\sqrt{x}}{12x}, \quad x \in \mathbb{R}, \quad x > 0,$$

where k is a non zero constant.

The tangent to the curve at P is parallel to the straight line with equation

$$6x + y = 17.$$

Determine the value of k.

$\boxed{\text{SYN-T}}$, $\boxed{k = 5}$

START BY OBTAINING THE GRADIENT FUNCTION IN TERMS OF k

$$y = \frac{k + 8x\sqrt{x}}{12x} = \frac{k + 8x^{\frac{3}{2}}}{12x} = \frac{k}{12x} + \frac{8x^{\frac{3}{2}}}{12x} = \frac{k}{12}x^{-1} + \frac{2}{3}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{k}{12}x^{-2} + \frac{1}{3}x^{-\frac{1}{2}} = \frac{1}{3\sqrt{x}} - \frac{k}{12x^2}$$

PROCEED AS FOLLOWS

$6x + y = 17$

$y = -6x + 17$

$\therefore \left.\frac{dy}{dx}\right|_{y=2} = -6$

$\frac{1}{3\sqrt{x}} - \frac{k}{12x^2} = -6 \quad \times 12x^2$

$4x^{\frac{3}{2}} - k = -72x^2$

$k = 4x^{\frac{3}{2}} + 72x^2$

ALSO WE HAVE $y = 2$

$2 = \frac{k + 8x\sqrt{x}}{12x}$

$24x = k + 8x\sqrt{x}$

$k = 24x - 8x\sqrt{x}$

$k = 24x - 8x^{\frac{3}{2}}$

SOLVING SIMULTANEOUSLY

$$\left.\begin{aligned} k &= 4x^{\frac{3}{2}} + 72x^2 \\ k &= 24x - 8x^{\frac{3}{2}} \end{aligned}\right\}$$

$72x^2 + 4x^{\frac{3}{2}} = 24x - 8x^{\frac{3}{2}}$

$72x^2 + 12x^{\frac{3}{2}} - 24x = 0$

$12x\left(6x + x^{\frac{1}{2}} - 2\right) = 0$

$12x\left(3x^{\frac{1}{2}} + 2\right)\left(2x^{\frac{1}{2}} - 1\right) = 0$

AS $x \neq 0$

$x^{\frac{1}{2}} = \cancel{-\frac{2}{3}}, \ \frac{1}{2}$

$\therefore x = \frac{1}{4}$

$\therefore k = 24\left(\frac{1}{4}\right) - 8\left(\frac{1}{4}\right)^{\frac{3}{2}} = 6 - 8 \times \frac{1}{8} = 6 - 1$

$\therefore k = 5$

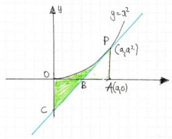
Question 294 (*****)

The point P lies on the curve with equation $y = x^2$, $x > 0$.

The finite region bounded by the curve, the tangent to the curve at P and the y axis has area of 72 square units.

Determine the x coordinate of P.

$\boxed{}$, $\boxed{6}$



$y = x^2$

$\frac{dy}{dx} = 2x$

$\left.\frac{dy}{dx}\right|_{x=a} = 2a$

EQUATION OF THE TANGENT AT $P(a,a^2)$

$y - a^2 = 2a(x-a)$

$y - a^2 = 2ax - 2a^2$

$y = 2ax - a^2$

WHEN $x=0$ $y=-a^2$ $\Rightarrow$ $C(0,-a^2)$

WHEN $y=0$ $x=\frac{a}{2}$ $\Rightarrow$ $B(\frac{a}{2},0)$

HENCE THE TRIANGLES $\overset{\triangle}{OBC}$ & $\overset{\triangle}{BAP}$ ARE IDENTICAL

SO THE REQUIRED AREA IS THE SAME AS THE INTEGRAL

$\int_0^a x^2\,dx = \frac{1}{3}a^3$

SO $\frac{1}{3}a^3 = 72$

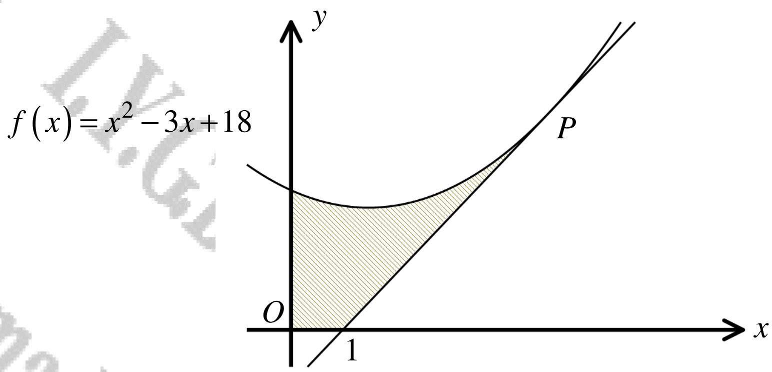
$a^3 = 3 \times 72$

$a^3 = 3 \times (3\times 3 \times 2 \times 2 \times 2)$

$a^3 = (3\times 3\times 3) \times (2\times 2\times 2)$

$a = 6$

Question 295 (*****)



A quadratic curve has equation

$$y = x^2 - 3x + 18, \quad x \in \mathbb{R}.$$

The tangent to the curve at the point P meets the x axis at the point with coordinates $(1,0)$, as shown in the figure above.

