INTEGRATION BY PARTS

Question 1

Carry out the following integrations:

1. $\int x e^{2x} \, dx = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$

2. $\int 3x \cos 2x \, dx = \frac{3}{2} x \sin 2x + \frac{3}{4} \cos 2x + C$

3. $\int x \sin 4x \, dx = -\frac{1}{4} x \cos 4x + \frac{1}{16} \sin 4x + C$

4. $\int -2x \sin 5x \, dx = \frac{2}{5} x \cos 5x - \frac{2}{25} \sin 5x + C$

5. $\int (1-2x) e^{-x} \, dx = (2x-1) e^{-x} + 2 e^{-x} + C$

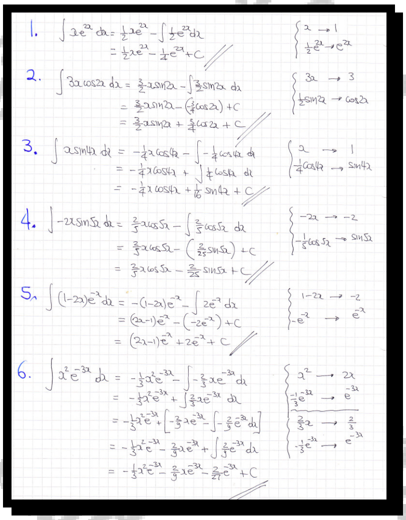
6. $\int x^2 e^{-3x} \, dx = -\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} - \frac{2}{27} e^{-3x} + C$

7. $\int 16x^3 \ln x \, dx = 4x^4 \ln|x| - x^4 + C$

8. $\int \ln x \, dx = x \ln x - x + C$

9. $\int x \cos\left(\frac{1}{2}x\right) dx = 2x \sin\left(\frac{1}{2}x\right) + 4\cos\left(\frac{1}{2}x\right) + C$

10. $\int (3x-1)\sin(3x-1) \, dx = -\frac{1}{3}(3x-1)\cos(3x-1) + \frac{1}{3}\sin(3x-1) + C$



1. $\int x e^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$
$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$
$\begin{cases} x \to 1 \\ \frac{1}{2}e^{2x} \to e^{2x} \end{cases}$

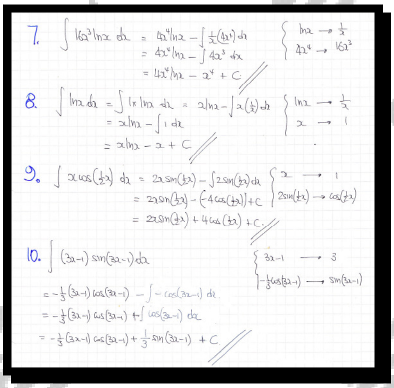
2. $\int 3x\cos 2x\,dx = \frac{3}{2}x\sin 2x - \int \frac{3}{2}\sin 2x\,dx$
$= \frac{3}{2}x\sin 2x - \left(-\frac{3}{4}\cos 2x\right) + C$
$= \frac{3}{2}x\sin 2x + \frac{3}{4}\cos 2x + C$
$\begin{cases} 3x \to 3 \\ \frac{1}{2}\sin 2x \to \cos 2x \end{cases}$

3. $\int x\sin 4x\,dx = -\frac{1}{4}x\cos 4x - \int -\frac{1}{4}\cos 4x\,dx$
$= -\frac{1}{4}x\cos 4x + \int \frac{1}{4}\cos 4x\,dx$
$= -\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C$
$\begin{cases} x \to 1 \\ -\frac{1}{4}\cos 4x \to \sin 4x \end{cases}$

4. $\int -2x\sin 5x\,dx = \frac{2}{5}x\cos 5x - \int \frac{2}{5}\cos 5x\,dx$
$= \frac{2}{5}x\cos 5x - \left(\frac{2}{25}\sin 5x\right) + C$
$= \frac{2}{5}x\cos 5x - \frac{2}{25}\sin 5x + C$
$\begin{cases} -2x \to -2 \\ -\frac{1}{5}\cos 5x \to \sin 5x \end{cases}$

5. $\int (1-2x)e^{-x}\,dx = -(1-2x)e^{-x} - \int 2e^{-x}\,dx$
$= (2x-1)e^{-x} - \left(-2e^{-x}\right) + C$
$= (2x-1)e^{-x} + 2e^{-x} + C$
$\begin{cases} 1-2x \to -2 \\ -e^{-x} \to e^{-x} \end{cases}$

6. $\int x^2 e^{-3x}\,dx = -\frac{1}{3}x^2e^{-3x} - \int -\frac{2}{3}xe^{-3x}\,dx$
$= -\frac{1}{3}x^2e^{-3x} + \int \frac{2}{3}xe^{-3x}\,dx$
$= -\frac{1}{3}x^2e^{-3x} + \left[-\frac{2}{9}xe^{-3x} - \int -\frac{2}{9}e^{-3x}\,dx\right]$
$= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} + \int \frac{2}{9}e^{-3x}\,dx$
$= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C$
$\begin{cases} x^2 \to 2x \\ -\frac{1}{3}e^{-3x} \to e^{-3x} \end{cases}$
$\begin{cases} \frac{2}{3}x \to \frac{2}{3} \\ -\frac{1}{3}e^{-3x} \to e^{-3x} \end{cases}$



7. $\int 16x^3\ln x\,dx = 4x^4\ln x - \int \frac{1}{x}(4x^4)\,dx$
$= 4x^4\ln x - \int 4x^3\,dx$
$= 4x^4\ln x - x^4 + C$
$\begin{cases} \ln x \to \frac{1}{x} \\ 4x^4 \to 16x^3 \end{cases}$

8. $\int \ln x\,dx = \int 1\times\ln x\,dx = x\ln x - \int x\left(\frac{1}{x}\right)dx$
$= x\ln x - \int 1\,dx$
$= x\ln x - x + C$
$\begin{cases} \ln x \to \frac{1}{x} \\ x \to 1 \end{cases}$

