GEOMETRIC SERIES

Question 1 **(**+)**

Miss Velibright started working as an accountant in a large law firm in the year 2001.

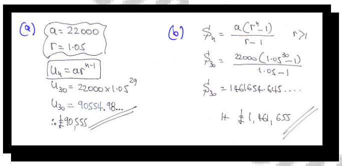
Her starting salary was £22,000 and her contract promised that she will be receiving a pay rise of 5% every year thereafter. Miss Velibright plans to retire in 2030.

Find to the nearest £, …

a) …her salary in the year 2030.

b) … her total earnings in employment for the years 2001 to 2030, inclusive.

$\boxed{£90,555}$, $\boxed{£1,461,655}$



(a) $a = 22000$
$r = 1.05$
$\boxed{u_n = ar^{n-1}}$
$u_{30} = 22000 \times 1.05^{29}$
$u_{30} = 90554.98\ldots$
$\therefore$ £90,555

(b) $S_n = \frac{a(r^n - 1)}{r - 1}$ $r > 1$
$S_{30} = \frac{22000(1.05^{30} - 1)}{1.05 - 1}$
$S_{30} = 1461654.645\ldots$
$\therefore$ £1,461,655

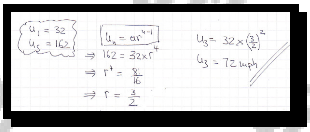
Question 2 **(***)**

The maximum speed, in mph, that can be achieved in each of the five gears of a sports car form a geometric progression.

The maximum speed obtained in first gear is 32 mph while the car can achieve a maximum speed of 162 mph in fifth gear.

Find the maximum speed that can be achieved in third gear.

$\boxed{72 \text{ mph}}$



$u_1 = 32$
$u_5 = 162$
$\boxed{u_n = ar^{n-1}}$
$\Rightarrow 162 = 32 \times r^4$
$\Rightarrow r^4 = \frac{81}{16}$
$\Rightarrow r = \frac{3}{2}$
$u_3 = 32 \times \left(\frac{3}{2}\right)^2$
$u_3 = 72$ mph

Question 3 (***)

Grandad gave Kevin £10 on his first birthday and he increased the amount by 20% on each subsequent birthday.

a) Calculate the amount of money that Kevin received from his grandad on his 10th birthday

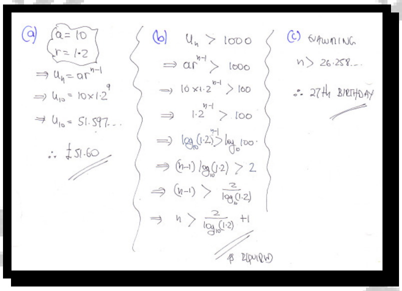
Kevin received the last birthday amount of money from his grandad on his n^{th} birthday and on that birthday the amount he received exceeded £1000 for the first time.

b) Show clearly that

$$n > \frac{2}{\log_{10}(1.2)} + 1.$$

c) State the value of n.

$\boxed{£51.60}$, $\boxed{n = 27}$



(a) $a = 10$, $r = 1.2$

$\Rightarrow u_n = ar^{n-1}$

$\Rightarrow u_{10} = 10 \times 1.2^9$

$\Rightarrow u_{10} = 51.597\ldots$

$\therefore$ £51.60

(b) $u_n > 1000$

$\Rightarrow ar^{n-1} > 1000$

$\Rightarrow 10 \times 1.2^{n-1} > 100$

$\Rightarrow 1.2^{n-1} > 100$

$\Rightarrow \log_{10}(1.2)^{n-1} > \log_{10} 100$

$\Rightarrow (n-1)\log_{10}(1.2) > 2$

$\Rightarrow (n-1) > \frac{2}{\log_{10}(1.2)}$

$\Rightarrow n > \frac{2}{\log_{10}(1.2)} + 1$

AS REQUIRED

(c) [illegible]

$n > 26.258\ldots$

$\therefore$ 27th BIRTHDAY

Question 4 (***+)

The manufacturer of a certain brand of washing machine is to replace an old model with a new model. There will be a "phase out" period for the old model and a "phase in" period for the new model, both lasting 24 months and starting at the same time.

On the first month of the phase out period 5000 old washing machines will be produced and each month thereafter, this figure will reduce by 20%.

a) Show that on the fifth month of the "phase out" period 2048 old washing machines will be produced.

b) Find how many old washing machines will be produced during the "phase out" period.

On the first month of the "phase in" period 1000 new washing machines will be produced and each month thereafter, this figure will increase by 5%.

c) Calculate how many new washing machines will be produced on the last month of the "phase in" period.

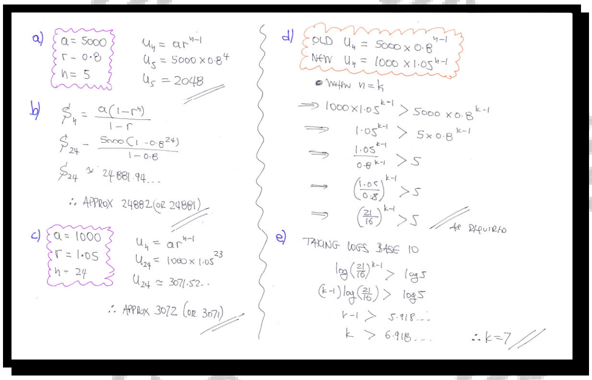
On the k^{th} month of the "phase in/phase out" period, for the first time more new washing machines will be produced than old washing machines.

d) Show that k satisfies

$$\left(\frac{21}{16}\right)^{k-1} > 5.$$

e) Use logarithms to determine the value of k.