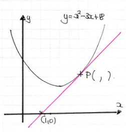
Find the area of the finite region bounded by the curve, the coordinates axes and the tangent to the curve at P, shown shaded in figure above.

$\boxed{}$, $\boxed{\text{area} = \frac{229}{6}}$

FIRSTLY WE REQUIRE TO FIND THE CO-ORDINATES OF THE POINT OF TANGENCY

METHOD A (PARAMETRIC APPROACH)

- LET THE COORDINATES OF P BE $(a, a^2-3a+18)$, FOR SOME $a>0$



- THEN WE HAVE

$\Rightarrow \frac{dy}{dx} = 2x-3$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=a} = 2a-3$

- EQUATION OF THE TANGENT AT P WILL BE

$\Rightarrow y-(a^2-3a+18) = (2a-3)(x-a)$

- AS THE TANGENT PASSES THROUGH $(1,0)$ WE HAVE

$\Rightarrow -(a^2-3a+18) = (2a-3)(1-a)$

$\Rightarrow a^2-3a+18 = (2a-3)(a-1)$

$\Rightarrow a^2-3a+18 = 2a^2-5a+3$

$\Rightarrow 0 = a^2-2a-15$

$\Rightarrow (a+3)(a-5)=0$

$\Rightarrow a = \cancel{-3}, \; 5$ IE $P(5,28)$

METHOD B (DISCRIMINANT APPROACH)

- LET THE EQUATION OF THE TANGENT BE $y=mx+c$
- USING THE CONSTRAINT $(1,0)$ WE HAVE $0=m+c$
- SOLVING SIMULTANEOUSLY

$\left.\begin{matrix} y=x^2-3x+18 \\ y=mx+c \end{matrix}\right\} \Rightarrow \left.\begin{matrix} y=x^2-3x+18 \\ y=mx-m \end{matrix}\right\} \Rightarrow x^2-3x+18 = mx-m$

$\Rightarrow x^2-(3+m)x+(m+18)=0$

LOOKING FOR REPEATED ROOTS

$\Rightarrow [-(3+m)]^2 - 4\times1\times(m+18) = 0$

$\Rightarrow m^2+6m+9-4m-72=0$

$\Rightarrow m^2+2m-63=0$

$\Rightarrow (m-7)(m+9)=0$

$\Rightarrow m = 7, \; \cancel{-9}$ (POSITIVE GRADIENT)

- HENCE WE HAVE

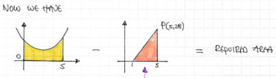
$x^2-(3+m)x+(m+18)=0$

$x^2-10x+25=0$

$(x-5)^2=0$

$x=5$ & $y=28$ AS BEFORE

- NOW WE HAVE



$\frac{1}{2}\times4\times28 = 56$

$\int_0^5 x^2-3x+18 \, dx = \left[\frac{1}{3}x^3-\frac{3}{2}x^2+18x\right]_0^5 = \left(\frac{125}{3}-\frac{75}{2}+90\right)-(0) = \frac{565}{6}$

- THE REQUIRED AREA IS

$\frac{565}{6} - 56 = \frac{229}{6}$

Question 296 (*****)

The curve with equation $y = f(x)$, lies entirely in the first quadrant. The point P, whose x coordinate is a lies on this curve.

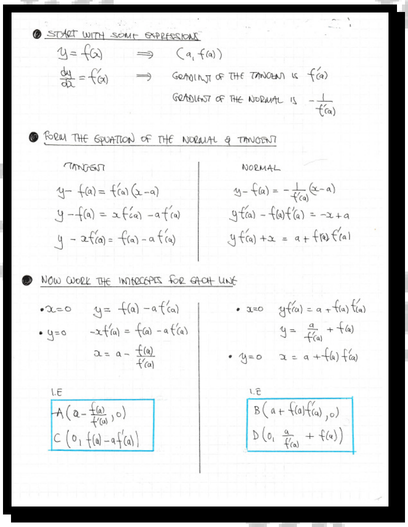
The tangent to the curve at P meets the x axis at the point A and the y axis at the point C.

The normal to the curve at P meets the x axis at the point B and the y axis at the point D.

Given further that the gradient at P is positive, show that the difference between the areas of the triangle PAB and the triangle PCD is given by

$$\frac{1+\left[f'(a)\right]^2}{2f'(a)}\left|\left[f(a)\right]^2 - a^2\right|.$$

[ ], [proof]



• START WITH SOME EXPRESSIONS

$y = f(x) \Longrightarrow (a, f(a))$

$\frac{dy}{dx} = f'(x) \Longrightarrow$ GRADIENT OF THE TANGENT IS $f'(a)$

GRADIENT OF THE NORMAL IS $-\frac{1}{f'(a)}$

• FORM THE EQUATION OF THE NORMAL & TANGENT

TANGENT

$y - f(a) = f'(a)(x-a)$

$y - f(a) = xf'(a) - af'(a)$

$y - xf'(a) = f(a) - af'(a)$

NORMAL

$y - f(a) = -\frac{1}{f'(a)}(x-a)$

$yf'(a) - f(a)f'(a) = -x + a$

$yf'(a) + x = a + f(a)f'(a)$

• NOW WORK THE INTERCEPTS FOR EACH LINE

• $x=0$ $\quad y = f(a) - af'(a)$

• $y=0$ $\quad -xf'(a) = f(a) - af'(a)$

$x = a - \frac{f(a)}{f'(a)}$

I.E

$\boxed{A\left(a - \frac{f(a)}{f'(a)}, 0\right) \quad C\left(0, f(a) - af'(a)\right)}$