9. $\int x\cos\left(\frac{1}{2}x\right)dx = 2x\sin\left(\frac{1}{2}x\right) - \int 2\sin\left(\frac{1}{2}x\right)dx$
$= 2x\sin\left(\frac{1}{2}x\right) - \left(-4\cos\left(\frac{1}{2}x\right)\right) + C$
$= 2x\sin\left(\frac{1}{2}x\right) + 4\cos\left(\frac{1}{2}x\right) + C$
$\begin{cases} x \to 1 \\ 2\sin\left(\frac{1}{2}x\right) \to \cos\left(\frac{1}{2}x\right) \end{cases}$

10. $\int (3x-1)\sin(3x-1)\,dx$
$\begin{cases} 3x-1 \to 3 \\ -\frac{1}{3}\cos(3x-1) \to \sin(3x-1) \end{cases}$
$= -\frac{1}{3}(3x-1)\cos(3x-1) - \int -\cos(3x-1)\,dx$
$= -\frac{1}{3}(3x-1)\cos(3x-1) + \int \cos(3x-1)\,dx$
$= -\frac{1}{3}(3x-1)\cos(3x-1) + \frac{1}{3}\sin(3x-1) + C$

Question 2

Carry out the following integrations:

1. $\int 6x\mathrm{e}^{3x}\,dx = 2x\mathrm{e}^{3x} - \frac{2}{3}\mathrm{e}^{3x} + C$

2. $\int 12x\cos 3x\,dx = 4x\sin 3x + \frac{4}{3}\cos 3x + C$

3. $\int x\sin 6x\,dx = -\frac{1}{6}x\cos 6x + \frac{1}{36}\sin 6x + C$

4. $\int -x\sin 2x\,dx = \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + C$

5. $\int (2-x)\mathrm{e}^{-3x}\,dx = -\frac{1}{3}(2-x)\mathrm{e}^{-3x} + \frac{1}{9}\mathrm{e}^{-3x} + C$

6. $\int x^2\mathrm{e}^{4x}\,dx = \frac{1}{4}x^2\mathrm{e}^{4x} - \frac{1}{8}x\mathrm{e}^{4x} + \frac{1}{32}\mathrm{e}^{4x} + C$

7. $\int x^2\mathrm{e}^{-\frac{1}{2}x}\,dx = -2x^2\mathrm{e}^{-\frac{1}{2}x} - 8x\mathrm{e}^{-\frac{1}{2}x} - 16\mathrm{e}^{-\frac{1}{2}x} + C$

8. $\int 25x^4\ln x\,dx = 5x^5\ln x - x^5 + C$

9. $\int 24x\cos\left(\frac{2}{3}x\right)dx = 36x\sin\left(\frac{2}{3}x\right) + 54\cos\left(\frac{2}{3}x\right) + C$

10. $\int x^2\sin(1-x)\,dx = x^2\cos(1-x) + 2x\sin(1-x) - 2\cos(1-x) + C$

1. $\int 6xe^{3x}\,dx = 2xe^{3x} - \int 2e^{3x}\,dx = 2xe^{3x} - \frac{2}{3}e^{3x} + C$

$$\begin{array}{l} 6x \to 6 \\ \frac{1}{3}e^{3x} \to e^{3x} \end{array}$$

2. $\int 12x\cos 3x\,dx = 4x\sin 3x - \int 4\sin 3x\,dx$
$= 4x\sin 3x - \left(-\frac{4}{3}\cos 3x\right) + C$
$= 4x\sin 3x + \frac{4}{3}\cos 3x + C$

$$\begin{array}{l} 12x \to 12 \\ \frac{1}{3}\sin 3x \to \cos 3x \end{array}$$

3. $\int x\sin 6x\,dx = -\frac{1}{6}x\cos 6x - \int -\frac{1}{6}\cos 6x\,dx$
$= -\frac{1}{6}x\cos 6x + \int \frac{1}{6}\cos 6x\,dx$
$= -\frac{1}{6}x\cos 6x + \frac{1}{36}\sin 6x + C$

$$\begin{array}{l} x \to 1 \\ -\frac{1}{6}\cos 6x \to \sin 6x \end{array}$$

4. $\int -x\sin 2x\,dx = \frac{1}{2}x\cos 2x - \int \frac{1}{2}\cos 2x\,dx$
$= \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + C$

$$\begin{array}{l} -x \to -1 \\ -\frac{1}{2}\cos 2x \to \sin 2x \end{array}$$

5. $\int (2-x)e^{-3x}\,dx = -\frac{1}{3}(2-x)e^{-3x} - \int \frac{1}{3}e^{-3x}\,dx$
$= -\frac{1}{3}(2-x)e^{-3x} - \left(-\frac{1}{9}e^{-3x}\right) + C$
$= -\frac{1}{3}(2-x)e^{-3x} + \frac{1}{9}e^{-3x} + C$

$$\begin{array}{l} 2-x \to -1 \\ -\frac{1}{3}e^{-3x} \to e^{-3x} \end{array}$$

6. $\int x^2e^{4x}\,dx = \frac{1}{4}x^2e^{4x} - \int \frac{1}{2}xe^{4x}\,dx$ BY PARTS AGAIN
$= \frac{1}{4}x^2e^{4x} - \left[\frac{1}{8}xe^{4x} - \int \frac{1}{8}e^{4x}\,dx\right]$
$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \int \frac{1}{8}e^{4x}\,dx$
$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C$

$$\begin{array}{l} x^2 \to 2x \\ \frac{1}{4}e^{4x} \to e^{4x} \end{array}$$

$$\begin{array}{l} \frac{1}{2}x \to \frac{1}{2} \\ \frac{1}{4}e^{4x} \to e^{4x} \end{array}$$

