DEFINITE INTEGRATION MIX

Part 1

1. $$\int_0^2 \frac{1}{\sqrt{4x+1}}\ dx = 1$$

$$\boxed{\begin{aligned}\int_0^2 \frac{1}{\sqrt{4x+1}}\,dx &= \int_0^2 (4x+1)^{-\frac{1}{2}}\,dx = \text{by inspection} = \left[\tfrac{1}{2}(4x+1)^{\frac{1}{2}}\right]_0^2\\ &= \tfrac{1}{2}\left[\sqrt{4x+1}\right]_0^2 = \tfrac{1}{2}\left[3-1\right] = \tfrac{1}{2}\times 2 = 1\end{aligned}}$$

2. $$\int_0^{\frac{\pi}{2}} \sin 2x\ \ dx = 1$$

$$\boxed{\begin{aligned}\int_0^{\frac{\pi}{2}} \sin 2x\,dx &= \left[-\tfrac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} = \tfrac{1}{2}\left[\cos 2x\right]_{\frac{\pi}{2}}^{0}\\ &= \tfrac{1}{2}\left(\cos 0 - \cos\pi\right) = \tfrac{1}{2}(1+1) = 1\end{aligned}}$$

3. $$\int_0^{\frac{\pi}{6}} \sin\left(4x+\frac{\pi}{6}\right)\ dx = \frac{\sqrt{3}}{4}$$

$$\boxed{\begin{aligned}\int_0^{\frac{\pi}{6}} \sin\left(4x+\tfrac{\pi}{6}\right)dx &= \left[-\tfrac{1}{4}\cos\left(4x+\tfrac{\pi}{6}\right)\right]_0^{\frac{\pi}{6}} = \tfrac{1}{4}\left[\cos\left(4x+\tfrac{\pi}{6}\right)\right]_{\frac{\pi}{6}}^{0}\\ &= \tfrac{1}{4}\left[\cos\tfrac{\pi}{6} - \cos\tfrac{5\pi}{6}\right] = \tfrac{1}{4}\left[\tfrac{\sqrt{3}}{2}+\tfrac{\sqrt{3}}{2}\right] = \tfrac{\sqrt{3}}{4}\end{aligned}}$$

4. $$\int_0^{\frac{\pi}{2}} \sin^2 x\ \ dx = \frac{\pi}{4}$$

$$\boxed{\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx &= \int_0^{\frac{\pi}{2}} \tfrac{1}{2} - \tfrac{1}{2}\cos 2x\,dx = \left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}\\ &= \left[\tfrac{\pi}{4} - \cancel{\tfrac{1}{4}\sin\pi}\right] - \left[0 - \cancel{\tfrac{1}{4}\sin 0}\right] = \tfrac{\pi}{4}\end{aligned}}$$

5. $$\int_1^2 x^3 \ln x \; dx = 4\ln 2 - \frac{15}{16}$$

$\displaystyle\int_1^2 x^3 \ln x \, dx$ = IGNORING LIMITS FOR THE TIME BEING

$$\begin{array}{ll} \ln x & \frac{1}{x} \\ \frac{1}{4}x^4 & x^3 \end{array}$$

$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^{\cancel{4}\,3}\left(\frac{1}{\cancel{x}}\right) dx$

$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 \, dx$

$= \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right]_1^2$

$= (4\ln 2 - 1) - \left(\cancel{\frac{1}{4}\ln 1} - \frac{1}{16}\right)$

$= 4\ln 2 - 1 + \frac{1}{16}$

$= 4\ln 2 - \frac{15}{16}$

6. $$\int_0^{\frac{1}{2}} \frac{x}{(2-x)^2} \; dx = \frac{1}{3} + \ln\frac{3}{4}$$

$\displaystyle\int_0^{\frac{1}{2}} \frac{x}{(2-x)^2}\,dx$ = by substitution = $\displaystyle\int_2^{\frac{3}{2}} \frac{2-u}{u^2}\,(-du)$

$$\begin{array}{c} u = 2-x \\ \frac{du}{dx} = -1 \\ dx = -du \\ \hline x=0 \mapsto u=2 \\ x=\frac{1}{2} \mapsto u=\frac{3}{2} \\ \hline x = 2-u \end{array}$$

$= \displaystyle\int_2^{\frac{3}{2}} \frac{u-2}{u^2}\,du = \int_2^{\frac{3}{2}} \frac{u}{u^2} - \frac{2}{u^2}\,du = \int_2^{\frac{3}{2}} \frac{1}{u} - 2u^{-2}\,du$

$= \left[\ln|u| + 2u^{-1}\right]_2^{\frac{3}{2}} = \left[\ln u + \frac{2}{u}\right]_2^{\frac{3}{2}}$

$= \left(\ln\frac{3}{2} + \frac{4}{3}\right) - (\ln 2 + 1) = \ln\frac{3}{2} - \ln 2 + \frac{1}{3} = \ln\frac{3}{4} + \frac{1}{3}$

• BY PARTIAL FRACTIONS

$\displaystyle\frac{x}{(2-x)^2} \equiv \frac{A}{(2-x)^2} + \frac{B}{2-x} \Rightarrow$

$$\begin{array}{ll} \boxed{x \equiv A + B(2-x)} & \\ \text{If } x=2 & 2=A \\ \text{If } x=0 & 0 = A+2B \Rightarrow B=-1 \end{array}$$

HENCE

$\displaystyle\int_0^{\frac{1}{2}} \frac{2}{(2-x)^2} - \frac{1}{2-x}\,dx = \int_0^{\frac{1}{2}} 2(2-x)^{-2} - \frac{1}{2-x}\,dx$

$= \left[2(2-x)^{-1} + \ln|2-x|\right]_0^{\frac{1}{2}} = \left[\frac{2}{2-x} + \ln|2-x|\right]_0^{\frac{1}{2}}$

$= \left(\frac{4}{3} + \ln\frac{3}{2}\right) - (1 - \ln 2) = \frac{4}{3} + \ln\frac{3}{2} - 1 + \ln 2 = \frac{1}{3} + \ln\frac{3}{4}$ AS BEFORE

7. $$\int_1^2 \frac{x}{(2x-1)^2} \; dx = \frac{2+\ln 27}{12}$$

$\displaystyle\int_1^2 \frac{x}{(2x-1)^2}\,dx$ = by substitution ....

$$\begin{array}{c} u = 2x-1 \\ \frac{du}{dx} = 2 \\ dx = \frac{du}{2} \\ \hline x=1 \mapsto u=1 \\ x=2 \mapsto u=3 \\ \hline 2x = u+1 \end{array}$$

$= \displaystyle\frac{1}{2}\int_1^3 \frac{2x}{(2x-1)^2}\,dx = \frac{1}{2}\int_1^3 \frac{u+1}{u^2}\,\frac{du}{2}$

$= \displaystyle\frac{1}{4}\int_1^3 \frac{u}{u^2} + \frac{1}{u^2}\,du = \frac{1}{4}\int_1^3 \frac{1}{u} + u^{-2}\,du$

$= \frac{1}{4}\left[\ln|u| - u^{-1}\right]_1^3 = \frac{1}{4}\left[\ln|u| - \frac{1}{u}\right]_1^3$

$= \frac{1}{4}\left[\left(\ln 3 - \frac{1}{3}\right) - (\cancel{\ln 1} - 1)\right] = \frac{1}{4}\left[\ln 3 + \frac{2}{3}\right] = \frac{1}{4}\ln 3 + \frac{1}{6}$

$= \frac{3}{4}\ln 3 + \frac{2}{12} = \frac{3\ln 3 + 2}{12} = \frac{\ln 27 + 2}{12}$

ALTERNATIVE BY PARTIAL FRACTIONS

$\displaystyle\frac{x}{(2x-1)^2} \equiv \frac{A}{(2x-1)^2} + \frac{B}{2x-1} \Rightarrow$

$$\begin{array}{l} x \equiv A + (2x-1) \\ \text{If } x=\frac{1}{2} \Rightarrow \frac{1}{2} = A \\ \text{If } x=0 \Rightarrow 0 = A - B \Rightarrow B = \frac{1}{2} \end{array}$$