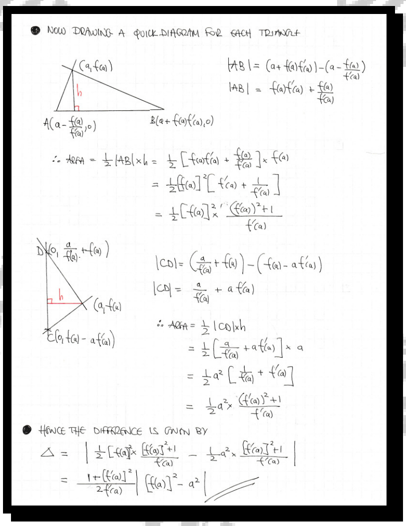
• $x=0$ $\quad yf'(a) = a + f(a)f'(a)$

$y = \frac{a}{f'(a)} + f(a)$

• $y=0$ $\quad x = a + f(a)f'(a)$

I.E

$\boxed{B\left(a + f(a)f'(a), 0\right) \quad D\left(0, \frac{a}{f'(a)} + f(a)\right)}$



• NOW DRAWING A QUICK DIAGRAM FOR EACH TRIANGLE

$|AB| = \left(a + f(a)f'(a)\right) - \left(a - \frac{f(a)}{f'(a)}\right)$

$|AB| = f(a)f'(a) + \frac{f(a)}{f'(a)}$

$\therefore \text{AREA} = \frac{1}{2}|AB| \times h = \frac{1}{2}\left[f(a)f'(a) + \frac{f(a)}{f'(a)}\right] \times f(a)$

$= \frac{1}{2}\left[f(a)\right]^2\left[f'(a) + \frac{1}{f'(a)}\right]$

$= \frac{1}{2}\left[f(a)\right]^2 \times \frac{(f'(a))^2+1}{f'(a)}$

$|CD| = \left(\frac{a}{f'(a)} + f(a)\right) - \left(f(a) - af'(a)\right)$

$|CD| = \frac{a}{f'(a)} + af'(a)$

$\therefore \text{AREA} = \frac{1}{2}|CD| h$

$= \frac{1}{2}\left[\frac{a}{f'(a)} + af'(a)\right] \times a$

$= \frac{1}{2}a^2\left[\frac{1}{f'(a)} + f'(a)\right]$

$= \frac{1}{2}a^2 \times \frac{(f'(a))^2+1}{f'(a)}$

• HENCE THE DIFFERENCE IS GIVEN BY

$\Delta = \left|\frac{1}{2}\left[f(a)\right]^2 \times \frac{\left[f'(a)\right]^2+1}{f'(a)} - \frac{1}{2}a^2 \times \frac{\left[f'(a)\right]^2+1}{f'(a)}\right|$

$= \frac{1+\left(f'(a)\right)^2}{2f'(a)}\left|\left[f(a)\right]^2 - a^2\right|$

Question 297 (*****)

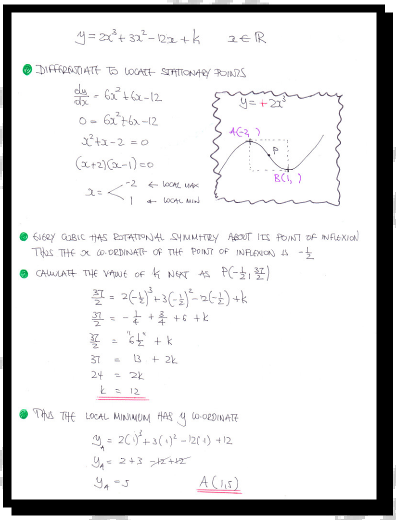
The points A and B are stationary points of the curve with equation

$$y = 2x^3 + 3x^2 - 12x + k, \ x \in \mathbb{R},$$

where k is a constant.

Given that $y = \frac{37}{2}$ at the point of inflexion of the curve, show that the area of the region bounded by the curve and the straight line through A and B is $5\frac{1}{16}$.

$\boxed{}$, $\boxed{\text{proof}}$



$y = 2x^3 + 3x^2 - 12x + k \quad x \in \mathbb{R}$

- DIFFERENTIATE TO LOCATE STATIONARY POINTS

$\frac{dy}{dx} = 6x^2 + 6x - 12$

$0 = 6x^2 + 6x - 12$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = \begin{cases} -2 & \leftarrow \text{LOCAL MAX} \\ 1 & \leftarrow \text{LOCAL MIN} \end{cases}$

- EVERY CUBIC HAS ROTATIONAL SYMMETRY ABOUT ITS POINT OF INFLEXION. THUS THE x CO-ORDINATE OF THE POINT OF INFLEXION IS $-\frac{1}{2}$
- CALCULATE THE VALUE OF k NEXT AS $P(-\frac{1}{2}, \frac{37}{2})$