7. $\int x^2e^{-\frac{1}{2}x}\,dx = -2x^2e^{-\frac{1}{2}x} - \int -4xe^{-\frac{1}{2}x}\,dx$
$= -2x^2e^{-\frac{1}{2}x} + \int 4xe^{-\frac{1}{2}x}\,dx$ BY PARTS AGAIN
$= -2x^2e^{-\frac{1}{2}x} + \left[-8xe^{-\frac{1}{2}x} - \int -8e^{-\frac{1}{2}x}\,dx\right]$
$= -2x^2e^{-\frac{1}{2}x} - 8xe^{-\frac{1}{2}x} + \int 8e^{-\frac{1}{2}x}\,dx$
$= -2x^2e^{-\frac{1}{2}x} - 8xe^{-\frac{1}{2}x} - 16e^{-\frac{1}{2}x} + C$

$$\begin{array}{l} x^2 \to 2x \\ -2e^{-\frac{1}{2}x} \to e^{-\frac{1}{2}x} \end{array}$$

$$\begin{array}{l} 4x \to 4 \\ -2e^{-\frac{1}{2}x} \to e^{-\frac{1}{2}x} \end{array}$$

8. $\int 25x^4\ln x\,dx = 5x^5\ln x - \int 5x^4\,dx$
$= 5x^5\ln x - x^5 + C$

$$\begin{array}{l} \ln x \to \frac{1}{x} \\ 5x^5 \to 25x^4 \end{array}$$

9. $\int 24x\cos\left(\frac{2}{3}x\right)dx = 36x\sin\left(\frac{2}{3}x\right) - \int 36\sin\left(\frac{2}{3}x\right)dx$
$= 36x\sin\left(\frac{2}{3}x\right) - \left(-54\cos\left(\frac{2}{3}x\right)\right) + C$
$= 36x\sin\left(\frac{2}{3}x\right) + 54\cos\left(\frac{2}{3}x\right) + C$

$$\begin{array}{l} 24x \to 24 \\ \frac{3}{2}\sin\left(\frac{2}{3}x\right) \to \cos\left(\frac{2}{3}x\right) \end{array}$$

10. $\int x^2\sin(1-x)\,dx = x^2\cos(1-x) - \int 2x\cos(1-x)\,dx$
$= x^2\cos(1-x) - \left[-2x\sin(1-x) - \int -2\sin(1-x)\,dx\right]$
$= x^2\cos(1-x) + 2x\sin(1-x) - \int 2\sin(1-x)\,dx$
$= x^2\cos(1-x) + 2x\sin(1-x) - \left[2\cos(1-x)\right]$
$= x^2\cos(1-x) + 2x\sin(1-x) - 2\cos(1-x) + C$

$$\begin{array}{l} x^2 \to 2x \\ \cos(1-x) \to \sin(1-x) \end{array}$$

BY PARTS AGAIN

$$\begin{array}{l} 2x \to 2 \\ -\sin(1-x) \to \cos(1-x) \end{array}$$

Question 3

Carry out the following integrations:

1. $\int \frac{1}{2} x \mathrm{e}^{4x} \, dx = \frac{1}{8} x \mathrm{e}^{4x} - \frac{1}{32} \mathrm{e}^{4x} + C$

2. $\int 5x \sin 4x \, dx = -\frac{5}{4} x \cos 4x + \frac{5}{16} \sin 4x + C$

3. $\int (2x+1) \cos 2x \, dx = \frac{1}{2}(2x+1) \sin 2x + \frac{1}{2} \cos 2x + C$

4. $\int -3x \cos 4x \, dx = -\frac{3}{4} x \sin 4x - \frac{3}{16} \cos 4x + C$

5. $\int x^2 \mathrm{e}^{-2x} \, dx = -\frac{1}{2} x^2 \mathrm{e}^{-2x} - \frac{1}{2} x \mathrm{e}^{-2x} - \frac{1}{4} \mathrm{e}^{-2x} + C$

6. $\int x^2 \sin 5x \, dx = -\frac{1}{5} x^2 \cos 5x + \frac{2}{25} x \sin 5x + \frac{2}{125} \cos 5x + C$

7. $\int x^2 \cos \frac{1}{3} x \, dx = 3x^2 \sin \frac{1}{3} x + 18x \cos \frac{1}{3} x - 54 \sin \frac{1}{3} x + C$

8. $\int \frac{1}{2} x^3 \ln x \, dx = \frac{1}{8} x^4 \ln x - \frac{1}{32} x^4 + C$

9. $\int x \ln 3x \, dx = \frac{1}{2} x^2 \ln 3x - \frac{1}{4} x^2 + C$

10. $\int \frac{\ln x}{x^3} \, dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$

1. $\int \frac{1}{2}xe^{4x}\,dx = \frac{1}{8}xe^{4x} - \int \frac{1}{8}e^{4x}\,dx$

$= \frac{1}{8}xe^{4x} - \frac{1}{32}e^{4x} + C$

$\frac{1}{2}x \to \frac{1}{2}$
$\frac{1}{4}e^{4x} \to e^{4x}$

2. $\int 5x\sin 4x\,dx = -\frac{5}{4}x\cos 4x - \int -\frac{5}{4}\cos 4x\,dx$

$= -\frac{5}{4}x\cos 4x + \int \frac{5}{4}\cos 4x\,dx$

$= -\frac{5}{4}x\cos 4x + \frac{5}{16}\sin 4x + C.$

$5x \quad 5$
$-\frac{1}{4}\cos 4x \quad \sin 4x$

3. $\int (2x+1)\cos 2x\,dx = \frac{1}{2}(2x+1)\sin 2x - \int \sin 2x\,dx$

$= \frac{1}{2}(2x+1)\sin 2x + \frac{1}{2}\cos 2x + C$

$2x+1 \to 2$
$\frac{1}{2}\sin 2x \to \cos 2x$

4. $\int -3x\cos 4x\,dx = -\frac{3}{4}x\sin 4x - \int -\frac{3}{4}\sin 4x\,dx$

$= -\frac{3}{4}x\sin 4x + \int \frac{3}{4}\sin 4x$

$= -\frac{3}{4}x\sin 4x - \frac{3}{16}\cos 4x + C$

$-3x \to -3$
$\frac{1}{4}\sin 4x \quad \cos 4x$

5. $\int x^2 e^{-2x}\,dx = -\frac{1}{2}x^2e^{-2x} - \int -xe^{-2x}\,dx$

$= -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}\,dx$

BY PARTS AGAIN

$= -\frac{1}{2}x^2e^{-2x} + \left[-\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx\right]$

$= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\,dx$

$= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

$x^2 \quad 2x$
$-\frac{1}{2}e^{-2x} \quad e^{-2x}$

$x \quad 1$
$-\frac{1}{2}e^{-2x} \quad e^{-2x}$

6. $\int x^2\sin 5x\,dx = -\frac{1}{5}x^2\cos 5x - \int -\frac{2}{5}x\cos 5x\,dx$

$= -\frac{1}{5}x^2\cos 5x + \int \frac{2}{5}x\cos 5x\,dx$

BY PARTS AGAIN

$= -\frac{1}{5}x^2\cos 5x + \left[\frac{2}{25}\sin 5x - \int \frac{2}{25}\sin 5x\,dx\right]$

$= -\frac{1}{5}x^2\cos 5x + \frac{2}{25}\sin 5x + \frac{2}{125}\cos 5x + C$

$x^2 \to 2x$
$-\frac{1}{5}\cos 5x \to \sin 5x$

$\frac{2}{5}x \to \frac{2}{5}$
$\frac{1}{5}\sin 5x \to \cos 5x$

7. $\int x^2\cos\frac{1}{3}x\,dx = 3x^2\sin\frac{1}{3}x - \int 6x\sin\frac{1}{3}x\,dx$

$= 3x^2\sin\frac{1}{3}x - \left[-18x\cos\frac{1}{3}x - \int -18\cos\frac{1}{3}x\,dx\right]$

$= 3x^2\sin\frac{1}{3}x + 18x\cos\frac{1}{3}x - \int 18\cos\frac{1}{3}x\,dx$

PARTS AGAIN

$= 3x^2\sin\frac{1}{3}x + 18x\cos\frac{1}{3}x - 54\sin\frac{1}{3}x + C$

$x^2 \quad 2x$
$3\sin\frac{1}{3}x \quad \cos\frac{1}{3}x$

$6x \to 6$
$-3\cos\frac{1}{3}x \to \sin\frac{1}{3}x$

8. $\int \frac{1}{2}x^3\ln x\,dx = \frac{1}{8}x^4\ln x - \int \frac{1}{8}x^4\left(\frac{1}{x}\right)dx$

$= \frac{1}{8}x^4\ln x - \int \frac{1}{8}x^3\,dx$

$= \frac{1}{8}x^4\ln x - \frac{1}{32}x^4 + C$

$\ln x \to \frac{1}{x}$
$\frac{1}{8}x^4 \to \frac{1}{2}x^3$

9. $\int x\ln 3x\,dx = \frac{1}{2}x^2\ln 3x - \int \frac{1}{2}x^2\left(\frac{1}{x}\right)dx$

$= \frac{1}{2}x^2\ln 3x - \int \frac{1}{2}x\,dx$

$= \frac{1}{2}x^2\ln 3x - \frac{1}{4}x^2 + C$

$\ln 3x \to \frac{1}{x}$
$\frac{1}{2}x^2 \quad x$

10 $\int \frac{\ln x}{x^3}\,dx = \int x^{-3}\ln x\,dx$

$= -\frac{1}{2}x^{-2}\ln x - \int -\frac{1}{2}x^{-2}\left(\frac{1}{x}\right)dx$

$= -\frac{\ln x}{2x^2} + \int \frac{1}{2}x^{-3}\,dx$

$= -\frac{\ln x}{2x^2} - \frac{1}{4}x^{-2} + C$

$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$

$\ln x \to \frac{1}{x}$
$-\frac{1}{2}x^{-2} \quad x^{-3}$

Question 4

Carry out the following integrations:

1. $\displaystyle\int x e^{5x} \, dx = \frac{1}{5} x e^{5x} - \frac{1}{25} e^{5x} + C$

2. $\displaystyle\int 2x \cos 3x \, dx = \frac{2}{3} x \sin 3x + \frac{2}{9} \cos 3x + C$

3. $\displaystyle\int x \sin 3x \, dx = -\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x + C$

4. $\displaystyle\int x \sin 4x \, dx = \frac{1}{16} \sin 4x - \frac{1}{4} x \cos 4x + C$

5. $\displaystyle\int 2 \ln x \, dx = 2x \ln x - 2x + C$

6. $\displaystyle\int x^2 \ln x \, dx = \frac{1}{3} x^3 \ln|x| - \frac{1}{9} x^3 + C$

7. $\displaystyle\int x \sin\left(\tfrac{1}{2} x\right) dx = 4 \sin\left(\tfrac{1}{2} x\right) - 2x \cos\left(\tfrac{1}{2} x\right) + C$

8. $\displaystyle\int x \sin(2x-1) \, dx = -\frac{1}{2} x \cos(2x-1) + \frac{1}{4} \sin(2x-1) + C$

9. $\displaystyle\int \frac{\ln x}{x^2} \, dx = -\frac{\ln x}{x} - \frac{1}{x} + C$

10. $\displaystyle\int x \sec^2 x \, dx = x \tan x - \ln|\sec x| + C$

1. $$\int xe^{5x}\,dx = \frac{1}{5}xe^{5x} - \int \frac{1}{5}e^{5x}\,dx = \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C$$
   $x \longrightarrow 1$, $\quad \frac{1}{5}e^{5x} \longrightarrow e^{5x}$

2. $$\int 2x\cos 3x\,dx = \frac{2}{3}x\sin x - \int \frac{2}{3}\sin 3x\,dx = \frac{2}{3}x\sin x + \frac{2}{9}\cos 3x + C$$
   $2x \longrightarrow 2$, $\quad \frac{1}{3}\sin 3x \longrightarrow \cos 3x$