$\displaystyle\int_1^2 \frac{\frac{1}{2}}{(2x-1)^2} + \frac{\frac{1}{2}}{2x-1}\,dx$

$= \displaystyle\int_1^2 \frac{1}{2}(2x-1)^{-2} + \frac{\frac{1}{2}}{2x-1}\,dx = \left[-\frac{1}{4}(2x-1)^{-1} + \frac{1}{4}\ln|2x-1|\right]_1^2$

$= \frac{1}{4}\left[\ln|2x-1| - \frac{1}{2x-1}\right]_1^2 = \frac{1}{4}\left[\left(\ln 3 - \frac{1}{3}\right) - (\cancel{\ln 1} - 1)\right]$

$= \frac{1}{4}\left(\ln 3 + \frac{2}{3}\right) = \frac{1}{12}(3\ln 3 + 2) = \frac{1}{12}(\ln 27 + 2)$ AS BEFORE

8. $$\int_0^1 \frac{3x}{(x+1)(x-2)}\,dx = -\ln 2$$

$$\int_0^1 \frac{3x}{(x+1)(x-2)}\,dx$$
$$= \int_0^1 \frac{1}{x+1} + \frac{2}{x-2}\,dx$$
$$= \Big[\ln|x+1| + 2\ln|x-2|\Big]_0^1$$
$$= \Big[\ln 2 - 2\ln|-1|\Big] - \Big[\cancel{\ln 1} + 2\ln|-2|\Big]$$
$$= \ln 2 - \cancel{2\ln 1} - 2\ln 2$$
$$= -\ln 2$$

PARTIAL FRACTIONS
$$\frac{3x}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$
$$3x \equiv A(x-2) + B(x+1)$$
If $x=2 \Rightarrow 6 = 3B \Rightarrow B=2$
If $x=-1 \Rightarrow -3 = -3A \Rightarrow A=1$

9. $$\int_0^{\frac{\pi}{4}} \tan^2 x\,dx = \frac{1}{4}(4-\pi)$$

$$\int_0^{\frac{\pi}{4}} \tan^2 x\,dx = \int_0^{\frac{\pi}{4}} \sec^2 x - 1\,dx = \Big[\tan x - x\Big]_0^{\frac{\pi}{4}}$$
$$\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - \left(\cancel{\tan 0} - 0\right) = 1 - \frac{\pi}{4} = \frac{1}{4}(4-\pi)$$

10. $$\int_0^2 \frac{x+2}{\sqrt{4x+1}}\,dx = \frac{17}{6}$$

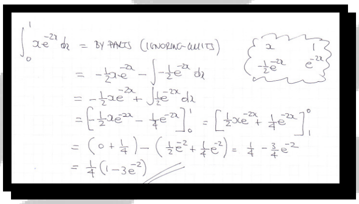
$$\int_0^2 \frac{x+2}{\sqrt{4x+1}}\,dx = \text{by substitution} = \frac{1}{4}\int_1^9 \frac{4x+8}{u^{\frac{1}{2}}}\,\frac{du}{4}$$
$$= \frac{1}{16}\int_1^9 \frac{(u-1)+8}{u^{\frac{1}{2}}}\,du = \frac{1}{16}\int_1^9 \frac{u}{u^{\frac{1}{2}}} + \frac{7}{u^{\frac{1}{2}}}\,du$$
$$= \frac{1}{16}\int_1^9 u^{\frac{1}{2}} + 7u^{-\frac{1}{2}}\,du = \frac{1}{16}\Big[\frac{2}{3}u^{\frac{3}{2}} + 14u^{\frac{1}{2}}\Big]_1^9$$
$$= \frac{1}{16}\Big[(18+42) - \left(\frac{2}{3}+14\right)\Big] = \frac{1}{\cancel{16}_{2}} \times \frac{\cancel{136}^{17}}{3} = \frac{17}{6}$$

$u = 4x+1$
$\frac{du}{dx} = 4$
$dx = \frac{du}{4}$
$x=0 \mapsto u=1$
$x=2 \mapsto u=9$
$4x = u-1$

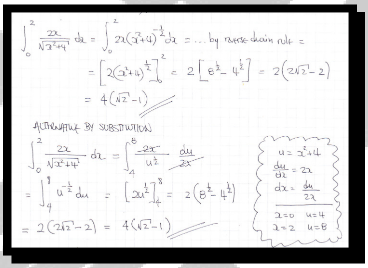
ALTERNATIVE BY SUBSTITUTION $u=\sqrt{4x+1}$
$$\int_0^2 \frac{x+2}{\sqrt{4x+1}}\,dx = \frac{1}{4}\int_1^3 \frac{4x+8}{\cancel{u}}\,\frac{1}{2}\cancel{u}\,du$$
$$= \frac{1}{4}\int_1^3 (u^2-1+8) \times \frac{1}{2}\,du = \frac{1}{8}\int_1^3 u^2+7\,du$$
$$= \frac{1}{8}\Big[\frac{1}{3}u^3 + 7u\Big]_1^3 = \frac{1}{8}\Big[(9+21) - \left(\frac{1}{3}+7\right)\Big]$$
$$= \frac{17}{6}$$

$u = \sqrt{4x+1}$
$u^2 = 4x+1$
$2u\frac{du}{dx} = 4$
$\frac{1}{2}u\,du = dx$
$x=0 \longmapsto u=1$
$x=2 \longmapsto u=3$
$4x = u^2-1$

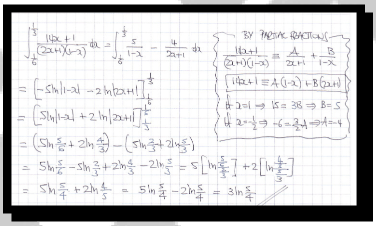
11. $$\int_{0}^{1} x\mathrm{e}^{-2x}\ dx = \tfrac{1}{4}\left(1-3\mathrm{e}^{-2}\right)$$



12. $$\int_{0}^{2} \frac{2x}{\sqrt{x^2+4}}\ dx = 4\left(\sqrt{2}-1\right)$$



13. $$\int_{\frac{1}{6}}^{\frac{1}{3}} \frac{14x+1}{(2x+1)(1-x)}\ dx = 3\ln\left(\frac{5}{4}\right)$$



14. $$\int_0^{36} \frac{1}{\sqrt{x}\left(\sqrt{x}+2\right)}\ dx = \ln 16$$

$$\int_0^{36} \frac{dx}{\sqrt{x}\left(\sqrt{x}+2\right)} = \int_0^{36} x^{-\frac{1}{2}} \times \frac{1}{x^{\frac{1}{2}}+2}\ dx \quad \text{BY REVERSE CHAIN RULE}$$

$$= \left[2\ln\left|x^{\frac{1}{2}}+2\right|\right]_0^{36} = 2\left[\ln 8 - \ln 2\right]$$

$$= 2(\ln 4) = \ln 16$$

OR
BY SUBSTITUTION

$$\int_0^{36} \frac{1}{\sqrt{x}\left(\sqrt{x}+2\right)}\ dx = \int_0^{6} \frac{1}{u(u+2)}\ 2u\ du$$

$$= \int_0^{6} \frac{2}{u+2}\ du = \left[2\ln|u+2|\right]_0^{6}$$

$$= 2\ln 8 - 2\ln 2 = 2(\ln 8 - \ln 2)$$

$$= 2\ln 4 = \ln 16$$

AS BEFORE

$$u = \sqrt{x}$$
$$u^2 = x$$
$$2u\frac{du}{dx} = 1$$
$$2u\ du = dx$$
$$x = 0 \mapsto u = 0$$
$$x = 36 \mapsto u = 6$$