$\frac{37}{2} = 2(-\frac{1}{2})^3 + 3(-\frac{1}{2})^2 - 12(-\frac{1}{2}) + k$

$\frac{37}{2} = -\frac{1}{4} + \frac{3}{4} + 6 + k$

$\frac{37}{2} = \text{"}6\frac{1}{2}\text{"} + k$

$37 = 13 + 2k$

$24 = 2k$

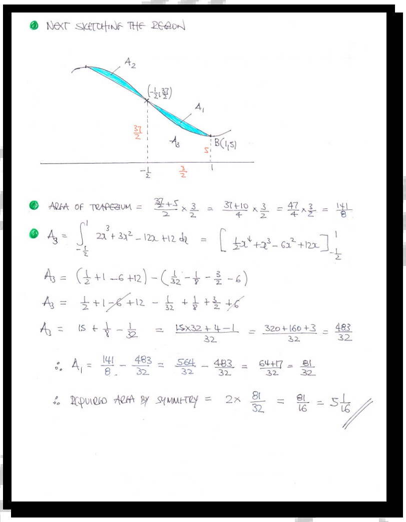
$k = 12$

- THUS THE LOCAL MINIMUM HAS y CO-ORDINATE

$y_A = 2(1)^3 + 3(1)^2 - 12(1) + 12$

$y_A = 2 + 3 \cancel{-12+12}$

$y_A = 5 \qquad A(1,5)$



- NEXT SKETCHING THE REGION

- AREA OF TRAPEZIUM $= \frac{\frac{37}{2}+5}{2} \times \frac{3}{2} = \frac{37+10}{4} \times \frac{3}{2} = \frac{47}{4} \times \frac{3}{2} = \frac{141}{8}$
- $A_3 = \int_{-\frac{1}{2}}^{1} 2x^3 + 3x^2 - 12x + 12 \, dx = \left[\frac{1}{2}x^4 + x^3 - 6x^2 + 12x\right]_{-\frac{1}{2}}^{1}$

$A_3 = (\frac{1}{2} + 1 - 6 + 12) - (\frac{1}{32} - \frac{1}{8} - \frac{3}{2} - 6)$

$A_3 = \frac{1}{2} + 1 \cancel{-6} + 12 - \frac{1}{32} + \frac{1}{8} + \frac{3}{2} \cancel{+6}$

$A_3 = 15 + \frac{1}{8} - \frac{1}{32} = \frac{15 \times 32 + 4 - 1}{32} = \frac{320 + 160 + 3}{32} = \frac{483}{32}$

$\therefore A_1 = \frac{141}{8} - \frac{483}{32} = \frac{564}{32} - \frac{483}{32} = \frac{64+17}{32} = \frac{81}{32}$

$\therefore$ REQUIRED AREA BY SYMMETRY $= 2 \times \frac{81}{32} = \frac{81}{16} = 5\frac{1}{16}$

Question 298 (*****)

A quadratic curve has equation

$$y = ax^2 + bx + c\,,\ a \neq 0\,,$$

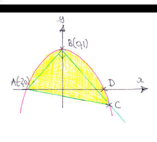
where a, b and c are constants.

The curve meets the x axis at $A(-2,0)$ and has a maximum point at $B(0,1)$.

The point C lies on the curve so that AB is perpendicular to BC.

Determine the area of the finite region bounded by the curve and the straight line segment AC.

SP U, proof



- THE MAXIMUM LIES ON THE y AXIS $B(0,1)$
- BY SYMMETRY $D(2,0)$, AS THE y AXIS MUST BE A LINE OF SYMMETRY FOR THE PARABOLA
- THE EQUATION OF THE PARABOLA MUST BE

$y = A(x+2)(x-2)\,,\ A<0$

USING $B(0,1)$

$1 = A(-4)$

$A = -\frac{1}{4}$

$\therefore \boxed{\begin{array}{l} y = -\frac{1}{4}(x-2)(x+2) \\ y = -\frac{1}{4}(x^2-4) \\ y = 1 - \frac{1}{4}x^2 \end{array}}$

- GRADIENT OF AB $= \frac{1-0}{0-(-2)} = \frac{1}{2}$
- GRADIENT OF BC $= -2$
- EQUATION OF STRAIGHT LINE BC, USING $m=-2$ & $B(0,1)$

$y - 1 = -2(x-0)$

$y = 1-2x$

- SOLVING SIMULTANEOUSLY WITH $y = 1-\frac{1}{4}x^2$

$1-2x = 1-\frac{1}{4}x^2$

$\frac{1}{4}x^2 - 2x = 0$

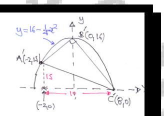
$\frac{1}{4}x(x-8) = 0$

$x = 8$ ($x=0$ AT B)

$y = -15$

$\therefore \boxed{C(8,-15)}$

- NEXT TRANSLATE EVERYTHING "UPWARDS" BY 15 UNITS



AREA UNDER THE CURVE BETWEEN -2 & 8 IS GIVEN BY:

$$\int_{-2}^{8} 16 - \frac{1}{4}x^2 \, dx = \left[16x - \frac{1}{12}x^3\right]_{-2}^{8}$$

$$= \left(128 - \frac{512}{12}\right) - \left(-32 + \frac{8}{12}\right)$$

$$= 160 - \frac{512}{12} - \frac{8}{12} = 160 - \frac{520}{12} = 160 - \frac{130}{3}$$

$$= \frac{480-130}{3} = \frac{350}{3}$$

- AREA OF THE RIGHT ANGLED TRIANGLE IS $\frac{1}{2} \times 10 \times 15 = 75$
- REQUIRED AREA $= \frac{350}{3} - 75 = \frac{350-225}{3} = \frac{125}{3}$