3. $$\int x\sin 3x\,dx = -\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x\,dx = -\frac{1}{3}x\cos 3x + \int \frac{1}{3}\cos 3x\,dx = -\frac{1}{3}x\cos 3x + \sin 3x + C$$
   $x \longrightarrow 1$, $\quad -\frac{1}{3}\cos 3x \longrightarrow \sin 3x$

4. $$\int x\sin 4x\,dx = -\frac{1}{4}x\cos 4x - \int -\frac{1}{4}\cos 4x\,dx = -\frac{1}{4}x\cos 4x + \int \frac{1}{4}\cos 4x\,dx = -\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C$$
   $x \longmapsto 1$, $\quad -\frac{1}{4}\cos 4x \longrightarrow \sin 4x$

5. $$\int 2\ln x\,dx = 2x\ln x - \int (2x)\frac{1}{x}\,dx = 2x\ln x - \int 2\,dx = 2x\ln x - 2x + C$$
   $\ln x \longrightarrow \frac{1}{x}$, $\quad 2x \longrightarrow 2$

6. $$\int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \int \left(\frac{1}{3}x^3\right)\frac{1}{x}\,dx = \frac{1}{3}x^3\ln x - \int \frac{1}{3}x^2\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$$
   $\ln x \longrightarrow \frac{1}{x}$, $\quad \frac{1}{3}x^3 \longrightarrow x^2$

7. $$\int x\sin\left(\tfrac{1}{2}x\right)dx = -2x\cos\left(\tfrac{1}{2}x\right) - \int -2\cos\left(\tfrac{1}{2}x\right)dx = -2x\cos\left(\tfrac{1}{2}x\right) + \int 2\cos\left(\tfrac{1}{2}x\right)dx = -2x\cos\left(\tfrac{1}{2}x\right) + 4\sin\left(\tfrac{1}{2}x\right) + C$$
   $x \longrightarrow 1$, $\quad -2\cos\left(\frac{1}{2}x\right) \longrightarrow \sin\left(\frac{1}{2}x\right)$

8. $$\int x\sin(2x-1)\,dx = -\frac{1}{2}x\cos(2x-1) - \int -\frac{1}{2}\cos(2x-1)\,dx = -\frac{1}{2}x\cos(2x-1) + \int \frac{1}{2}\cos(2x-1)\,dx = -\frac{1}{2}x\cos(2x-1) + \frac{1}{4}\sin(2x-1) + C$$
   $x \longrightarrow 1$, $\quad -\frac{1}{2}\cos(2x-1) \longrightarrow \sin(2x-1)$

9. $$\int \frac{\ln x}{x^2}\,dx = \int (\ln x)\times(x^{-2})\,dx = -x^{-1}\ln x - \int -x^{-1}\times\frac{1}{x}\,dx = -\frac{\ln x}{x} + \int x^{-2}\,dx = -\frac{\ln x}{x} - x^{-1} + C = -\frac{\ln x}{x} - \frac{1}{x} + C$$
   $\ln x \longrightarrow \frac{1}{x}$, $\quad -x^{-1} \longrightarrow x^{-2}$

10. $$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x - \ln|\sec x| + C$$
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    $x \longrightarrow 1$, $\quad \tan x \longrightarrow \sec^2 x$

Question 5

Carry out the following integrations:

1. $\displaystyle\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$

2. $\displaystyle\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$

3. $\displaystyle\int \sin x \ln(\sec x) \, dx = -\cos x\left(1 + \ln|\sec x|\right) + C$

4. $\displaystyle\int x \cos 5x \, dx = \frac{1}{5}x \sin 5x + \frac{1}{25}\cos 5x + C$

5. $\displaystyle\int x^2 \sin 3x \, dx = -\frac{1}{3}x^2 \cos 3x + \frac{2}{9}x \sin 3x + \frac{2}{27}\cos 3x + C$

6. $\displaystyle\int 4x e^{-\frac{2}{3}x} \, dx = -3(2x+3)e^{-\frac{2}{3}x} + C$

7. $\displaystyle\int x^2 \cos\left(\frac{1}{3}x\right) dx = 3x^2 \sin\left(\frac{1}{3}x\right) + 18x \cos\left(\frac{1}{3}x\right) - 54 \sin\left(\frac{1}{3}x\right) + C$

8. $\displaystyle\int 2x^2 \sec^2 x \tan x \, dx = x^2 \sec^2 x - 2x \tan x + 2\ln|\sec x| + C$

9. $\displaystyle\int x^2 e^{\frac{1}{2}x} \, dx = \left(2x^2 - 8x + 16\right)e^{\frac{1}{2}x} + C$

10. $\displaystyle\int x \sec x \tan x \, dx = x \sec x - \ln|\sec x + \tan x| + C$

Question 6

Carry out the following integrations:

1. $\displaystyle\int x^2 \, e^{-\frac{1}{4}x} \; dx = -4e^{-\frac{1}{4}x}\left(x^2+8x+32\right)+C$

2. $\displaystyle\int x^2 \, e^{-x} \; dx = -e^{-x}\left(x^2+2x+2\right)+C$

3. $\displaystyle\int e^x \cos x \; dx = \frac{1}{2}e^x\left(\cos x+\sin x\right)+C$

4. $\displaystyle\int (\ln x)^2 \; dx = x(\ln x)^2 - 2x\ln x + 2x + C$

5. $\displaystyle\int e^x \sin x \; dx = \frac{1}{2}e^x\left(\sin x-\cos x\right)+C$

6. $\displaystyle\int \left(x^3+5x^2-2\right)e^{2x} \; dx = \frac{1}{8}e^{2x}\left(4x^3+14x^2-14x-1\right)+C$

7. $\displaystyle\int x\cos^2 x \; dx = \frac{1}{8}x^2+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+C$

8. $\displaystyle\int x\ln 2x^3 \; dx = \frac{3}{4}x^2\ln 2\left(2\ln x-1\right)+C$

Question 7

Carry out the following integrations, to the answer given:

1. $\int_0^{\ln 2} x e^{2x} \, dx = \ln 4 - \frac{3}{4}$

2. $\int_0^{\frac{\pi}{3}} 6x \sin 3x \, dx = \frac{2\pi}{3}$

3. $\int_0^{\frac{\pi}{2}} x^2 \cos x \, dx = \frac{1}{4}\left(\pi^2 - 8\right)$

4. $\int_1^{e} x \ln x \, dx = \frac{1}{4}\left(e^2 + 1\right)$

5. $\int_0^{1} 4x e^{3x} \, dx = \frac{4}{9}\left(2e^3 + 1\right)$

6. $\int_0^{\frac{\pi}{4}} x \sin 4x \, dx = \frac{\pi}{16}$

7. $\int_1^{2} x^3 \ln x \, dx = 4\ln 2 - \frac{15}{16}$

8. $\int_0^{1} x e^{-2x} \, dx = \frac{1}{4}\left(1 - 3e^{-2}\right)$

9. $\int_0^{\frac{\pi}{4}} 12x \cos 2x \, dx = \frac{3}{2}\left(\pi - 2\right)$

10. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4x \sin 2x \, dx = \pi - 1$

1. $\displaystyle\int_0^{\ln 2} xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$

$= \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^{\ln 2}$

$= \left[\left(\frac{1}{2}\ln 2\right)e^{2\ln 2} - \frac{1}{4}e^{2\ln 2}\right] - \left[0 - \frac{1}{4}\right]$

$= 2\ln 2 - 1 + \frac{1}{4} = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$

$\left\{\begin{array}{l} x \to 1 \\ \frac{1}{2}e^{2x} \to e^{2x} \end{array}\right.$

2. $\displaystyle\int_0^{\frac{\pi}{3}} 6x\sin 3x\,dx = -2x\cos 3x + \int 2\cos 3x\,dx$

$= \left[-2x\cos 3x - \frac{2}{3}\sin 3x\right]_0^{\pi/3}$

$= \left[-\frac{2\pi}{3}\cos\pi - \cancel{\frac{2}{3}\sin\pi}\right] - \left[0 - \cancel{\frac{2}{3}\sin 0}\right]$

$= \left(-\frac{2\pi}{3}\right)(-1) = \frac{2\pi}{3}$

$\left\{\begin{array}{l} 6x \to 6 \\ -\frac{1}{3}\cos 3x \to \sin 3x \end{array}\right.$

3. $\displaystyle\int_0^{\frac{\pi}{2}} x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx$

$= x^2\sin x - \left[-2x\cos x - \int -2\cos x\,dx\right]$

$= x^2\sin x + 2x\cos x - \int 2\cos x\,dx$

$= \left[x^2\sin x + 2x\cos x - 2\sin x\right]_0^{\pi/2}$

$= \left(\frac{\pi^2}{4}\sin\frac{\pi}{2} + \cancel{\pi\cos\frac{\pi}{2}} - 2\sin\frac{\pi}{2}\right) - \left(\cancel{0 - 2\sin 0}\right) = \frac{\pi^2}{4} - 2 = \frac{1}{4}(\pi^2 - 8)$

$\left\{\begin{array}{l} x^2 \to 2x \\ \sin x \to \cos x \\ 2x \to 2 \\ -\cos x \quad \sin x \end{array}\right.$

4. $\displaystyle\int_1^{e} x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx$

$= \left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_1^{e}$

$= \left(\frac{1}{2}e^2\ln e - \frac{1}{4}e^2\right) - \left(\cancel{\frac{1}{2}\ln 1} - \frac{1}{4}\right) = \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}$

$= \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}(e^2 + 1)$

$\left\{\begin{array}{l} \ln x \to \frac{1}{x} \\ \frac{1}{2}x^2 \to x \end{array}\right.$

5. $\displaystyle\int_0^{1} 4xe^{3x}\,dx = \frac{4}{3}xe^{3x} - \int \frac{4}{3}e^{3x}\,dx = \left[\frac{4}{3}xe^{3x} - \frac{4}{9}e^{3x}\right]_0^1$

$= \left(\frac{4}{3}e^3 - \frac{4}{9}e^3\right) - \left(0 - \frac{4}{9}\right)$

$= \frac{8}{9}e^3 + \frac{4}{9} = \frac{4}{9}(2e^3 + 1)$

$\left\{\begin{array}{l} 4x \to 4 \\ \frac{1}{3}e^{3x} \to e^{3x} \end{array}\right.$

6. $\displaystyle\int_0^{\frac{\pi}{4}} x\sin 4x\,dx = -\frac{1}{4}x\cos 4x - \int -\frac{1}{4}\cos 4x\,dx$

$= -\frac{1}{4}x\cos 4x + \int \frac{1}{4}\cos 4x\,dx$

$= \left[-\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x\right]_0^{\frac{\pi}{4}}$

$= \left(-\frac{1}{4}\frac{\pi}{4}\cos\pi + \cancel{\frac{1}{16}\sin\pi}\right) - \left(0 + \cancel{\frac{1}{16}\sin 0}\right) = \frac{\pi}{16}$

$\left\{\begin{array}{l} x \to 1 \\ -\frac{1}{4}\cos 4x \to \sin 4x \end{array}\right.$

7. $\displaystyle\int_1^{2} x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \int \frac{1}{4}x^3\,dx = \left[\frac{1}{4}x^4\ln x - \frac{1}{16}x^4\right]_1^2$

$= (4\ln 2 - 1) - \left(0 - \frac{1}{16}\right) = 4\ln 2 - \frac{15}{16}$

$\left\{\begin{array}{l} \ln x \to \frac{1}{x} \\ \frac{1}{4}x^4 \to x^3 \end{array}\right.$

8. $\displaystyle\int_0^{1} xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$

$= -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\,dx = \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^1$

$= \left[\frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x}\right]_1^0 = \left(0 + \frac{1}{4}\right) - \left(\frac{1}{2}e^{-2} + \frac{1}{4}e^{-2}\right)$