$\boxed{24881 \text{ or } 24882}$, $\boxed{3071 \text{ or } 3072}$, $\boxed{k=7}$



a) $a = 5000$
$r - 0.8$
$n = 5$

$u_n = ar^{n-1}$
$u_5 = 5000 \times 0.8^4$
$u_5 = 2048$

b) $S_n = \frac{a(1-r^n)}{1-r}$
$S_{24} - \frac{5000(1-0.8^{24})}{1-0.8}$
$S_{24} \approx 24881.94\ldots$

∴ APPROX 24882 (OR 24881)

c) $a = 1000$
$r = 1.05$
$n = 24$

$u_n = ar^{n-1}$
$u_{24} = 1000 \times 1.05^{23}$
$u_{24} \simeq 3071.52\ldots$

∴ APPROX 3072 (OR 3071)

d) OLD $u_n = 5000 \times 0.8^{n-1}$
NEW $u_n = 1000 \times 1.05^{n-1}$

• WHEN $n = k$

$\Rightarrow 1000 \times 1.05^{k-1} > 5000 \times 0.8^{k-1}$
$\Rightarrow 1.05^{k-1} > 5 \times 0.8^{k-1}$
$\Rightarrow \frac{1.05^{k-1}}{0.8^{k-1}} > 5$
$\Rightarrow \left(\frac{1.05}{0.8}\right)^{k-1} > 5$
$\Rightarrow \left(\frac{21}{16}\right)^{k-1} > 5$ AS REQUIRED

e) TAKING LOGS BASE 10

$\log\left(\frac{21}{16}\right)^{k-1} > \log 5$
$(k-1)\log\left(\frac{21}{16}\right) > \log 5$
$k-1 > 5.918\ldots$
$k > 6.918\ldots$

$\therefore k = 7$

Question 5 (***+)

In a certain quiz game, contestants answer questions consecutively until they get a question wrong.

They win £10 for answering the first question correctly, £20 for answering the second question correctly, £40 for answering the third question correctly, and so on so that the amounts won for each successive question is a term of a geometric series.

When contestants answer a question wrong their game is over and they get to keep $\frac{1}{10}$ of their **total** winnings up to that point.

Connor answers 5 questions correctly.

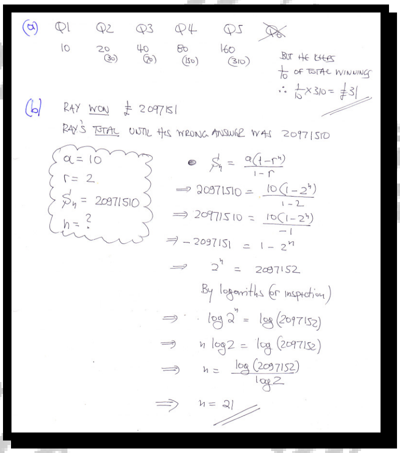
a) Show that Connor won £31.

The highest prize won in this game, by a contestant called Ray, was £2,097,151.

b) Use algebra to find the number of questions that Ray answered correctly.

Full workings, justifying every step in the calculations, must be shown in this part of the question.

$\boxed{21}$



(a)

Q1	Q2	Q3	Q4	Q5	~~Q6~~
10	20 (30)	40 (70)	80 (150)	160 (310)	

BUT HE KEEPS $\frac{1}{10}$ OF TOTAL WINNINGS

$\therefore \frac{1}{10} \times 310 = £31$

(b) RAY WON £2097151

RAY'S TOTAL UNTIL HIS WRONG ANSWER WAS 20971510

$a = 10$
$r = 2$
$S_n = 20971510$
$n = ?$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\Rightarrow 20971510 = \frac{10(1-2^n)}{1-2}$$

$$\Rightarrow 20971510 = \frac{10(1-2^n)}{-1}$$

$$\Rightarrow -2097151 = 1 - 2^n$$

$$\Rightarrow 2^n = 2097152$$

By logarithms (or inspection)

$$\Rightarrow \log 2^n = \log(2097152)$$

$$\Rightarrow n\log 2 = \log(2097152)$$

$$\Rightarrow n = \frac{\log(2097152)}{\log 2}$$

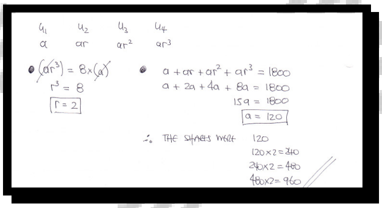
$$\Rightarrow n = 21$$

Question 6 (*+)**

Four brothers shared £1800 so that their shares formed the terms of a geometric progression.

Given that the largest share was 8 times as large as the smallest share, determine the individual amounts each brother got.

$\boxed{£120,\ £240,\ £480,\ £960}$



u_1	u_2	u_3	u_4
a	ar	ar^2	ar^3

- $(\cancel{a}r^3) = 8\times(\cancel{a})$

  $r^3 = 8$

  $\boxed{r = 2}$

- $a + ar + ar^2 + ar^3 = 1800$

  $a + 2a + 4a + 8a = 1800$

  $15a = 1800$

  $\boxed{a = 120}$

$\therefore$ THE SHARES WERE 120

$120\times2 = 240$

$240\times2 = 480$

$480\times2 = 960$ //

Question 7 (***+)

A steamboat uses 5 tonnes of coal to cover a standard journey designed for tourists.

Due to the engines becoming less efficient, the steamboat requires in each journey 2% more coal than the previous journey.

a) Calculate, in tonnes correct to three decimal places, …

i. … the amount of coal the steamboat will use on the tenth journey.

ii. … the total amount of coal the steamboat will use in the first ten journeys.

The company that owns the steamboat has stocked up with 360 tonnes of coal and plans to use all the coal during a single tourist season.