ALTERNATIVE – WITHOUT TRANSLATION

$A(-2,0)$ & $C(8,-15)$

GRADIENT AC $= \frac{-15-0}{8-(-2)} = -\frac{15}{10} = -\frac{3}{2}$

EQUATION OF LINE AC

$y - 0 = -\frac{3}{2}(x+2)$

$y = -\frac{3}{2}x - 3$

REQUIRED AREA IS GIVEN BY

$$\int_{-2}^{8} \left(1 - \frac{1}{4}x^2\right) - \left(-\frac{3}{2}x - 3\right) dx$$

$$= \int_{-2}^{8} 4 + \frac{3}{2}x - \frac{1}{4}x^2 \, dx$$

$$= \left[4x + \frac{3}{4}x^2 - \frac{1}{12}x^3\right]_{-2}^{8}$$

$$= \left(32 + 48 - \frac{512}{12}\right) - \left(-8 + 3 + \frac{8}{12}\right)$$

$$= 80 - \frac{512}{12} + 8 - 3 - \frac{8}{12}$$

$$= 85 - \frac{520}{12}$$

$$= 85 - \frac{130}{3}$$

$$= \frac{255-130}{3}$$

$$= \frac{125}{3}$$

Question 299 (*****)

The rate of change, with respect to x, of the gradient of a curve is constant.

The curve passes through the points with coordinates $(1,2)$ and $(-3,0)$, the gradient at the former point being $-\frac{1}{2}$.

Show that the area of the finite region bounded between the curve and the straight line with equation $y = 2x$ is $\frac{125}{3}$.

$\boxed{}$, $\boxed{\text{proof}}$

- THE RATE OF CHANGE OF THE GRADIENT W.R.T x IS $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

$$\Rightarrow \frac{d^2y}{dx^2} = k \quad \text{(CONSTANT)}$$
$$\Rightarrow \frac{dy}{dx} = kx + C$$
$$\Rightarrow y = \frac{1}{2}kx^2 + Cx + D$$
$$\Rightarrow y = Px^2 + Qx + R \qquad \frac{dy}{dx} = 2Px + Q$$

- APPLY THE CONDITIONS GIVEN

$$\left.\begin{array}{l} ① \ (1,2) \Rightarrow P+Q+R=2 \\ ② \ (-3,0) \Rightarrow 9P-3Q+R=0 \\ ③ \ \left.\frac{dy}{dx}\right|_{x=1} = -\frac{1}{2} \Rightarrow 2P+Q=-\frac{1}{2} \end{array}\right\} \Rightarrow \boxed{Q = -\frac{1}{2} - 2P}$$

$$\left\{\begin{array}{l} P + \left(-\frac{1}{2}-2P\right) + R = 2 \\ 9P - 3\left(-\frac{1}{2}-2P\right) + R = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} R - P = \frac{5}{2} \\ R + 15P = -\frac{3}{2} \end{array}\right\} \Rightarrow$$

$$-16P = 4$$
$$P = -\frac{1}{4}$$

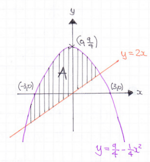
$$R - P = \frac{5}{2}$$
$$R + \frac{1}{4} = \frac{5}{2}$$
$$R = \frac{9}{4}$$

$$Q = -\frac{1}{2} - 2\left(-\frac{1}{4}\right)$$
$$Q = 0$$

$$\therefore \ y = -\frac{1}{4}x^2 + \frac{9}{4}$$
$$4y = 9 - x^2$$
$$4y = (3-x)(3+x)$$

- SKETCHING THE CURVE AND THE LINE

$$\left.\begin{array}{l} 4y = 9 - x^2 \\ y = 2x \end{array}\right\} \Rightarrow 8x = 9 - x^2$$
$$\Rightarrow x^2 + 8x - 9 = 0$$
$$\Rightarrow (x-1)(x+9) = 0$$
$$\Rightarrow x = \begin{cases} 1 \\ -9 \end{cases}$$



- HENCE THE REQUIRED AREA IS

$$\int_{x=-9}^{1} \left(\frac{9}{4} - \frac{1}{4}x^2\right) - (2x) \, dx = \int_{-9}^{1} \frac{9}{4} - \frac{1}{4}x^2 - 2x \, dx$$

$$= \left[\frac{9}{4}x - \frac{1}{12}x^3 - x^2\right]_{-9}^{1} = \left(\frac{9}{4} - \frac{1}{12} - 1\right) - \left[\frac{9}{4}(-9) - \frac{1}{12}(-9)^3 - (-9)^2\right]$$

$$= \frac{9}{4} - \frac{1}{12} - 1 - \left[\frac{-81}{4} + \frac{9 \times 9 \times 9}{12} - 81\right]$$

$$= \frac{9}{4} - \frac{1}{12} - 1 + \frac{81}{4} - \frac{3 \times 81}{4} + 81 = 80 + \frac{90}{4} - \frac{243}{4} - \frac{1}{12}$$

$$= 80 - \frac{153}{4} - \frac{1}{12} = 80 - \frac{144+8+1}{4} - \frac{1}{12} = 80 - 36 - 2 - \frac{1}{4} - \frac{1}{12}$$

$$= 42 - \left(\frac{3}{12} + \frac{1}{12}\right) = 42 - \frac{1}{3} = 41\frac{2}{3} = \frac{125}{3}$$