15. $$\int_0^{\frac{\pi}{4}} 12x\cos 2x\ \ dx = \frac{3}{2}(\pi - 2)$$

$$\int_0^{\frac{\pi}{4}} 12x\cos 2x\ dx = \text{BY PARTS \& IGNORING LIMITS}$$

$$= 6x\sin 2x - \int 6\sin 2x$$

$$= \left[6x\sin 2x + 3\cos 2x\right]_0^{\pi/4}$$

$$= \left[6\left(\tfrac{\pi}{4}\right)\sin\tfrac{\pi}{2} + \cancel{3\cos\tfrac{\pi}{2}}\right] - \left[0 + 3\cos 0\right]$$

$$= \frac{3\pi}{2} - 3 = \frac{3}{2}(\pi - 2)$$

$$12x \mapsto 12$$
$$\tfrac{1}{2}\sin 2x \mapsto \cos 2x$$

16. $$\int_0^{\frac{\pi}{2}} \left(2\sin x - 3\cos x\right)^2\ \ dx = \frac{1}{4}(13\pi - 24)$$

$$\int_0^{\frac{\pi}{2}} (2\sin x - 3\cos x)^2\ dx = \int_0^{\frac{\pi}{2}} 4\sin^2 x - 12\sin x\cos x + 9\cos^2 x\ dx$$

$$= \int_0^{\frac{\pi}{2}} 4\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) - 6(2\sin x\cos x) - 9\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)\ dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{13}{2} - 6\sin 2x + \frac{5}{2}\cos 2x\ \ dx = \left[\frac{13}{2}x + 3\cos 2x + \frac{5}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left[\tfrac{13}{2}\left(\tfrac{\pi}{2}\right) + 3\cos\pi + \cancel{\tfrac{5}{4}\sin\pi}\right] - \left[0 + 3\cos 0 + \cancel{\tfrac{5}{4}\sin 0}\right]$$

$$= \frac{13\pi}{4} - 3 - 3 = \frac{13\pi}{4} - 6 = \frac{1}{4}(13\pi - 24)$$

17. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4x\sin 2x \ dx = \pi - 1$$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4x\sin 2x \, dx =$ BY PARTS AND IGNORING LIMITS

$4x \longrightarrow 4$

$-\frac{1}{2}\cos 2x \longrightarrow \sin 2x$

$= -2x\cos 2x - \int -2\cos 2x \, dx$

$= -2x\cos 2x + \int 2\cos 2x \, dx$

$= \left[-2x\cos 2x + \sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \left(-2\left(\frac{\pi}{2}\right)\cos\pi + \cancel{\sin\pi}\right) - \left(\cancel{-2\left(\frac{\pi}{4}\right)\cos\frac{\pi}{2}} + \sin\frac{\pi}{2}\right)$

$= -\pi(-1) - 1 = \pi - 1$

18. $$\int_{-6}^{\frac{3}{2}} \frac{x}{\sqrt{4-2x}} \ dx = -\frac{9}{2}$$

$\int_{-6}^{\frac{3}{2}} \frac{x}{\sqrt{4-2x}} \, dx = \int_{16}^{1} \frac{x}{u^{\frac{1}{2}}} \, \frac{du}{-2} = \frac{1}{2}\int_{1}^{16} \frac{x}{u^{\frac{1}{2}}} \, du$

$= \frac{1}{4}\int_{1}^{16} \frac{2x}{u^{\frac{1}{2}}} \, du = \frac{1}{4}\int_{1}^{16} \frac{4-u}{u^{\frac{1}{2}}} \, du = \frac{1}{4}\int_{1}^{16} 4u^{-\frac{1}{2}} - u^{\frac{1}{2}} \, du$

$= \frac{1}{4}\left[8u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_{1}^{16} = \frac{1}{4}\left[\left(32 - \frac{128}{3}\right) - \left(8 - \frac{2}{3}\right)\right]$

$= \frac{1}{4}\left[32 - \frac{128}{3} - 8 + \frac{2}{3}\right] = \frac{1}{4}\left[24 - \frac{126}{3}\right]$

$= \frac{1}{4}(-18) = -\frac{9}{2}$

BY SUBSTITUTION

$u = 4 - 2x$

$\frac{du}{dx} = -2$

$dx = \frac{du}{-2}$

$x = -6 \mapsto u = 16$

$x = \frac{3}{2} \mapsto u = 1$

$2x = 4 - u$

19. $$\int_{1}^{5} \frac{x+1}{(2x-1)^{\frac{3}{2}}} \ dx = 2$$

$\int_{1}^{5} \frac{x+1}{(2x-1)^{\frac{3}{2}}} \, dx = \int_{1}^{9} \frac{x+1}{u^{3/2}} \, \frac{du}{2}$

$= \frac{1}{2}\int_{1}^{9} \frac{\frac{u+1}{2}+1}{u^{3/2}} \, du =$ MULTIPLY TOP/BOTTOM BY 2

$= \frac{1}{2}\int_{1}^{9} \frac{u+1+2}{2u^{3/2}} \, du = \frac{1}{4}\int_{1}^{9} \frac{u+3}{u^{3/2}} \, du =$

$= \frac{1}{4}\int_{1}^{9} u^{-\frac{1}{2}} + 3u^{-\frac{3}{2}} \, du = \frac{1}{4}\left[2u^{\frac{1}{2}} - 6u^{-\frac{1}{2}}\right]_{1}^{9}$

$= \frac{1}{4}\left[(6-2) - (2-6)\right] = \frac{1}{4}(4+4) = 2$

$u = 2x - 1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$x = 1 \mapsto u = 1$

$x = 5 \mapsto u = 9$

$2x = u + 1$

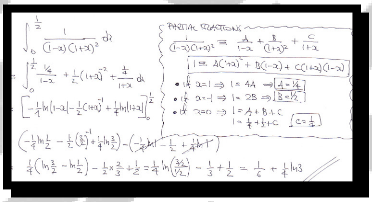
$x = \frac{u+1}{2}$

20. $$\int_{0}^{\frac{1}{12}\pi} 6\sin^2\theta \ d\theta = \frac{1}{4}(\pi - 3)$$

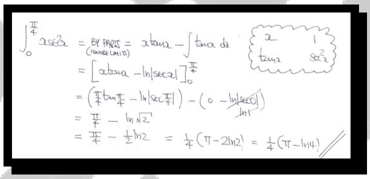
$\int_{0}^{\frac{\pi}{12}} 6\sin^2\theta \, d\theta = \int_{0}^{\frac{\pi}{12}} 6\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta = \int_{0}^{\frac{\pi}{12}} 3 - 3\cos 2\theta \, d\theta$

$= \left[3\theta - \frac{3}{2}\sin 2\theta\right]_{0}^{\frac{\pi}{12}} = \left(\frac{\pi}{4} - \frac{3}{2}\sin\frac{\pi}{6}\right) - \left(0 - \cancel{\frac{3}{2}\sin 0}\right) = \frac{\pi}{4} - \frac{3}{4} = \frac{1}{4}(\pi - 3)$

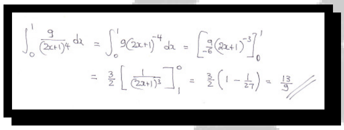
21. $$\int_0^{\frac{1}{2}} \frac{1}{(1-x)(1+x)^2}\,dx = \frac{1}{6}+\frac{1}{4}\ln 3$$



22. $$\int_0^{\frac{1}{4}\pi} x\sec^2 x \;\; dx = \tfrac{1}{4}(\pi - \ln 4)$$



23. $$\int_0^{1} \frac{9}{(2x+1)^4}\;\; dx = \frac{13}{9}$$



24. $$\int_0^{\frac{1}{2}\pi} 8x\sin^2 x \ dx = \frac{1}{2}\left(\pi^2+4\right)$$

$\int_0^{\frac{\pi}{2}} 8x\sin^2 x \, dx = \int_0^{\frac{\pi}{2}} 8x\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)dx = \int_0^{\frac{\pi}{2}} 4x - 4x\cos 2x \, dx$