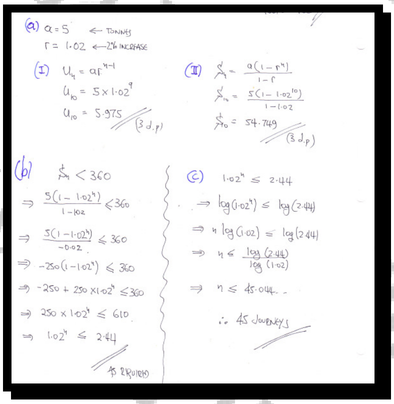
b) Assuming that in the first journey the steamboat used 5 tonnes of coal, and the consumption of coal increased by 2% in each subsequent journey, show clearly that

$$1.02^n \le 2.44,$$

where n is the total number of journeys during a single tourist season.

c) Hence, or otherwise, determine the maximum number of journeys that the steamboat can make a single tourist season.

$\boxed{5.975}$, $\boxed{54.749}$, $\boxed{45}$



(a) $a = 5$ ← TONNES

$r = 1.02$ ← 2% INCREASE

(I) $u_n = ar^{n-1}$

$u_{10} = 5 \times 1.02^9$

$u_{10} = 5.975$ (3 d.p)

(II) $S_n = \frac{a(1-r^n)}{1-r}$

$S_{10} = \frac{5(1-1.02^{10})}{1-1.02}$

$S_{10} = 54.749$ (3 d.p)

(b) $S_n < 360$

$\Rightarrow \frac{5(1-1.02^n)}{1-1.02} \leqslant 360$

$\Rightarrow \frac{5(1-1.02^n)}{-0.02} \leqslant 360$

$\Rightarrow -250(1-1.02^n) \leqslant 360$

$\Rightarrow -250 + 250 \times 1.02^n \leqslant 360$

$\Rightarrow 250 \times 1.02^n \leqslant 610$

$\Rightarrow 1.02^n \leqslant 2.44$

AS REQUIRED

(c) $1.02^n \leqslant 2.44$

$\Rightarrow \log(1.02^n) \leqslant \log(2.44)$

$\Rightarrow n\log(1.02) \leqslant \log(2.44)$

$\Rightarrow n \leqslant \frac{\log(2.44)}{\log(1.02)}$

$\Rightarrow n \leqslant 45.044\ldots$

$\therefore$ 45 JOURNEYS

Question 8 **(****)**

Max is revising for an exam by practicing papers.

He takes 3 hours and 20 minutes to complete the first paper and 3 hours and 15 minutes to complete the second paper.

It is assumed that the times Max takes to complete each successive paper are consecutive terms of a geometric progression.

a) Assuming this model, show that Max will take approximately …

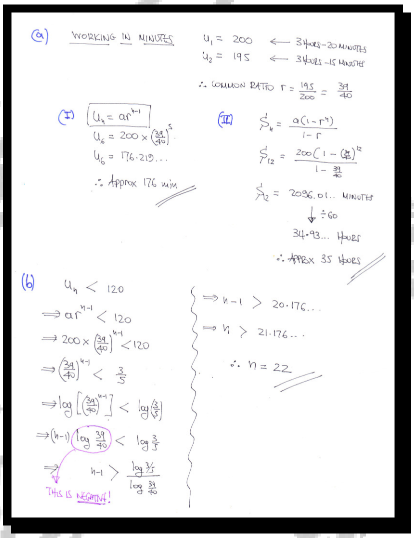
i. … 176 minutes to complete the sixth paper.

ii. … 35 hours to complete the first 12 papers.

Max aims to be able to complete a paper in under two hours.

b) Determine, by using logarithms, the minimum number of papers he needs to practice in order to achieve his target according to this model.

$\boxed{22}$



(a) WORKING IN MINUTES

$u_1 = 200 \leftarrow$ 3 HOURS – 20 MINUTES

$u_2 = 195 \leftarrow$ 3 HOURS – 15 MINUTES

$\therefore$ COMMON RATIO $r = \frac{195}{200} = \frac{39}{40}$

(I) $\boxed{u_n = ar^{n-1}}$

$u_6 = 200 \times \left(\frac{39}{40}\right)^5$

$u_6 = 176.219\ldots$

$\therefore$ Approx 176 min

(II) $S_n = \frac{a(1-r^n)}{1-r}$

$S_{12} = \frac{200\left(1-\left(\frac{39}{40}\right)^{12}\right)}{1-\frac{39}{40}}$

$S_{12} = 2096.01\ldots$ MINUTES

$\downarrow \div 60$

$34.93\ldots$ HOURS

$\therefore$ APPROX 35 HOURS

(b) $u_n < 120$

$\Rightarrow ar^{n-1} < 120$

$\Rightarrow 200 \times \left(\frac{39}{40}\right)^{n-1} < 120$

$\Rightarrow \left(\frac{39}{40}\right)^{n-1} < \frac{3}{5}$

$\Rightarrow \log\left[\left(\frac{39}{40}\right)^{n-1}\right] < \log\left(\frac{3}{5}\right)$

$\Rightarrow (n-1)\log\frac{39}{40} < \log\frac{3}{5}$

$\Rightarrow n-1 > \frac{\log 3/5}{\log\frac{34}{40}}$

THIS IS NEGATIVE!

$\Rightarrow n-1 > 20.176\ldots$

$\Rightarrow n > 21.176\ldots$

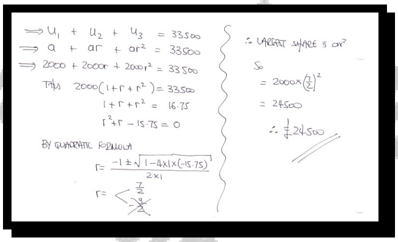
$\therefore n = 22$

Question 9 (****)

The amount of £33500 is to be divided into three shares, so that the three shares form the terms of a geometric progression.

Given that the value of the smallest share is £2000 , find the value of the largest share.