Question 300 (*****)

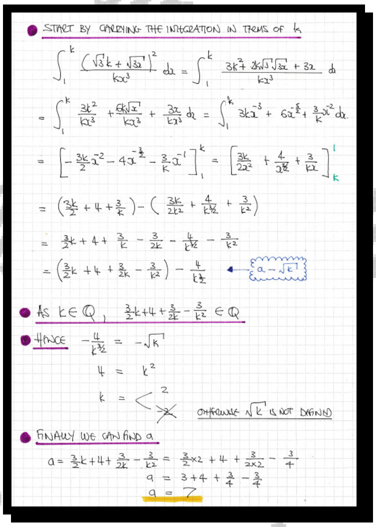
It is given that

$$\int_1^k \frac{\left(\sqrt{3}\,k+\sqrt{3x}\right)^2}{kx^3}\,dx = a-\sqrt{k}\,,$$

where a and k are integers.

Use algebra to determine the value of a.

$\boxed{}$, $\boxed{a=7}$


- START BY CARRYING THE INTEGRATION IN TERMS OF k

$$\int_1^k \frac{(\sqrt{3}k+\sqrt{3x})^2}{kx^3}\,dx = \int_1^k \frac{3k^2+2k\sqrt{3}\sqrt{3x}+3x}{kx^3}\,dx$$

$$= \int_1^k \frac{3k^2}{kx^3}+\frac{6k\sqrt{x}}{kx^3}+\frac{3x}{kx^3}\,dx = \int_1^k 3kx^{-3}+6x^{-\frac{5}{2}}+\frac{3}{k}x^{-2}\,dx$$

$$= \left[-\frac{3k}{2}x^{-2}-4x^{-\frac{3}{2}}-\frac{3}{k}x^{-1}\right]_1^k = \left[\frac{3k}{2x^2}+\frac{4}{x^{3/2}}+\frac{3}{kx}\right]_k^1$$

$$= \left(\frac{3k}{2}+4+\frac{3}{k}\right)-\left(\frac{3k}{2k^2}+\frac{4}{k^{3/2}}+\frac{3}{k^2}\right)$$

$$= \frac{3}{2}k+4+\frac{3}{k}-\frac{3}{2k}-\frac{4}{k^{3/2}}-\frac{3}{k^2}$$

$$= \left(\frac{3}{2}k+4+\frac{3}{2k}-\frac{3}{k^2}\right)-\frac{4}{k^{3/2}} \longleftarrow a-\sqrt{k}$$

- AS $k\in\mathbb{Q}$, $\frac{3}{2}k+4+\frac{3}{2k}-\frac{3}{k^2}\in\mathbb{Q}$
- HENCE $-\frac{4}{k^{3/2}} = -\sqrt{k}$

$$4 = k^2$$

$$k = 2 \quad (\cancel{-2}) \quad \text{OTHERWISE } \sqrt{k} \text{ IS NOT DEFINED}$$

- FINALLY WE CAN FIND a

$$a = \frac{3}{2}k+4+\frac{3}{2k}-\frac{3}{k^2} = \frac{3}{2}\times2+4+\frac{3}{2\times2}-\frac{3}{4}$$

$$a = 3+4+\frac{3}{4}-\frac{3}{4}$$

$$a = 7$$

Question 301 (*****)

A cubic curve C has equation

$$y = ax^3 + bx, \quad x \in \mathbb{R},$$

where a and b are non zero constants with $a > 0$.

The curve C' is the reflection of C about the straight line with equation $y = x$.

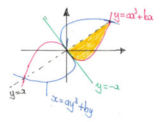
The straight line with equation $y = -x$ is a tangent to both C and C' at the origin O.

Given that the finite region bounded by C and C' has area 9, find the value of a.

$\boxed{a = \frac{4}{9}}$

$y = ax^3 + bx \qquad a > 0$

- START BY SKETCHING TO VISUALISE THE REGION BY SKETCHING THE CURVE $y = ax^3 + bx$ & ITS REFLECTION IN THE LINE $y = x$



IF $y = ax^3 + bx$

$\frac{dy}{dx} = 3ax^2 + b$

$-1 = b$

- USING SYMMETRY ARGUMENTS THE REQUIRED AREA IS 4 TIMES THE AREA OF THE YELLOW REGION

- FIND THE INTERSECTION BETWEEN THE TWO CURVES, USING THE FACT THAT THEY MEET ON THE LINE $y = x$

$$\left.\begin{array}{l} y = ax^3 - x \\ y = x \end{array}\right\} \implies ax^3 - x = x$$

$$ax^3 - 2x = 0$$

$$x(ax^2 - 2) = 0$$

$$x = +\sqrt{\frac{2}{a}} \quad \text{(TAKING THE INTERSECTION IN THE FIRST QUADRANT)}$$