BY PARTS AND IGNORING LIMITS

PARTS: $-4x \to -4$, $\frac{1}{2}\sin 2x \to \cos 2x$

$\dots = -2x\sin 2x - \int -2\sin 2x \, dx$

$= -2x\sin 2x + \int 2\sin 2x \, dx$

$= -2x\sin 2x - \cos 2x + C$

$\therefore \int_0^{\frac{\pi}{2}} 4x - 4x\cos 2x \, dx = \left[2x^2 - 2x\sin 2x - \cos 2x\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{\pi^2}{2} - 0 - (-1)\right) - \left(0 - 0 - 1\right)$

$= \frac{\pi^2}{2} + 1 + 1$

$= \frac{\pi^2}{2} + 2$

$= \frac{1}{2}(\pi^2+4)$

ALTERNATIVE

$\int_0^{\frac{\pi}{2}} 8x\sin^2 x \, dx = \dots$ BY PARTS & IGNORING LIMITS

$8x \Rightarrow 8$, $\frac{1}{2}x - \frac{1}{4}\sin 2x \Rightarrow \sin^2 x$

$\int \sin^2 x \, dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

$= 8x\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) - \int 8\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) dx$

$= 4x^2 - 2x\sin 2x - \int 4x - 2\sin 2x \, dx$

$= 4x^2 - 2x\sin 2x - (2x^2 + \cos 2x) + C$

$= \left[2x^2 - 2x\sin 2x - \cos 2x\right]_0^{\pi/2}$

$= \dots \left[\frac{\pi^2}{2} - 0 - (-1)\right] - \left[0 - 0 - 1\right]$

$= \frac{\pi^2}{2} + 1 + 1$

$= \frac{\pi^2}{2} + 2 = \frac{1}{2}(\pi^2+4)$

25. $$\int_0^{\frac{1}{4}\pi} 2x\cos 4x \ dx = -\frac{1}{4}$$

$\int_0^{\frac{\pi}{4}} 2x\cos 4x \, dx =$ BY PARTS (IGNORE LIMITS)

$2x \to 2$, $\frac{1}{4}\sin 4x \to \cos 4x$

$= \frac{1}{2}x\sin 4x - \int \frac{1}{2}\sin 4x \, dx$

$= \left[\frac{1}{2}x\sin 4x + \frac{1}{8}\cos 4x\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{1}{2}\left(\frac{\pi}{4}\right)\sin\pi + \frac{1}{8}\cos\pi\right] - \left[0 + \frac{1}{8}\cos 0\right]$

$= -\frac{1}{8} - \frac{1}{8}$

$= -\frac{1}{4}$

26. $$\int_0^{\frac{1}{4}\pi} \left(\cos x + \sec x\right)^2 \ dx = \frac{5}{8}\left(\pi+2\right)$$

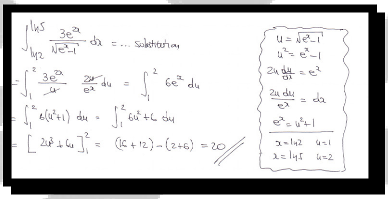
$\int_0^{\frac{\pi}{4}} (\cos x + \sec x)^2 \, dx = \int_0^{\frac{\pi}{4}} \cos^2 x + 2\cos x\sec x + \sec^2 x \, dx$

$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) + 2\cos x \times \frac{1}{\cos x} + \sec^2 x \, dx = \int_0^{\frac{\pi}{4}} \frac{5}{2} + \frac{1}{2}\cos 2x + \sec^2 x \, dx$

$= \left[\frac{5}{2}x + \frac{1}{4}\sin 2x + \tan x\right]_0^{\frac{\pi}{4}} = \left(\frac{5\pi}{8} + \frac{1}{4}\sin\frac{\pi}{2} + \tan\frac{\pi}{4}\right) - \left(0 + \cancel{\frac{1}{4}\sin 0} + \cancel{\tan 0}\right)$

$= \frac{5\pi}{8} + \frac{1}{4} + 1 = \frac{5}{8}\pi + \frac{5}{4} = \frac{5}{8}(\pi+2)$

27. $$\int_{\ln 2}^{\ln 5} \frac{3e^{2x}}{\sqrt{e^x - 1}} \, dx = 20$$



$$\int_{\ln 2}^{\ln 5} \frac{3e^{2x}}{\sqrt{e^x-1}} \, dx = \dots \text{substitution}$$

$$= \int_1^2 \frac{3e^{2x}}{\cancel{u}} \, \frac{2\cancel{u}}{e^x} \, du = \int_1^2 6e^x \, du$$

$$= \int_1^2 6(u^2+1) \, du = \int_1^2 6u^2+6 \, du$$

$$= \left[2u^3 + 6u \right]_1^2 = (16+12)-(2+6) = 20$$

$u = \sqrt{e^x - 1}$

$u^2 = e^x - 1$

$2u \frac{du}{dx} = e^x$

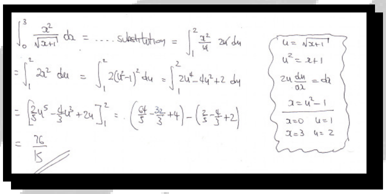
$\frac{2u \, du}{e^x} = dx$

$e^x = u^2 + 1$

$x = \ln 2 \quad u = 1$

$x = \ln 5 \quad u = 2$

28. $$\int_0^3 \frac{x^2}{\sqrt{x+1}} \, dx = \frac{76}{15}$$



$$\int_0^3 \frac{x^2}{\sqrt{x+1}} \, dx = \dots \text{substitution} = \int_1^2 \frac{x^2}{u} \, 2u \, du$$

$$= \int_1^2 2x^2 \, du = \int_1^2 2(u^2-1)^2 \, du = \int_1^2 2u^4 - 4u^2 + 2 \, du$$

$$= \left[\frac{2}{5}u^5 - \frac{4}{3}u^3 + 2u \right]_1^2 = \left(\frac{64}{5} - \frac{32}{3} + 4 \right) - \left(\frac{2}{5} - \frac{4}{3} + 2 \right)$$

$$= \frac{76}{15}$$

$u = \sqrt{x+1}$

$u^2 = x+1$

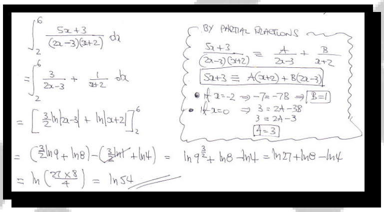
$2u \frac{du}{dx} = dx$

$x = u^2 - 1$

$x = 0 \quad u = 1$

$x = 3 \quad u = 2$

29. $$\int_2^6 \frac{5x+3}{(2x-3)(x+2)} \, dx = \ln 54$$



$$\int_2^6 \frac{5x+3}{(2x-3)(x+2)} \, dx$$

$$= \int_2^6 \frac{3}{2x-3} + \frac{1}{x+2} \, dx$$

$$= \left[\frac{3}{2}\ln|2x-3| + \ln|x+2| \right]_2^6$$

$$= \left(\frac{3}{2}\ln 9 + \ln 8 \right) - \left(\frac{3}{2}\ln 1 + \ln 4 \right) = \ln 9^{\frac{3}{2}} + \ln 8 - \ln 4 = \ln 27 + \ln 8 - \ln 4$$

$$= \ln\left(\frac{27 \times 8}{4} \right) = \ln 54$$

BY PARTIAL FRACTIONS

$\frac{5x+3}{(2x-3)(x+2)} \equiv \frac{A}{2x-3} + \frac{B}{x+2}$

$\boxed{5x+3 \equiv A(x+2) + B(2x-3)}$

- If $x = -2 \Rightarrow -7 = -7B \Rightarrow \boxed{B=1}$
- If $x = 0 \Rightarrow 3 = 2A - 3B$