$\boxed{£24500}$



$\Longrightarrow u_1 + u_2 + u_3 = 33500$

$\Longrightarrow a + ar + ar^2 = 33500$

$\Longrightarrow 2000 + 2000r + 2000r^2 = 33500$

Thus $2000(1+r+r^2) = 33500$

$1+r+r^2 = 16.75$

$r^2+r-15.75 = 0$

BY QUADRATIC FORMULA

$r = \frac{-1 \pm \sqrt{1-4\times1\times(-15.75)}}{2\times1}$

$r = \begin{cases} \frac{7}{2} \\ \cancel{-\frac{9}{2}} \end{cases}$

$\therefore$ LARGEST SHARE IS ar^2

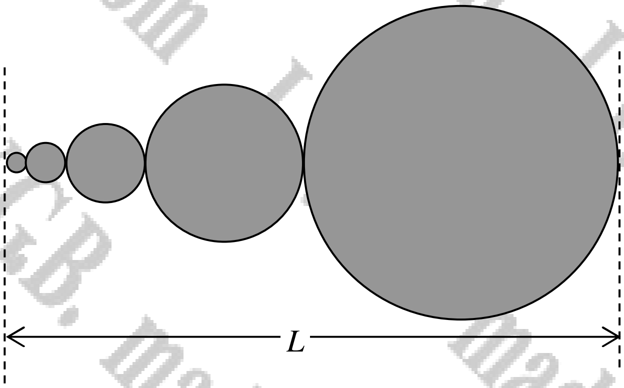
So

$= 2000\times\left(\frac{7}{2}\right)^2$

$= 24500$

$\therefore £24500$

Question 10 (****)



The figure above shows a pattern of 5 circles, touching externally, whose centres lie on a straight line of length L units.

The radii of these circles form a geometric progression, where the radius of the smaller circle is 3 units and that of the fifth (larger) circle is 48 units.

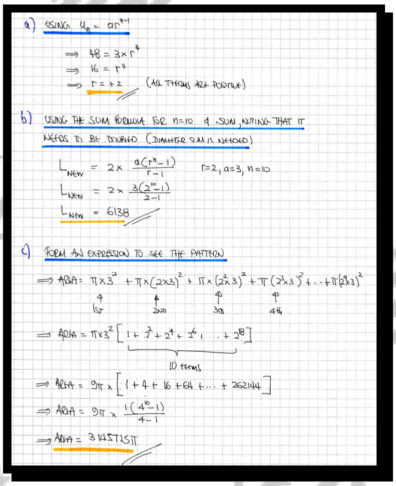
a) Find the common ratio of the geometric progression.

The pattern is extended by 5 more circles to 10 circles.

b) Determine the new value of L.

c) Calculate, in terms of π, the total area of the 10 circles of the new pattern.

$\boxed{r=2}$, $\boxed{L=6138}$, $\boxed{\text{area}=3{,}145{,}725\pi}$



a) USING $u_n = ar^{n-1}$

$\Rightarrow 48 = 3 \times r^4$

$\Rightarrow 16 = r^4$

$\Rightarrow r = +2$ (ALL TERMS ARE POSITIVE)

b) USING THE SUM FORMULA FOR $n=10$ & SUM, NOTING THAT IT NEEDS TO BE DOUBLED (DIAMETER SUM IS NEEDED)

$L_{NEW} = 2 \times \frac{a(r^n-1)}{r-1}$ $r=2, a=3, n=10$

$L_{NEW} = 2 \times \frac{3(2^{10}-1)}{2-1}$

$L_{NEW} = 6138$

c) FORM AN EXPRESSION TO SEE THE PATTERN

$\Rightarrow \text{AREA} = \underset{1\text{ST}}{\underset{\uparrow}{\pi \times 3^2}} + \underset{2\text{ND}}{\underset{\uparrow}{\pi \times (2\times3)^2}} + \underset{3\text{RD}}{\underset{\uparrow}{\pi \times (2^2\times3)^2}} + \underset{4\text{th}}{\underset{\uparrow}{\pi (2^3\times3)^2}} + \dots + \pi(2^9\times3)^2$

$\Rightarrow \text{AREA} = \pi \times 3^2 [\underbrace{1 + 2^2 + 2^4 + 2^6 + \dots + 2^{18}}_{10 \text{ terms}}]$

$\Rightarrow \text{AREA} = 9\pi \times [1 + 4 + 16 + 64 + \dots + 262144]$

$\Rightarrow \text{AREA} = 9\pi \times \frac{1(4^{10}-1)}{4-1}$

$\Rightarrow \text{AREA} = 3145725\pi$

Question 11 (****)

Liquid is kept in containers, which due to evaporation and ongoing chemical reactions, at the end of each month the volume of the liquid in these containers reduces by 10% compared with the volume at the start of the same month.

One such container is filled up with 250 litres of liquid.

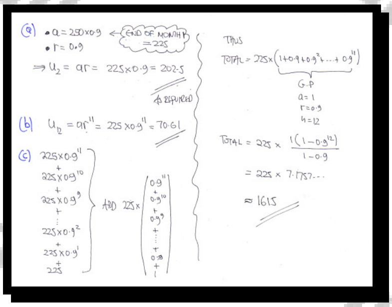
a) Show that the volume of the liquid in the container at the end of the second month is 202.5 litres.

b) Find the volume of the liquid in the container at the end of the twelfth month.

At the start of each month a new container is filled up with 250 litres of liquid, so that at the end of twelve months there are 12 containers with liquid.

c) Use an algebraic method to calculate the total amount of liquid in the 12 containers at the end of 12 months.