- NOW THE TOTAL AREA IS GIVEN BY THE FOLLOWING EXPRESSION AND EQUALS 9

$$4\left[\int_0^{\sqrt{\frac{2}{a}}} x \, dx - \int_0^{\sqrt{\frac{2}{a}}} ax^3 - x \, dx\right] = 9$$

$$\implies 4\int_0^{\sqrt{\frac{2}{a}}} 2x - ax^3 \, dx = 9$$

$$\implies 4\left[x^2 - \frac{1}{4}ax^4\right]_0^{\sqrt{\frac{2}{a}}} = 9$$

$$\implies \left[4x^2 - ax^4\right]_0^{\sqrt{\frac{2}{a}}} = 9$$

$$\implies 4\left(\frac{2}{a}\right) - a\left(\frac{4}{a^2}\right) = 9$$

$$\implies \frac{8}{a} - \frac{4}{a} = 9$$

$$\implies \frac{4}{a} = 9$$

$$\implies a = \frac{4}{9}$$

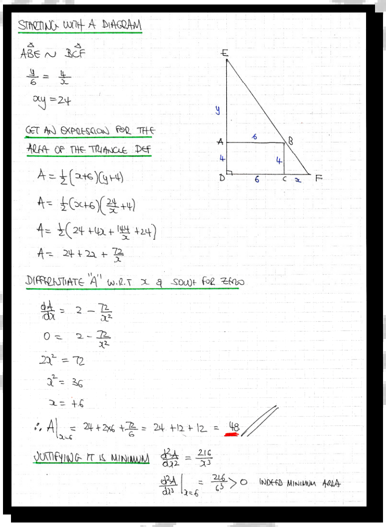
Question 302 (*****)

A rectangle $ABCD$ is such so that $|DC| = 6$ and $|DA| = 4$.

The side DA is extended to the point E and the side DC is extended to the point F so that EBF is a straight line.

Determine, with full justification, the minimum area of the triangle EDF.

$\boxed{}$, $\boxed{A_{\text{min}} = 48}$



STARTING WITH A DIAGRAM

$\triangle ABE \sim \triangle BCF$

$\frac{y}{6} = \frac{4}{x}$

$xy = 24$

GET AN EXPRESSION FOR THE AREA OF THE TRIANGLE DEF

$A = \frac{1}{2}(x+6)(y+4)$

$A = \frac{1}{2}(x+6)\left(\frac{24}{x}+4\right)$

$A = \frac{1}{2}\left(24 + 4x + \frac{144}{x} + 24\right)$

$A = 24 + 2x + \frac{72}{x}$

DIFFERENTIATE "A" W.R.T x & SOLVE FOR ZERO

$\frac{dA}{dx} = 2 - \frac{72}{x^2}$

$0 = 2 - \frac{72}{x^2}$

$2x^2 = 72$

$x^2 = 36$

$x = +6$

$\therefore A\big|_{x=6} = 24 + 2\times 6 + \frac{72}{6} = 24 + 12 + 12 = 48$

JUSTIFYING IT IS MINIMUM $\frac{d^2A}{dx^2} = \frac{216}{x^3}$

$\frac{d^2A}{dx^2}\Big|_{x=6} = \frac{216}{6^3} > 0$ INDEED MINIMUM AREA

Question 303 (*****)

A curve C and a straight line L have respective equations

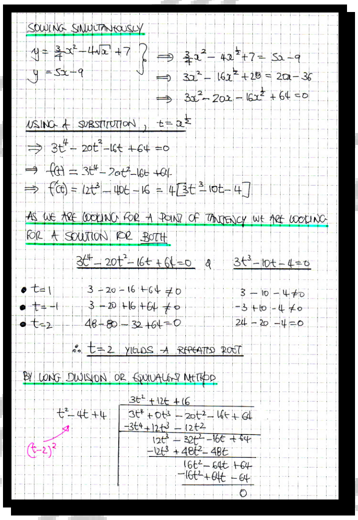
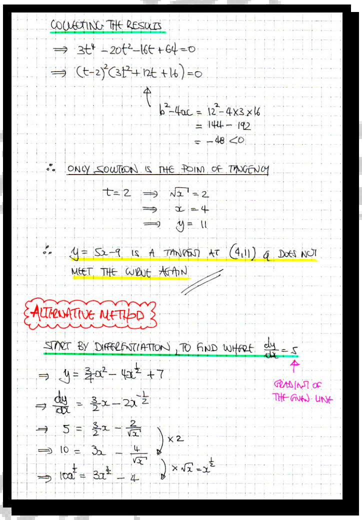
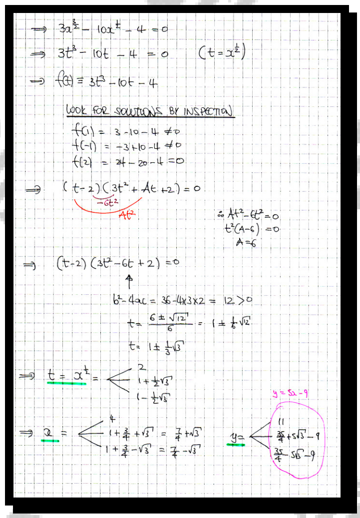
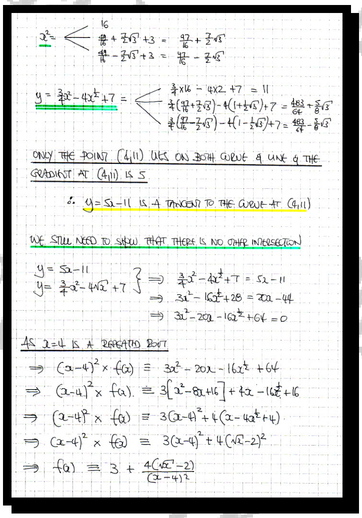
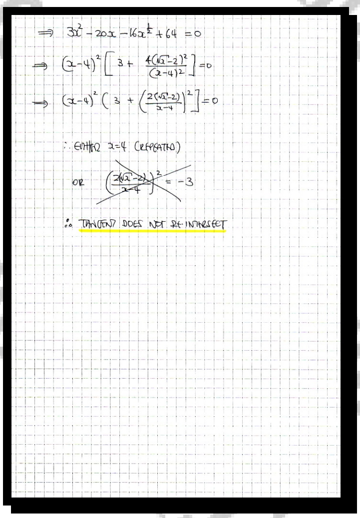
$$C: \ y = \frac{3}{4}x^2 - 4\sqrt{x} + 7, \ x \geq 0$$