$3 = 2A - 3$

$\boxed{A=3}$

30. $\displaystyle\int_{0}^{\ln 2} 4x\mathrm{e}^{-x} \; dx = 2 - \ln 4$

$\displaystyle\int_0^{\ln 2} 4xe^{-x}\,dx =$ by parts & ignoring limits

$\begin{matrix} 4x & 4 \\ -e^{-x} & e^{-x} \end{matrix}$

$= -4xe^{-x} - \int -4e^{-x}\,dx$

$= -4xe^{-x} + \int 4e^{-x}\,dx$

$= \left[-4xe^{-x} - 4e^{-x}\right]_0^{\ln 2}$

$= \left[4xe^{-x} + 4e^{-x}\right]_{\ln 2}^{0}$

$= (0+4) - \left(4\ln 2\, e^{-\ln 2} + 4e^{-\ln 2}\right)$

$= 4 - (2\ln 2 + 2)$

$= 2 - 2\ln 2$ $(= 2 - \ln 4)$

31. $\displaystyle\int_{0}^{3} \frac{x}{x^2+9} \; dx = \frac{1}{2}\ln 2$

$\displaystyle\int_0^3 \frac{x}{x^2+9}\,dx = \frac{1}{2}\int_0^3 \frac{2x}{x^2+9}\,dx = \left[\frac{1}{2}\ln|x^2+9|\right]_0^3$

OR BY SUBSTITUTION $u = x^2+9$

$= \frac{1}{2}\left[\ln 18 - \ln 9\right] = \frac{1}{2}\ln 2$

32. $\displaystyle\int_{0}^{\frac{1}{4}\pi} \sin\left(2x + \frac{\pi}{4}\right) \; dx = \frac{\sqrt{2}}{2}$

$\displaystyle\int_0^{\frac{\pi}{4}} \sin\left(2x+\frac{\pi}{4}\right)dx = \left[-\frac{1}{2}\cos\left(2x+\frac{\pi}{4}\right)\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\cos\left(2x+\frac{\pi}{4}\right)\right]_{\frac{\pi}{4}}^{0}$

$= \frac{1}{2}\left[\cos\frac{\pi}{4} - \cos\frac{3\pi}{4}\right] = \frac{1}{2}\left(\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right)$

$= \frac{1}{2}\sqrt{2}$

33. $\displaystyle\int_{\frac{2}{3}}^{1} \frac{x}{2x-1} \; dx = \frac{1}{6} + \frac{1}{4}\ln 3$

$\displaystyle\int_{\frac{2}{3}}^{1} \frac{x}{2x-1}\,dx = \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{2x}{2x-1}\,dx = \frac{1}{2}\int_{\frac{2}{3}}^{1} \frac{(2x-1)+1}{(2x-1)}\,dx$

$= \frac{1}{2}\int_{\frac{2}{3}}^{1} 1 + \frac{1}{2x-1}\,dx = \frac{1}{2}\left[x + \frac{1}{2}\ln|2x-1|\right]_{\frac{2}{3}}^{1}$

$= \frac{1}{2}\left[\left(1 + \cancel{\frac{1}{2}\ln 1}\right) - \left(\frac{2}{3} + \frac{1}{2}\ln\frac{1}{3}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{2}\ln\frac{1}{3}\right] = \frac{1}{2}\left[\frac{1}{3} + \frac{1}{2}\ln 3\right]$

$= \frac{1}{6} + \frac{1}{4}\ln 3$

ALTERNATIVE BY SUBSTITUTION

$\displaystyle\int_{\frac{2}{3}}^{1} \frac{x}{2x-1}\,dx = \int_{\frac{2}{3}}^{1} \frac{x}{u}\,\frac{du}{2} = \int_{\frac{1}{3}}^{1} \frac{x}{2u}\,du$

$u = 2x-1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$x = \frac{2}{3},\ u = \frac{1}{3}$

$x = 1,\ u = 1$

$2x = u+1$

$\displaystyle= \int_{\frac{1}{3}}^{1} \frac{2x}{4u}\,du = \int_{\frac{1}{3}}^{1} \frac{u+1}{4u}\,du = \int_{\frac{1}{3}}^{1} \frac{1}{4} + \frac{1}{4u}\,du$

$= \left[\frac{1}{4}u + \frac{1}{4}\ln|u|\right]_{\frac{1}{3}}^{1} = \left(\frac{1}{4} + \cancel{\frac{1}{4}\ln 1}\right) - \left(\frac{1}{12} + \frac{1}{4}\ln\frac{1}{3}\right)$

$= \frac{1}{6} - \frac{1}{4}\ln\frac{1}{3} = \frac{1}{6} + \frac{1}{4}\ln 3$

AS BEFORE

34. $$\int_0^4 \frac{13-2x}{(x+4)(2x+1)}\ dx = 4\ln 3 - 3\ln 2$$

$$\int_0^4 \frac{13-2x}{(x+4)(2x+1)}\,dx = \int_0^4 \frac{4}{2x+1} - \frac{3}{x+4}\,dx$$

$$= \Big[2\ln|2x+1| - 3\ln|x+4|\Big]_0^4$$

$$= (2\ln 9 - 3\ln 8) - (\cancel{2\ln 1} - 3\ln 4)$$

$$= 2\ln 9 - 3\ln 8 + 3\ln 4 = 2\ln 3^2 - 3\ln 2^3 + 3\ln 2^2$$

$$= 4\ln 3 - 9\ln 2 + 6\ln 2 = 4\ln 3 - 3\ln 2$$

BY PARTIAL FRACTIONS

$$\frac{13-2x}{(x+4)(2x+1)} \equiv \frac{A}{x+4} + \frac{B}{2x+1}$$

$$\boxed{13-2x \equiv A(2x+1) + B(x+4)}$$

- If $x=-4$, $21 = -7A \Rightarrow \boxed{A=-3}$
- If $x=0$, $13 = A+4B$, $13 = -3+4B \Rightarrow \boxed{B=4}$

35. $$\int_1^e \ln x\ \ dx = 1$$

$$\int_1^e \ln x\,dx = \int_1^e 1\times\ln x = \text{BY PARTS IGNORING LIMITS}$$

$$\begin{array}{cc} \ln x & \frac{1}{x} \\ x & 1 \end{array}$$

$$= x\ln x - \int 1\,dx$$

$$= \Big[x\ln x - x\Big]_1^e$$

$$= (e\ln e - e) - (\cancel{\ln 1} - 1)$$

$$= \cancel{e} - \cancel{e} + 1$$

$$= 1$$

36. $$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \cos 3x\ \ dx = -\frac{1}{3}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3x\,dx = \Big[\tfrac{1}{3}\sin 3x\Big]_{\pi/6}^{\pi/3} = \cancel{\tfrac{1}{3}\sin\pi} - \tfrac{1}{3}\sin\tfrac{\pi}{2} = -\tfrac{1}{3}$$

37. $$\int_0^{\frac{1}{2}\pi} 4\cos x\,(1+\sin x)^3\ dx = 15$$

$$\int_0^{\frac{\pi}{2}} 4\cos x(1+\sin x)^3\,dx = \text{BY REVERSE CHAIN RULE} = \Big[(1+\sin x)^4\Big]_0^{\frac{\pi}{2}}$$

$$= \left(1+\sin\tfrac{\pi}{2}\right)^4 - (1+\sin 0)^4 = 16 - 1 = 15$$

ALTERNATIVE BY SUBSTITUTION

$$\int_0^{\frac{\pi}{2}} 4\cos x(1+\sin x)^3\,dx = \int_1^2 4\cancel{\cos x}\times u^3\times\frac{du}{\cancel{\cos x}}$$

$$= \int_1^2 4u^3\,du = \Big[u^4\Big]_1^2$$

$$= 16 - 1 = 15$$

$u = 1+\sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$x = \frac{\pi}{2}$, $u = 2$

$x = 0$, $u = 1$

38. $$\int_{\frac{1}{8}}^{\frac{1}{6}\pi} \cot^2 2x \ dx = \frac{1}{2} - \frac{1}{6}\sqrt{3} - \frac{1}{24}\pi$$