$\boxed{\approx 70.6}$, $\boxed{\approx 1615}$



(a) • $a = 250 \times 0.9$ ← END OF MONTH 1 $= 225$

• $r = 0.9$

$\Rightarrow u_2 = ar = 225 \times 0.9 = 202.5$

AS REQUIRED

(b) $u_{12} = ar^{11} = 225 \times 0.9^{11} = 70.61$

(c) $\left.\begin{matrix} 225\times0.9^{11} \\ + \\ 225\times0.9^{10} \\ + \\ 225\times0.9^{9} \\ + \\ \vdots \\ 225\times0.9^{2} \\ + \\ 225\times0.9^{1} \\ + \\ 225 \end{matrix}\right\}$ ADD $225 \times \begin{pmatrix} 0.9^{11} \\ + \\ 0.9^{10} \\ + \\ 0.9^{9} \\ + \\ \vdots \\ + \\ 0.9 \\ + \\ 1 \end{pmatrix}$

THUS

$\text{TOTAL} = 225 \times \underbrace{\left(1 + 0.9 + 0.9^2 + \dots + 0.9^{11}\right)}_{\substack{\text{G.P} \\ a=1 \\ r=0.9 \\ n=12}}$

$\text{TOTAL} = 225 \times \frac{1(1-0.9^{12})}{1-0.9}$

$= 225 \times 7.1757\dots$

≈ 1615

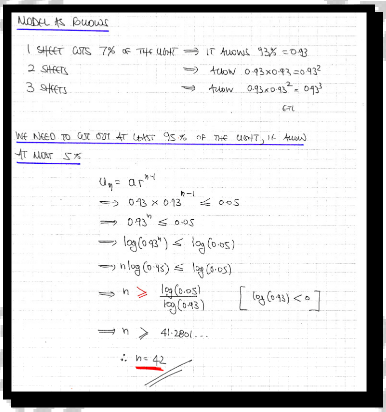
Question 12 (****)

A certain type of plastic sheet blocks 7% of the sunlight.

It is required to block at least 95% of the sunlight by placing N of these plastic sheets on top of each other.

Use algebra, to determine the least value of N.

$\boxed{N = 42}$



MODEL AS FOLLOWS

1 SHEET CUTS 7% OF THE LIGHT $\Rightarrow$ IT ALLOWS $93\% = 0.93$

2 SHEETS $\Rightarrow$ ALLOW $0.93 \times 0.93 = 0.93^2$

3 SHEETS $\Rightarrow$ ALLOW $0.93 \times 0.93^2 = 0.93^3$

ETC

WE NEED TO CUT OUT AT LEAST 95% OF THE LIGHT, IE ALLOW AT MOST 5%

$$u_n = ar^{n-1}$$

$$\Rightarrow 0.93 \times 0.93^{n-1} \leqslant 0.05$$

$$\Rightarrow 0.93^n \leqslant 0.05$$

$$\Rightarrow \log(0.93^n) \leqslant \log(0.05)$$

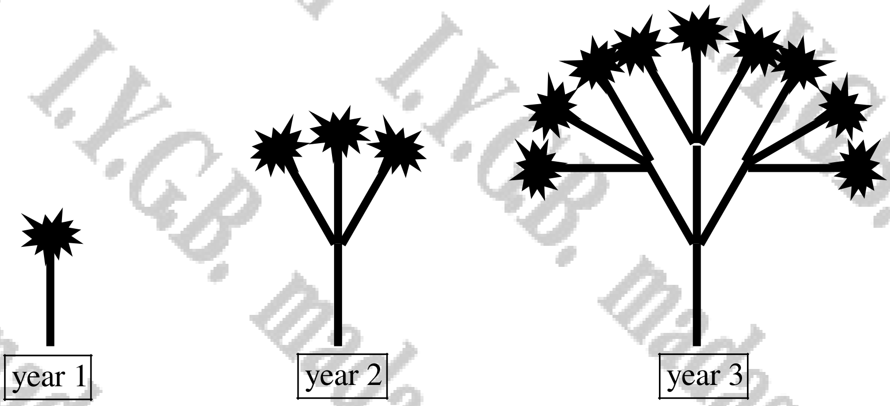
$$\Rightarrow n\log(0.93) \leqslant \log(0.05)$$

$$\Rightarrow n \geqslant \frac{\log(0.05)}{\log(0.93)} \qquad \left[\log(0.93) < 0\right]$$

$$\Rightarrow n \geqslant 41.2801\ldots$$

$$\therefore \underline{n = 42}$$

Question 13 (****+)



The figure above shows a flowering plant. In year 1 it produces a single stem with a flower at the end.

In year 2, the flower withers and in its place three more stems are produced, with each new stem having a new flower at its end, i.e. 4 stems in total.

In year 3, the flowers wither again and in each of their places a new stems is produced, with each new stem having a new flower at its end, i.e. 13 stems in total.

This flowering pattern continues every year.

a) Find an expression for …

i. … the number of flowers in the n^{th} year.

ii. … the number of stems in the n^{th} year.

One such plant has 1093 stems.

b) Determine the number of flowers of this plant.

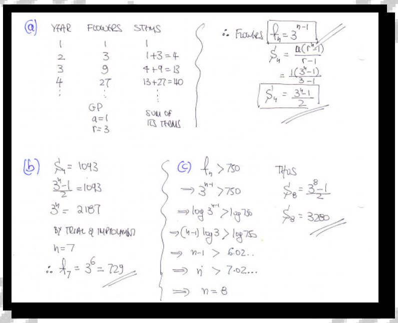
[continues overleaf]

[continued from overleaf]

A different plant of the above variety has over 750 flowers.

c) Determine the **least** number of stems of this plant.