$$L: \ y = 5x - 9$$

a) Show that L is a tangent to C at some point P, further determining the coordinates of P.

b) Show further that L does not meet C again.

$\boxed{}$, $\boxed{P(4,11)}$

Question 304 (*****)

If a and b such that $a > b > 0$, find with justification the minimum value of

$$\frac{4a^2+9b^2}{5ab}.$$

$\boxed{\text{X-P}}$, $\boxed{\frac{12}{5}}$

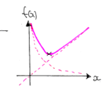
REWRITE THE EXPRESSION AS FOLLOWS

$$\frac{4a^2+9b^2}{5ab} = \frac{4a^2}{5ab} + \frac{9b^2}{5ab} = \frac{4}{5}\left(\frac{a}{b}\right) + \frac{9}{5}\left(\frac{b}{a}\right)$$

LET $x = \frac{a}{b}$, $x > 1$

$$f(x) = \frac{4}{5}x + \frac{9}{5}\left(\frac{1}{x}\right)$$

$$f'(x) = \frac{4}{5} - \frac{9}{5x^2}$$



DEFINITELY A MINIMUM

SOLVE FOR ZERO

$$0 = \frac{4}{5} - \frac{9}{5x^2}$$

$$0 = 4 - \frac{9}{x^2}$$

$$\frac{9}{x^2} = 4$$

$$x^2 = \frac{9}{4}$$

$$x = +\frac{3}{2}$$

AS THIS DEFINITELY YIELDS A MINIMUM

$$f\left(\frac{3}{2}\right) = \frac{4}{5}\left(\frac{3}{2}\right) + \frac{9}{5}\left(\frac{2}{3}\right)$$

$$f\left(\frac{3}{2}\right) = \frac{6}{5} + \frac{6}{5}$$

$$f\left(\frac{3}{2}\right) = \frac{12}{5}$$

$$\therefore \left[\frac{4a^2+9b^2}{5ab}\right]_{\text{MIN}} = \frac{12}{5}$$

Question 305 (*****)

A convergent geometric progression has positive first term and positive common ratio.

Show that the sum to infinity of the progression is at least four times as large as its second term.

V, , proof

STARTING WITH THE "GIVENS"

$$S_{\infty} = \frac{a}{1-r} \qquad 0 < r < 1 \quad a > 0 \qquad u_2 = ar$$

NOW CONSIDER THE "RATIO" BELOW

$$\frac{S_{\infty}}{u_2} = \frac{\frac{a}{1-r}}{ar} = \frac{a}{ar(1-r)} = \frac{1}{r(1-r)}$$

PARTIAL FRACTIONS (BY COVER UP)

$$\frac{S_{\infty}}{u_2} = \frac{1}{r} + \frac{1}{1-r} = \frac{1}{r} - \frac{1}{r-1}$$

NOW CONSIDER A NEW FUNCTION & CALCULUS

$$\frac{S_{\infty}}{u_2} = f(r) = \frac{1}{r} - \frac{1}{r-1}$$

$$f'(r) = -\frac{1}{r^2} + \frac{1}{(r-1)^2}$$

$$f''(r) = \frac{2}{r^3} - \frac{2}{(r-1)^3}$$

LOOK FOR STATIONARY POINTS

$$\Rightarrow -\frac{1}{r^2} + \frac{1}{(r-1)^2} = 0$$

$$\Rightarrow \frac{1}{(r-1)^2} = \frac{1}{r^2}$$

$$\Rightarrow (r-1)^2 = r^2$$

$$\Rightarrow r^2 - 2r\ 1\ \ r^2$$

$$\Rightarrow 1 = 2r$$

$$\Rightarrow r = \frac{1}{2}$$

HENCE WE HAVE

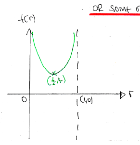
$$f\left(\tfrac{1}{2}\right) = \frac{S_{\infty}}{u_2} = \frac{1}{\frac{1}{2}\left(1-\frac{1}{2}\right)} = \frac{1}{\frac{1}{2} \times \frac{1}{2}} = 4$$

$$f''\left(\tfrac{1}{2}\right) = \frac{2}{1/8} - \frac{2}{-1/8} = 16 + 16 = 32 > 0$$

IE LOCAL MINIMUM

NEED A SKETCH OF $f(r)$ FOR $0 < 0 < 1$

OR SOME OTHER ARGUMENT



$f(0) = f(1) = +\infty$

$$\therefore \quad f(r) \geqslant 4 \qquad 0 < r < 1$$

$$\frac{S_{\infty}}{u_2} \geqslant 4$$

$$S_{\infty} \geqslant 4u_2$$

AS REQUIRED