$= \frac{1}{4} - \frac{3}{4}e^{-2} = \frac{1}{4}(1 - 3e^{-2})$

$\left\{\begin{array}{l} x \to 1 \\ -\frac{1}{2}e^{-2x} \to e^{-2x} \end{array}\right.$

9. $\displaystyle\int_0^{\frac{\pi}{4}} 12x\cos 2x\,dx = 6x\sin 2x - \int 6\sin 2x\,dx$

$= \left[6x\sin 2x + 3\cos 2x\right]_0^{\frac{\pi}{4}}$

$= \left[6\left(\frac{\pi}{4}\right)\sin\frac{\pi}{2} + \cancel{3\cos\frac{\pi}{2}}\right] - \left[0 + 3\cos 0\right]$

$= \frac{3\pi}{2} - 3 = \frac{3}{2}(\pi - 2)$

$\left\{\begin{array}{l} 12x \to 12 \\ \frac{1}{2}\sin 2x \to \cos 2x \end{array}\right.$

10. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4x\sin 2x\,dx = -2x\cos 2x - \int -2\cos 2x\,dx$

$= -2x\cos 2x + \int 2\cos 2x\,dx$

$= \left[-2x\cos 2x + \sin 2x\right]_{\pi/4}^{\pi/2}$

$= \left[-2\left(\frac{\pi}{2}\right)\cos\pi + \cancel{\sin\pi}\right] - \left[-2\left(\frac{\pi}{4}\right)\cancel{\cos\frac{\pi}{2}} + \sin\frac{\pi}{2}\right]$

$= -2\left(\frac{\pi}{2}\right)(-1) - 1$

$= \pi - 1$

$\left\{\begin{array}{l} 4x \to 4 \\ -\frac{1}{2}\cos 2x \to \sin 2x \end{array}\right.$

Question 8

Carry out the following integrations, to the answer given:

1. $\displaystyle\int_0^{\frac{\pi}{3}} x\sin 3x \ \ dx = \frac{\pi}{9}$

2. $\displaystyle\int_0^{\frac{\pi}{4}} 2x\cos 4x \ \ dx = -\frac{1}{4}$

3. $\displaystyle\int_0^{\ln 2} 4x\mathrm{e}^{-x} \ \ dx = 2 - \ln 4$

4. $\displaystyle\int_1^{\mathrm{e}} \ln x \ \ dx = 1$

5. $\displaystyle\int_0^{\frac{\pi}{2}} x\sin 2x \ \ dx = \frac{\pi}{4}$

6. $\displaystyle\int_0^{\ln 4} x\mathrm{e}^{\frac{1}{2}x} \ \ dx = 8\ln 2 - 4$

7. $\displaystyle\int_0^{\pi} x\cos\left(\frac{1}{4}x\right) \ \ dx = 2\sqrt{2}(\pi + 4) - 16$

8. $\displaystyle\int_0^{1} (2x+1)\mathrm{e}^{2x} \ \ dx = \mathrm{e}^2$

9. $\displaystyle\int_{\frac{1}{\mathrm{e}}}^{1} x\ln x \ \ dx = \frac{1}{4}\left(\frac{3}{\mathrm{e}^2} - 1\right)$

10. $\displaystyle\int_{-1}^{0} 3\ln(2x+3) \ \ dx = \tfrac{3}{2}(\ln 27 - 2)$ REQUIRES ADDITIONAL TECHNIQUES

1. $\displaystyle\int_0^{\frac{\pi}{3}} x\sin 3x = \text{IGNORE LIMITS} = -\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x \, dx$ $\quad \begin{array}{l} x \to 1 \\ -\frac{1}{3}\cos 3x \to \sin 3x \end{array}$

$= -\frac{1}{3}x\cos 3x + \int \frac{1}{3}\cos 3x \, dx$

$= \left[-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\right]_0^{\frac{\pi}{3}} = \left(-\frac{1}{3}\frac{\pi}{3}\cos\pi + \cancel{\frac{1}{9}\sin\pi}\right) - \left(0 + \cancel{\frac{1}{9}\sin 0}\right)$

$= -\frac{1}{3}\frac{\pi}{3}(-1) = \frac{\pi}{9}$

2. $\displaystyle\int_0^{\frac{\pi}{4}} 2x\cos 4x \, dx = \dots\text{IGNORE LIMITS}\dots = \frac{1}{2}x\sin 4x - \int \frac{1}{2}\sin 4x \, dx$ $\quad \begin{array}{l} 2x \to 2 \\ \frac{1}{4}\sin 4x \quad \cos 4x \end{array}$

$= \left[\frac{1}{2}x\sin 4x + \frac{1}{8}\cos 4x\right]_0^{\frac{\pi}{4}}$

$= \left(\cancel{\frac{1}{2}\frac{\pi}{4}\sin\pi} + \frac{1}{8}\cos\pi\right) - \left(0 + \frac{1}{8}\cos 0\right) = -\frac{1}{8} - \left(\frac{1}{8}\right) = -\frac{1}{4}$

3. $\displaystyle\int_0^{\ln 2} 4xe^{-x} \, dx = \dots\text{IGNORE LIMITS}\dots = -4xe^{-x} - \int -4e^{-x} \, dx$ $\quad \begin{array}{l} 4x \to 4 \\ -e^{-x} \to e^{-x} \end{array}$

$= -4xe^{-x} + \int 4e^{-x} \, dx = \left[-4xe^{-x} - 4e^{-x}\right]_0^{\ln 2}$

$= \left[4xe^{-x} + 4e^{-x}\right]_{\ln 2}^{0} = (0+4) - \left(4\ln 2 \times e^{-\ln 2} + 4e^{-\ln 2}\right)$

$= 4 - (2\ln 2 + 2) = 2 - 2\ln 2$ or $2 - \ln 4$

4. $\displaystyle\int_1^e \ln x \, dx = \int_1^e 1 \times \ln x \, dx = \dots\text{IGNORE LIMITS}\dots$ $\quad \begin{array}{l} \ln x \to \frac{1}{x} \\ x \to 1 \end{array}$