$\boxed{f_n = 3^{n-1}}$, $\boxed{S_n = \dfrac{3^n - 1}{2}}$, $\boxed{729}$, $\boxed{3280}$



(a)

YEAR	FLOWERS	STEMS
1	1	1
2	3	1+3=4
3	9	4+9=13
4	27	13+27=40
⋮	⋮	⋮
	GP a=1 r=3	SUM OF ITS TERMS

∴ Flowers $\boxed{f_n = 3^{n-1}}$

$S_n = \frac{a(r^n-1)}{r-1}$

$= \frac{1(3^n-1)}{3-1}$

$\boxed{S_n = \frac{3^n-1}{2}}$

(b) $S_n = 1093$

$\frac{3^n-1}{2} = 1093$

$3^n = 2187$

BY TRIAL & IMPROVEMENT

$n=7$

$\therefore f_7 = 3^6 = 729$

(c) $f_n > 750$

$\Rightarrow 3^{n-1} > 750$

$\Rightarrow \log 3^{n-1} > \log 750$

$\Rightarrow (n-1)\log 3 > \log 750$

$\Rightarrow n-1 > 6.02..$

$\Rightarrow n > 7.02...$

$\Rightarrow n = 8$

Thus

$S_8 = \frac{3^8-1}{2}$

$S_8 = 3280$

Question 14 (****+)

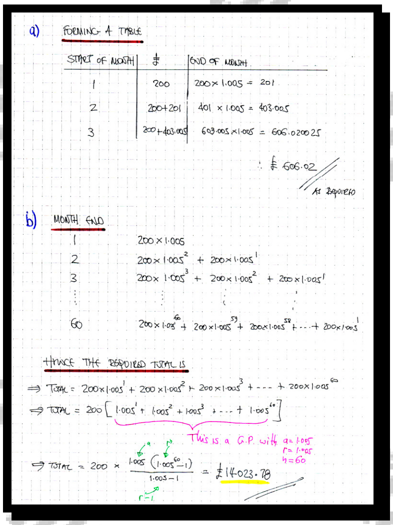
Anton is planning to save for a house purchase deposit over a period of 5 years.

He opens an account known as a "Homesaver" and plans to pay into this account £200 at the start of every month, and continue to do so for 5 years.

The account pays 0.5% compound interest **per month**, with the interest credited to the account at the end of every month.

- **a)** Show clearly that at the **end** of the third month the balance of the account will be £606.02 .
- **b)** Calculate the total amount in Anton's "Homesaver" account after 5 years.

$\boxed{£14023.78}$



a) FORMING A TABLE

START OF MONTH	£	END OF MONTH
1	200	$200 \times 1.005 = 201$
2	200+201	$401 \times 1.005 = 403.005$
3	200+403.005	$603.005 \times 1.005 = 606.020025$

$\therefore$ £ 606.02 // AS REQUIRED

b) MONTH END

1	200×1.005
2	$200 \times 1.005^2 + 200 \times 1.005^1$
3	$200 \times 1.005^3 + 200 \times 1.005^2 + 200 \times 1.005^1$
$\vdots$	$\vdots$
60	$200 \times 1.005^{60} + 200 \times 1.005^{59} + 200 \times 1.005^{58} + \dots + 200 \times 1.005^1$

HENCE THE REQUIRED TOTAL IS

$$\Rightarrow \text{TOTAL} = 200 \times 1.005^1 + 200 \times 1.005^2 + 200 \times 1.005^3 + \dots + 200 \times 1.005^{60}$$

$$\Rightarrow \text{TOTAL} = 200\Big[\underbrace{1.005^1 + 1.005^2 + 1.005^3 + \dots + 1.005^{60}}_{\text{This is a G.P. with } a = 1.005,\ r = 1.005,\ n = 60}\Big]$$

$$\Rightarrow \text{TOTAL} = 200 \times \frac{\overset{a}{1.005}\,(\overset{r}{1.005}^{60} - 1)}{\underset{r-1}{1.005 - 1}} = £14023.78$$

Question 15 (****+)

A pension contribution scheme is scheduled as follows.

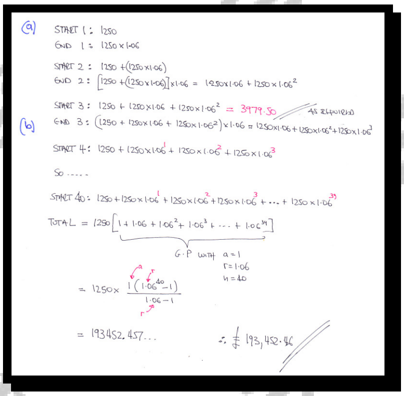
A £1250 contribution is made at the **start** of every year.

The total money in the scheme at the end of every year is re-invested at a constant compound interest rate of 6% per annum.

a) Show that at the start of the third year, after the annual contribution has been made, the amount in the pension scheme is £3979.50 .

b) Calculate the amount in the pension scheme at the start of the fortieth year, after the annual contribution is made.

$\boxed{£193452.46}$



(a)

START 1 : 1250

END 1 : 1250×1.06

START 2 : $1250 + (1250 \times 1.06)$

END 2 : $[1250 + (1250 \times 1.06)] \times 1.06 = 1250 \times 1.06 + 1250 \times 1.06^2$

START 3 : $1250 + 1250 \times 1.06 + 1250 \times 1.06^2 = 3979.50$ // AS REQUIRED

(b)

END 3 : $(1250 + 1250 \times 1.06 + 1250 \times 1.06^2) \times 1.06 = 1250 \times 1.06 + 1250 \times 1.06^2 + 1250 \times 1.06^3$

START 4 : $1250 + 1250 \times 1.06^1 + 1250 \times 1.06^2 + 1250 \times 1.06^3$

So .....