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \cot^2 2x \, dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \operatorname{cosec}^2 2x - 1 \, dx = \left[-\tfrac{1}{2}\cot 2x - x\right]_{\frac{\pi}{8}}^{\frac{\pi}{6}}$$

$$= \left[\tfrac{1}{2}\cot 2x + x\right]_{\frac{\pi}{6}}^{\frac{\pi}{8}} = \left(\tfrac{1}{2}\cot\tfrac{\pi}{4} + \tfrac{\pi}{8}\right) - \left(\tfrac{1}{2}\cot\tfrac{\pi}{3} + \tfrac{\pi}{6}\right)$$

$$= \left(\tfrac{1}{2} + \tfrac{\pi}{8}\right) - \left(\tfrac{1}{2}\times\tfrac{\sqrt{3}}{3} + \tfrac{\pi}{6}\right) = \tfrac{1}{2} + \tfrac{\pi}{8} - \tfrac{\sqrt{3}}{6} - \tfrac{\pi}{6}$$

$$= \tfrac{1}{2} - \tfrac{\sqrt{3}}{6} - \tfrac{\pi}{24}$$

39. $$\int_0^{\frac{1}{2}} \frac{3-5x}{(1-x)(2-3x)} \ dx = \frac{4}{3}\ln 2$$

$$\int_0^{\frac{1}{2}} \frac{3-5x}{(1-x)(2-3x)} \, dx$$

$$= \int_0^{\frac{1}{2}} \frac{2}{1-x} - \frac{1}{2-3x} \, dx$$

$$= \left[-2\ln|1-x| + \tfrac{1}{3}\ln|2-3x|\right]_0^{\frac{1}{2}}$$

$$= \left(-2\ln\tfrac{1}{2} + \tfrac{1}{3}\ln\tfrac{1}{2}\right) - \left(\cancel{2\ln 1} + \tfrac{1}{3}\ln 2\right)$$

$$= -2\ln\tfrac{1}{2} + \tfrac{1}{3}\ln\tfrac{1}{2} - \tfrac{1}{3}\ln 2$$

$$= 2\ln 2 - \tfrac{1}{3}\ln 2 - \tfrac{1}{3}\ln 2 = \tfrac{4}{3}\ln 2$$

BY PARTIAL FRACTIONS

$$\frac{3-5x}{(1-x)(2-3x)} \equiv \frac{A}{1-x} + \frac{B}{2-3x}$$

$$\boxed{3-5x \equiv A(2-3x) + B(1-x)}$$

If $x=1$, $-2 = -A \Rightarrow \boxed{A=2}$

If $x=0$, $3 = 2A + B$

$3 = 4 + B$

$\boxed{B=-1}$

40. $$\int_{\ln 2}^{\ln 4} \left(e^{2x} - 2\right)^2 \ dx = 4(9 + \ln 2)$$

$$\int_{\ln 2}^{\ln 4} (e^{2x}-2)^2 \, dx = \int_{\ln 2}^{\ln 4} (e^{2x})^2 - 2\times 2\times e^{2x} + 4 \, dx = \int_{\ln 2}^{\ln 4} e^{4x} - 4e^{2x} + 4 \, dx$$

$$= \left[\tfrac{1}{4}e^{4x} - 2e^{2x} + 4x\right]_{\ln 2}^{\ln 4}$$

$$= \left(\tfrac{1}{4}e^{4\ln 4} - 2e^{2\ln 4} + 4\ln 4\right) - \left(\tfrac{1}{4}e^{4\ln 2} - 2e^{2\ln 2} + 4\ln 2\right)$$

$$= (64 - 32 + 4\ln 4) - (4 - 8 + 4\ln 2)$$

$$= (32 + 8\ln 2) - (-4 + 4\ln 2)$$

$$= 36 + 4\ln 2$$

$$= 4(9 + \ln 2)$$

41. $$\int_0^{\frac{1}{2}\pi} x\sin 2x \ dx = \frac{1}{4}\pi$$

$\int_0^{\frac{\pi}{2}} x\sin 2x \, dx =$ BY PARTS (IGNORE LIMITS) $= -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x \, dx$

x	1
$-\frac{1}{2}\cos 2x$	$\sin 2x$

$$= -\tfrac{1}{2}x\cos 2x + \int \tfrac{1}{2}\cos 2x \, dx$$

$$= \left[-\tfrac{1}{2}x\cos 2x + \tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left(-\tfrac{1}{2}\times\tfrac{\pi}{2}\times\cos\pi + \cancel{\tfrac{1}{4}\sin\pi}\right) - \left(0 + \cancel{\tfrac{1}{4}\sin 0}\right)$$

$$= -\tfrac{1}{2}\times\tfrac{\pi}{2}\times(-1) = \tfrac{\pi}{4}$$

42. $$\int_{-1}^{1} \frac{9+4x^2}{9-4x^2}\,dx = -2+3\ln 5$$

$$\int_{-1}^{1} \frac{9+4x^2}{9-4x^2}\,dx$$

PARTIAL FRACTIONS ARE IMPROPER!

$$\frac{9+4x^2}{9-4x^2} = \frac{9+4x^2}{(3-2x)(3+2x)} \equiv A + \frac{B}{3-2x} + \frac{C}{3+2x}$$

$$\boxed{9+4x^2 \equiv A(3-2x)(3+2x) + B(3+2x) + C(3-2x)}$$

- If $x = \frac{3}{2} \Rightarrow 9+9 = 6B \Rightarrow \boxed{B=3}$
- If $x = -\frac{3}{2} \Rightarrow 9+9 = 6C \Rightarrow \boxed{C=3}$
- If $x = 0 \Rightarrow 9 = 9A + 3B + 3C$, $\cancel{9} = 9A + \cancel{9} + 9 \Rightarrow \boxed{A=-1}$

$$= \int_{-1}^{1} -1 + \frac{3}{3-2x} + \frac{3}{3+2x}\,dx$$

$$= \Big[-x - \tfrac{3}{2}\ln|3-2x| + \tfrac{3}{2}\ln|3+2x|\Big]_{-1}^{1}$$

$$= \left(-1 - \cancel{\tfrac{3}{2}\ln 1} + \tfrac{3}{2}\ln 5\right) - \left(1 - \tfrac{3}{2}\ln 5 + \cancel{\tfrac{3}{2}\ln 1}\right) = -2 + 3\ln 5$$

43. $$\int_{-1}^{7} \frac{x^2}{\sqrt{x+2}}\,dx = \frac{652}{15}$$

$$\int_{-1}^{7} \frac{x^2}{\sqrt{x+2}}\,dx = \int_{1}^{3} \frac{x^2}{\cancel{u}}\,2\cancel{u}\,du = \int_{1}^{3} 2x^2\,du$$

$u = \sqrt{x+2}$
$u^2 = x+2$
$2u\frac{du}{dx} = 1$
$2u\,du = dx$
$x=-1 \mapsto u=1$
$x=7 \mapsto u=3$
$x = u^2-2$

$$= \int_{1}^{3} 2(u^2-2)^2\,du = \int_{1}^{3} 2(u^4-4u^2+4)\,du = \int_{1}^{3} 2u^4 - 8u^2 + 8\,du$$

$$= \Big[\tfrac{2}{5}u^5 - \tfrac{8}{3}u^3 + 8u\Big]_{1}^{3} = \left(\tfrac{486}{5} - 72 + 24\right) - \left(\tfrac{2}{5} - \tfrac{8}{3} + 8\right)$$

$$= \frac{484}{5} - 56 + \frac{8}{3} = \frac{652}{15}$$

44. $$\int_{0}^{2} \frac{6}{3x+2}\,dx = \ln 16$$

$$\int_{0}^{2} \frac{6}{3x+2}\,dx = \Big[2\ln|3x+2|\Big]_{0}^{2} = 2\ln 8 - 2\ln 2 = \ln 64 - \ln 4$$

$$= \ln\left(\frac{64}{4}\right) = \ln 16$$

45. $$\int_{2}^{5} \frac{1}{4+\sqrt{x-1}}\,dx = 2+8\ln\left(\frac{5}{6}\right)$$

$$\int_{2}^{5} \frac{1}{4+\sqrt{x-1}}\,dx = \int_{5}^{6} \frac{1}{u} \times 2(u-4)\,du$$