$= x\ln x - \int 1 \, dx = \left[x\ln x - x\right]_1^e$

$= (e\ln e - e) - (0 - 1) = \cancel{(e - e)} - (-1) = 1$

5. $\displaystyle\int_0^{\frac{\pi}{2}} x\sin 2x \, dx = \dots\text{IGNORE LIMITS}\dots$ $\quad \begin{array}{l} x \to 1 \\ -\frac{1}{2}\cos 2x \to \sin 2x \end{array}$

$= -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x \, dx$

$= -\frac{1}{2}x\cos 2x + \int \frac{1}{2}\cos 2x \, dx$

$= \left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \left(-\frac{1}{2}\times\frac{\pi}{2}\cos\pi + \cancel{\frac{1}{4}\sin\pi}\right) - \left(0 + \cancel{\frac{1}{4}\sin 0}\right)$

$= -\frac{1}{2}\times\frac{\pi}{2}\times(-1) = \frac{\pi}{4}$

6. $\displaystyle\int_0^{\ln 4} xe^{\frac{1}{2}x} \, dx = \dots\text{IGNORE LIMITS}\dots = 2xe^{\frac{1}{2}x} - \int 2e^{\frac{1}{2}x} \, dx$ $\quad \begin{array}{l} x \to 1 \\ 2e^{\frac{1}{2}x} \quad e^{\frac{1}{2}x} \end{array}$

$= \left[2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x}\right]_0^{\ln 4}$

$= \left(2\ln 4 \times e^{\frac{1}{2}\ln 4} - 4e^{\frac{1}{2}\ln 4}\right) - (0 - 4) = 4\ln 4 - 8 + 4$

$= 4\ln 4 - 4 = 8\ln 2 - 4$

7. $\displaystyle\int_0^{\pi} x\cos\left(\tfrac{1}{4}x\right) dx = \dots\text{IGNORE LIMITS}\dots = 4x\sin\left(\tfrac{1}{4}x\right) - \int 4\sin\left(\tfrac{1}{4}x\right) dx$ $\quad \begin{array}{l} x \to 1 \\ 4\sin\left(\frac{1}{4}x\right) \to \cos\left(\frac{1}{4}x\right) \end{array}$

$= \left[4x\sin\left(\tfrac{1}{4}x\right) + 16\cos\left(\tfrac{1}{4}x\right)\right]_0^{\pi}$

$= \left(4\pi\sin\frac{\pi}{4} + 16\cos\frac{\pi}{4}\right) - (0 + 16\cos 0) = 2\sqrt{2}\pi + 8\sqrt{2} - 16$

$= 2\sqrt{2}(\pi + 4) - 16$

8. $\displaystyle\int_0^1 (2x+1)e^{2x} \, dx = \dots\text{IGNORE LIMITS}\dots = \frac{1}{2}(2x+1)e^{2x} - \int e^{2x} \, dx$ $\quad \begin{array}{l} 2x+1 \to 2 \\ \frac{1}{2}e^{2x} \to e^{2x} \end{array}$

$= \left[\frac{1}{2}(2x+1)e^{2x} - \frac{1}{2}e^{2x}\right]_0^1 = \left(\frac{3}{2}e^2 - \frac{1}{2}e^2\right) - \cancel{\left(\frac{1}{2} - \frac{1}{2}\right)}$

$= e^2$

9. $\displaystyle\int_{\frac{1}{e}}^1 x\ln x \, dx = \dots\text{IGNORE LIMITS}\dots = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x \, dx$ $\quad \begin{array}{l} \ln x \to \frac{1}{x} \\ \frac{1}{2}x^2 \to x \end{array}$

$= \left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_{\frac{1}{e}}^1$

$= \left(\cancel{\frac{1}{2}\ln 1} - \frac{1}{4}\right) - \left[\frac{1}{2}\left(\frac{1}{e^2}\right)\ln\left(\frac{1}{e}\right) - \frac{1}{4}\left(\frac{1}{e^2}\right)\right] = -\frac{1}{4} - \left[-\frac{1}{2e^2} - \frac{1}{4e^2}\right]$

$= -\frac{1}{4} + \frac{1}{2e^2} + \frac{1}{4e^2} = \frac{1}{4}\left[-1 + \frac{2}{e^2} + \frac{1}{e^2}\right] = \frac{1}{4}\left[\frac{3}{e^2} - 1\right]$

10. $\displaystyle\int_{-1}^0 3\ln(2x+3) \, dx = \dots\text{IGNORE LIMITS}\dots \int 3 \times \ln(2x+3) \, dx$ $\quad \begin{array}{l} \ln(2x+3) \to \frac{2}{2x+3} \\ 3x \to 3 \end{array}$

$= 3x\ln(2x+3) - \int \frac{6x}{2x+3} \, dx$

$= 3x\ln(2x+3) - \int \frac{3(2x+3) - 9}{2x+3} \, dx$

$= 3x\ln(2x+3) - \int 3 - \frac{9}{2x+3} \, dx$

$= \left[3x\ln(2x+3) - 3x + \frac{9}{2}\ln(2x+3)\right]_{-1}^0$

$= \left(0 + \frac{9}{2}\ln 3\right) - \left(\cancel{-3\ln 1} + 3 + \cancel{\frac{9}{2}\ln 1}\right) = \frac{9}{2}\ln 3 - 3$

or $\frac{3}{2}(3\ln 3 - 2) = \frac{3}{2}(\ln 27 - 2)$

Question 9

Carry out the following integrations, to the answer given:

1. $\int_0^{\frac{\pi}{4}} x\sec^2 x \; dx = \frac{1}{4}(\pi - \ln 4)$

2. $\int_1^2 \frac{\ln x}{x} \; dx = \frac{1}{2}(\ln 2)^2$

3. $\int_0^{\frac{\pi}{2}} x\sin^2 x \; dx = \frac{1}{16}(\pi^2 + 4)$