START 40 : $1250 + 1250 \times 1.06^1 + 1250 \times 1.06^2 + 1250 \times 1.06^3 + \dots + 1250 \times 1.06^{39}$

$$\text{TOTAL} = 1250\Big[\underbrace{1 + 1.06 + 1.06^2 + 1.06^3 + \dots + 1.06^{39}}_{\text{G.P WITH } a=1,\ r=1.06,\ n=40}\Big]$$

$$= 1250 \times \frac{\overset{a}{1}(\overset{r}{1.06}^{40} - 1)}{\underset{r}{1.06} - 1}$$

$= 193452.457\dots$ $\therefore$ £ 193,452.46 //

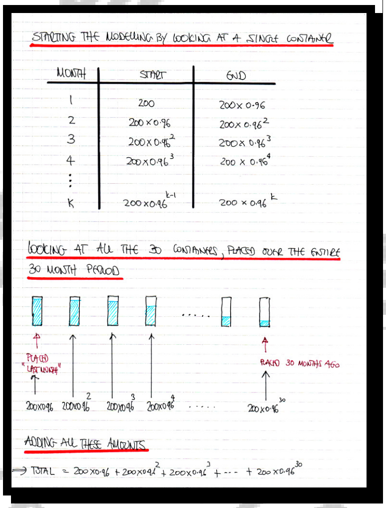
Question 16 (****+)

Liquid is kept in containers, which due to evaporation and ongoing chemical reactions, at the end of each month the volume of the liquid in these containers reduces by 4% compared with the volume at the start of the same month.

At the start of each month a new container is filled up with 200 litres of liquid, so that at the end of thirty months there are 30 containers with liquid.

Calculate the total amount of liquid in the 30 containers at the end of 30 months.

$\boxed{\approx 3389}$



STARTING THE MODELLING BY LOOKING AT A SINGLE CONTAINER

MONTH	START	END
1	200	200×0.96
2	200×0.96	200×0.96^2
3	200×0.96^2	200×0.96^3
4	200×0.96^3	200×0.96^4
⋮		
K	$200 \times 0.96^{k-1}$	200×0.96^{k}

LOOKING AT ALL THE 30 CONTAINERS, PLACED OVER THE ENTIRE 30 MONTH PERIOD

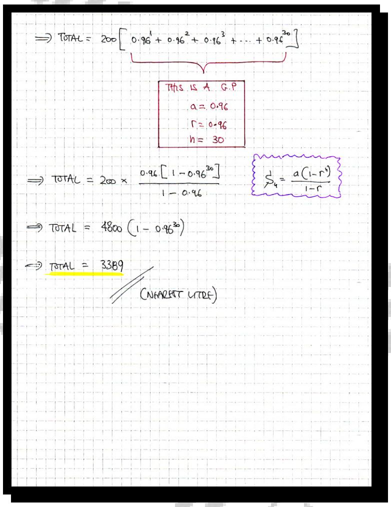
PLACED "LAST MONTH"

PLACED 30 MONTHS AGO

200×0.96 200×0.96^2 200×0.96^3 200×0.96^4 ..... 200×0.96^{30}

ADDING ALL THESE AMOUNTS

$\Rightarrow$ TOTAL $= 200 \times 0.96 + 200 \times 0.96^2 + 200 \times 0.96^3 + \dots + 200 \times 0.96^{30}$



$\Rightarrow$ TOTAL $= 200\underbrace{\left[0.96^1 + 0.96^2 + 0.96^3 + \dots + 0.96^{30}\right]}$

THIS IS A G.P
$a = 0.96$
$r = 0.96$
$n = 30$

$\Rightarrow$ TOTAL $= 200 \times \dfrac{0.96\left[1 - 0.96^{30}\right]}{1 - 0.96}$

$S_n = \dfrac{a(1-r^n)}{1-r}$

$\Rightarrow$ TOTAL $= 4800\left(1 - 0.96^{30}\right)$

$\Rightarrow$ TOTAL $= 3389$ //

(NEAREST LITRE)

Question 17 (*****)

An elastic ball is dropped from a height of 20 metres, and bounces repeatedly.

The ball bounces off the ground to a height which is $\frac{1}{2}$ the height from which it was last dropped.

a) Show that after the n^{th} bounce the ball reaches a height of $5\times2^{2-n}$ metres.

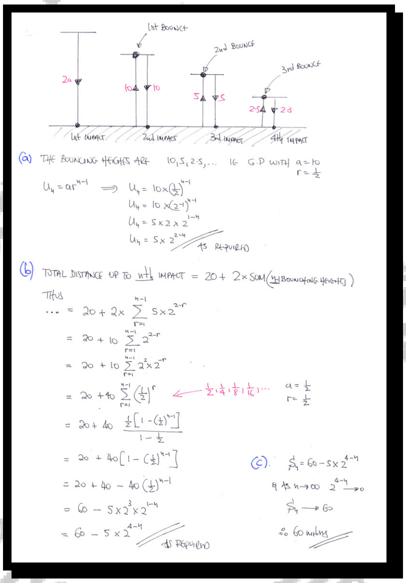
b) Show clearly that the total distance covered by the ball **up and including the n^{th} impact** is given by

$$60-5\times2^{4-n}.$$

The ball keeps bouncing off the ground in this fashion until it comes to rest.

c) Determine the total distance covered by the ball until it comes to rest.