$u = 4+\sqrt{x-1}$
$u-4 = \sqrt{x-1}$
$(u-4)^2 = x+1$
$x = (u-4)^2-1$
$\frac{dx}{du} = 2(u-4)$
$dx = 2(u-4)\,du$
$x=2 \mapsto u=5$
$x=5 \mapsto u=6$

$$= \int_{5}^{6} \frac{2u-8}{u}\,du = \int_{5}^{6} 2 - \frac{8}{u}\,du = \Big[2u - 8\ln|u|\Big]_{5}^{6}$$

$$= (12 - 8\ln 6) - (10 - 8\ln 5) = 2 + 8\ln 5 - 8\ln 6$$

$$= 2 + 8\ln\left(\frac{5}{6}\right)$$

46. $$\int_{0}^{\frac{1}{4}\pi} \frac{\cos 2x}{\cos^2 x}\, dx = \tfrac{1}{2}(\pi - 2)$$

$$\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{\cos^2 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{2\cos^2 x - 1}{\cos^2 x}\,dx = \int_0^{\frac{\pi}{4}} 2 - \frac{1}{\cos^2 x}\,dx = \int_0^{\frac{\pi}{4}} 2 - \sec^2 x\,dx$$
$$= \Big[2x - \tan x\Big]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{2} - 1\right) - (0 - 0) = \frac{\pi}{2} - 1$$
$$= \frac{1}{2}(\pi - 2)$$

47. $$\int_{0}^{\frac{\pi}{4}} \cos\left(3x + \frac{\pi}{4}\right) dx = -\frac{\sqrt{2}}{6}$$

$$\int_0^{\frac{\pi}{4}} \cos\left(3x + \frac{\pi}{4}\right)dx = \left[\frac{1}{3}\sin\left(3x + \frac{\pi}{4}\right)\right]_0^{\frac{\pi}{4}} = \frac{1}{3}\left[\sin\pi - \sin\frac{\pi}{4}\right]$$
$$= \frac{1}{3}\left[0 - \frac{\sqrt{2}}{2}\right] = -\frac{\sqrt{2}}{6}$$

48. $$\int_{2}^{4} \frac{8}{(3x-4)^3}\, dx = \frac{5}{16}$$

$$\int_2^4 \frac{8}{(3x-4)^3}\,dx = \int_2^4 8(3x-4)^{-3}\,dx = \left[\frac{8}{-6}(3x-4)^{-2}\right]_2^4$$
$$= \frac{4}{3}\left[\frac{1}{(3x-4)^2}\right]_4^2 = \frac{4}{3}\left[\frac{1}{4} - \frac{1}{64}\right]$$
$$= \frac{4}{3}\left[\frac{15}{64}\right] = \frac{5}{16}$$

49. $$\int_{1}^{\frac{5}{2}} \frac{4x}{\sqrt{2x-1}}\, dx = \frac{20}{3}$$

$$\int_1^{\frac{5}{2}} \frac{4x}{\sqrt{2x-1}}\,dx = \int_1^2 \frac{4x}{\cancel{u}}\,\cancel{u}\,du = \int_1^2 4x\,du$$
$$\int_1^2 2u^2 + 2\,du = \left[\frac{2}{3}u^3 + 2u\right]_1^2 = \left(\frac{16}{3} + 4\right) - \left(\frac{2}{3} + 2\right)$$
$$\frac{14}{3} + 2 = \frac{20}{3}$$

$u = \sqrt{2x-1}$
$u^2 = 2x - 1$
$\cancel{2}u\frac{du}{dx} = \cancel{2}$
$u\,du = dx$
$x = 1 \quad u = 1$
$x = \frac{5}{2} \quad u = 2$
$2x = u^2 + 1$

50. $$\int_{0}^{\frac{\pi}{3}} \cos\left(3x+\frac{\pi}{3}\right)\,dx = -\frac{\sqrt{3}}{3}$$

$$\int_{0}^{\frac{\pi}{3}} \cos\left(3x+\frac{\pi}{3}\right) dx = \left[\frac{1}{3}\sin\left(3x+\frac{\pi}{3}\right)\right]_{0}^{\frac{\pi}{3}} = \frac{1}{3}\left[\sin\frac{4\pi}{3} - \sin\frac{\pi}{3}\right]$$
$$= \frac{1}{3}\left[-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right] = -\frac{\sqrt{3}}{3}$$

51. $$\int_{0}^{1} \frac{18-4x-x^2}{(4-3x)(1+x)^2}\,dx = \frac{7}{3}\ln 2 + \frac{3}{2}$$

$$\int_{0}^{1} \frac{18-4x-x^2}{(4-3x)(1+x)^2}\,dx$$
$$= \int_{0}^{1} \frac{2}{4-3x} + 3(1+x)^{-2} + \frac{1}{1+x}\,dx$$
$$= \left[-\frac{2}{3}\ln|4-3x| - 3(1+x)^{-1} + \ln|1+x|\right]_{0}^{1}$$
$$= \left[\frac{2}{3}\ln|4-3x| + \frac{3}{1+x} - \ln|1+x|\right]_{1}^{0}$$
$$= \left(\frac{2}{3}\ln 4 + 3 - \cancel{\ln 1}\right) - \left(\frac{2}{3}\cancel{\ln 1} + \frac{3}{2} - \ln 2\right)$$
$$= \frac{2}{3}\ln 4 + 3 - \frac{3}{2} + \ln 2$$
$$= \frac{2}{3}\ln 2^2 + \frac{3}{2} + \ln 2$$
$$= \frac{4}{3}\ln 2 + \frac{3}{2} + \ln 2 = \frac{3}{2} + \frac{7}{3}\ln 2$$

PARTIAL FRACTIONS

$$\frac{18-4x-x^2}{(4-3x)(1+x)^2} \equiv \frac{A}{4-3x} + \frac{B}{(1+x)^2} + \frac{C}{1+x}$$
$$18-4x-x^2 \equiv A(1+x)^2 + B(4-3x) + C(1+x)(4-3x)$$

- If $x=-1 \Rightarrow 18+4-1 = 7B \Rightarrow \boxed{B=3}$
- If $x=\frac{4}{3} \Rightarrow 18 - \frac{16}{3} - \frac{16}{9} = \frac{49}{9}A$
  $\Rightarrow 162-48-16 = 49A$
  $\Rightarrow 49A = 98$
  $\Rightarrow \boxed{A=2}$
- If $x=0 \Rightarrow 18 = A + 4B + 4C$
  $18 = 2 + 12 + 4C$
  $4 = 4C$
  $\boxed{C=1}$

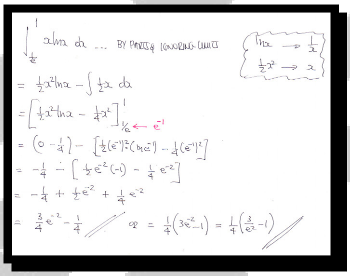
52. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x + \cot x)^2\,dx = \frac{1}{8}\left(26 - \pi - 4\sqrt{2}\right)$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x+\cot x)^2\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x + 2\sin x\cot x + \cot^2 x\,dx$$
$$= \int_{\pi/4}^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) + 2\cancel{\sin x}\frac{\cos x}{\cancel{\sin x}} + \left(\operatorname{cosec}^2 x - 1\right) dx$$
$$= \int_{\pi/4}^{\pi/2} -\frac{1}{2} - \frac{1}{2}\cos 2x + 2\cos x + \operatorname{cosec}^2 x = \left[-\frac{1}{2}x - \frac{1}{4}\sin 2x + 2\sin x - \cot x\right]_{\pi/4}^{\pi/2}$$
$$= \left(-\frac{\pi}{4} + 0 + 2 - 0\right) - \left(-\frac{\pi}{8} - \frac{1}{4} + \sqrt{2} - 1\right) = -\frac{\pi}{8} + \frac{13}{4} - \sqrt{2}$$