$\boxed{60\text{ metres}}$



(a) THE BOUNCING HEIGHTS ARE $10, 5, 2\cdot5, \ldots$ IE G.P WITH $a=10$, $r=\frac{1}{2}$

$$u_n = ar^{n-1} \Longrightarrow u_n = 10\times\left(\tfrac{1}{2}\right)^{n-1}$$
$$u_n = 10\times\left(2^{-1}\right)^{n-1}$$
$$u_n = 5\times2\times2^{1-n}$$
$$u_n = 5\times2^{2-n}$$

AS REQUIRED

(b) TOTAL DISTANCE UP TO $\underline{n\text{th}}$ IMPACT $= 20 + 2\times \text{SUM}(\underline{n-1}\text{ BOUNCING HEIGHTS})$

THUS

$$\ldots = 20 + 2\times\sum_{r=1}^{n-1} 5\times2^{2-r}$$
$$= 20 + 10\sum_{r=1}^{n-1} 2^{2-r}$$
$$= 20 + 10\sum_{r=1}^{n-1} 2^{2}\times2^{-r}$$
$$= 20 + 40\sum_{r=1}^{n-1}\left(\tfrac{1}{2}\right)^{r} \leftarrow \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots \quad a=\tfrac{1}{2},\ r=\tfrac{1}{2}$$
$$= 20 + 40\,\frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^{n-1}\right]}{1-\frac{1}{2}}$$
$$= 20 + 40\left[1-\left(\tfrac{1}{2}\right)^{n-1}\right]$$
$$= 20 + 40 - 40\left(\tfrac{1}{2}\right)^{n-1}$$
$$= 60 - 5\times2^{3}\times2^{1-n}$$
$$= 60 - 5\times2^{4-n}$$

AS REQUIRED

(c). $S_n = 60 - 5\times2^{4-n}$

& AS $n\to\infty$ $2^{4-n}\to0$

$S_n \to 60$

∴ 60 metres

Question 18 (*****)

An elastic ball is dropped from a height of h metres.

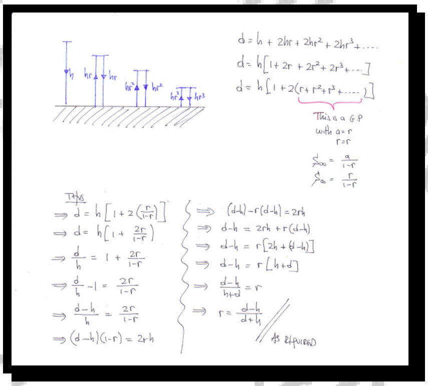
The ball bounces off the ground to a height which is r times the height from which it was dropped, where $0 < r < 1$.

The ball keeps bouncing off the ground in this fashion until it comes to rest.

Given the ball covers a total distance d show that

$$r = \frac{d-h}{d+h}.$$

proof



$d = h + 2hr + 2hr^2 + 2hr^3 + \dots$

$d = h\left[1 + 2r + 2r^2 + 2r^3 + \dots\right]$

$d = h\left[1 + 2\underbrace{(r + r^2 + r^3 + \dots)}\right]$

This is a G.P
with $a = r$
$r = r$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{r}{1-r}$

Thus

$\Rightarrow d = h\left[1 + 2\left(\frac{r}{1-r}\right)\right]$

$\Rightarrow d = h\left(1 + \frac{2r}{1-r}\right)$

$\Rightarrow \frac{d}{h} = 1 + \frac{2r}{1-r}$

$\Rightarrow \frac{d}{h} - 1 = \frac{2r}{1-r}$

$\Rightarrow \frac{d-h}{h} = \frac{2r}{1-r}$

$\Rightarrow (d-h)(1-r) = 2rh$

$\Rightarrow (d-h) - r(d-h) = 2rh$

$\Rightarrow d-h = 2rh + r(d-h)$

$\Rightarrow d-h = r[2h + (d-h)]$

$\Rightarrow d-h = r[h+d]$

$\Rightarrow \frac{d-h}{h+d} = r$

$\Rightarrow r = \frac{d-h}{d+h}$ //

AS REQUIRED

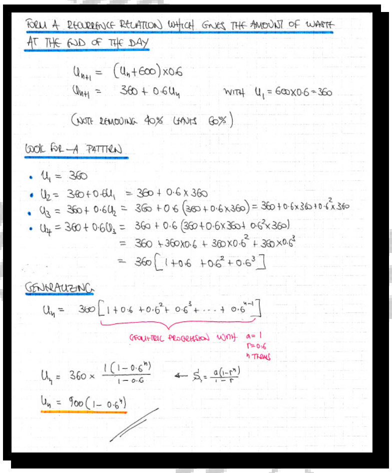
Question 19 (*****)

A factory gets permission to dispose, at the start of every day, 600 kg of waste into a stream of water.

The running stream removes 40% of the any waste present, by the end of the day.

Determine a simplified expression for the amount of waste present in the stream at the end of the n^{th} day.

$$\boxed{u_n = 900\left[1-\left(\frac{3}{5}\right)^n\right]}$$



FORM A RECURRENCE RELATION WHICH GIVES THE AMOUNT OF WASTE AT THE END OF THE DAY

$$u_{n+1} = (u_n+600)\times 0.6$$
$$u_{n+1} = 360 + 0.6u_n \qquad \text{WITH } u_1 = 600\times 0.6 = 360$$

(NOTE REMOVING 40% LEAVES 60%)

LOOK FOR A PATTERN

- $u_1 = 360$
- $u_2 = 360+0.6u_1 = 360 + 0.6\times 360$
- $u_3 = 360+0.6u_2 = 360+0.6(360+0.6\times360) = 360+0.6\times360+0.6^2\times360$
- $u_4 = 360+0.6u_3 = 360+0.6(360+0.6\times360+0.6^2\times360)$

$$= 360+360\times0.6+360\times0.6^2+360\times0.6^3$$
$$= 360\left[1+0.6+0.6^2+0.6^3\right]$$

GENERALIZING

$$u_n = 360\underbrace{\left[1+0.6+0.6^2+0.6^3+\dots+0.6^{n-1}\right]}_{\text{GEOMETRIC PROGRESSION WITH } a=1,\ r=0.6,\ n \text{ TERMS}}$$

$$u_n = 360\times\frac{1(1-0.6^n)}{1-0.6} \quad \leftarrow S_n = \frac{a(1-r^n)}{1-r}$$

$$u_n = 900(1-0.6^n)$$