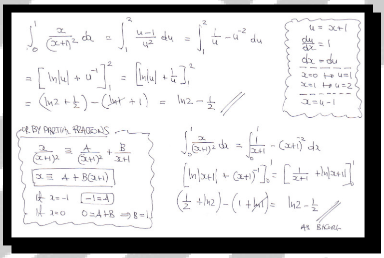
53. $$\int_{0}^{\frac{\pi}{3}} \tan^3 x\,dx = \frac{3}{2} - \ln 2$$

$$\int_{0}^{\frac{\pi}{3}} \tan^3 x\,dx = \int_{0}^{\frac{\pi}{3}} \tan x\tan^2 x\,dx = \int_{0}^{\frac{\pi}{3}} \tan x\left(\sec^2 x - 1\right) dx$$
$$= \int_{0}^{\frac{\pi}{3}} \tan x\sec^2 x - \tan x\,dx = \left[\frac{1}{2}\tan^2 x - \ln|\sec x|\right]_{0}^{\frac{\pi}{3}}$$
$$= \left(\frac{1}{2}\times 3 - \ln 2\right) - \left(0 - \cancel{\ln 1}\right) = \frac{3}{2} - \ln 2$$

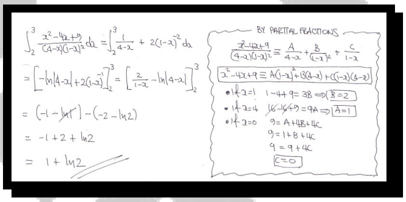
54. $$\int_{\frac{1}{e}}^{1} x\ln x \; dx = \frac{1}{4}\left(\frac{3}{e^2}-1\right)$$



55. $$\int_{0}^{1} \frac{x}{(1+x)^2} \, dx = \ln 2 - \frac{1}{2}$$



56. $$\int_{2}^{3} \frac{x^2-4x+9}{(4-x)(1-x)^2} \, dx = 1+\ln 2$$



57. $$\int_{0}^{\frac{\pi}{12}} 10\sin 8\theta \cos 2\theta \, d\theta = \frac{1}{12}\left(16+3\sqrt{3}\right)$$

$$\int_{0}^{\frac{\pi}{12}} 10\sin 8\theta\cos 2\theta \, d\theta = \int_{0}^{\pi/12} 5\sin 10\theta + 5\sin 6\theta \, d\theta$$
$$= \left[-\frac{1}{2}\cos 10\theta - \frac{5}{6}\cos 6\theta\right]_{0}^{\frac{\pi}{12}} = \left[\frac{1}{2}\cos 10\theta + \frac{5}{6}\cos 6\theta\right]_{\frac{\pi}{12}}^{0}$$
$$= \left(\frac{1}{2}+\frac{5}{6}\right) - \left(-\frac{\sqrt{3}}{4}+0\right) = \frac{1}{4}\sqrt{3} + \frac{4}{3} = \frac{1}{12}\left[3\sqrt{3}+16\right]$$

$$\sin(8\theta+2\theta) = \sin 8\theta\cos 2\theta + \cancel{\cos 8\theta \sin 2\theta}$$
$$\sin(8\theta-2\theta) = \sin 8\theta\cos 2\theta - \cancel{\cos 8\theta\sin 2\theta}$$
ADD $\boxed{\sin 10\theta + \sin 6\theta = 2\sin 8\theta\cos 2\theta}$

58. $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\left(4x+\frac{\pi}{6}\right) dx = -\frac{\sqrt{3}}{8}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\left(4x+\frac{\pi}{6}\right) dx = \left[-\frac{1}{4}\cos\left(4x+\frac{\pi}{6}\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{4}\left[\cos\left(4x+\frac{\pi}{6}\right)\right]_{\frac{\pi}{3}}^{\frac{\pi}{6}}$$
$$= \frac{1}{4}\left[\cos\frac{5\pi}{6} - \cancel{\cos\frac{3\pi}{2}}\right] = \frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{8}$$

59. $$\int_{1}^{e} \left(x^{2}+1\right)\ln x \; dx = \frac{2}{9}\left(e^{3}+5\right)$$

$$\int_{1}^{e}\left(x^2+1\right)\ln x = \dots \text{BY PARTS \& IGNORE LIMITS}$$
$$\ln x \longrightarrow \frac{1}{x}$$
$$\frac{1}{3}x^3+x \longrightarrow x^2+1$$
$$= \left(\frac{1}{3}x^3+x\right)\ln x - \int\left(\frac{1}{3}x^3+x\right)\times\frac{1}{x}\,dx$$
$$= \left(\frac{1}{3}x^3+x\right)\ln x - \int \frac{1}{3}x^2+1\,dx$$
$$= \left(\frac{1}{3}x^3+x\right)\ln x - \left(\frac{1}{9}x^3+x\right)+C$$
$$= \left[\left(\frac{1}{3}x^3+x\right)\ln|x| - \frac{1}{9}x^3 - x\right]_{1}^{e}$$
$$= \left[\left(\frac{1}{3}e^3+e\right)\overset{=1}{\ln e} - \frac{1}{9}e^3 - e\right] - \left[0-\frac{1}{9}-1\right]$$
$$= \frac{1}{3}e^3 + \cancel{e} - \frac{1}{9}e^3 - \cancel{e} + \frac{10}{9}$$
$$= \frac{2}{9}e^3 + \frac{10}{9} = \frac{2}{9}\left(e^3+5\right)$$

60. $$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos x)^2 \ dx = 4\pi + 3\sqrt{3}$$

$$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos x)^2 \, dx = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 - 4\cos x + 4\cos^2 x \, dx = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 - 4\cos x + 4\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2x\right) dx$$
$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 3 - 4\cos x + 2\cos 2x \, dx = \Big[3x - 4\sin x + \sin 2x\Big]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$
$$= \left(5\pi - 4\left(-\tfrac{\sqrt{3}}{2}\right) - \tfrac{\sqrt{3}}{2}\right) - \left(\pi - 4\left(\tfrac{\sqrt{3}}{2}\right) + \tfrac{\sqrt{3}}{2}\right)$$
$$= 5\pi + 2\sqrt{3} - \tfrac{\sqrt{3}}{2} - \pi + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}$$
$$= 4\pi + 3\sqrt{3}$$

61. $$\int_0^3 x\sqrt{x+1} \ dx = \frac{116}{15}$$

$$\int_0^3 x\sqrt{x+1} \, dx = \int_1^2 (u^2-1)u \ 2u \, du$$
$$= \int_1^2 2u^4 - 2u^2 \, du = \left[\tfrac{2}{5}u^5 - \tfrac{2}{3}u^3\right]_1^2$$
$$= \left(\tfrac{64}{5} - \tfrac{16}{3}\right) - \left(\tfrac{2}{5} - \tfrac{2}{3}\right) = \tfrac{112}{15} - \left(-\tfrac{4}{15}\right) = \tfrac{116}{15}$$

$u = \sqrt{x+1}$
$u^2 = x+1$
$2u\frac{du}{dx} = 1$
$2u\,du = dx$
$x = 0 \mapsto u = 1$
$x = 3 \mapsto u = 2$
$x = u^2 - 1$

62. $$\int_2^3 \frac{x^2+x+2}{x^2+2x-3} \ dx = 1 + \ln\left(\frac{25}{18}\right)$$

$$\int_2^3 \frac{x^2+x+2}{x^2+2x-3} \, dx \Leftarrow$$ THESE ARE IMPROPER PARTIAL FRACTIONS
$$= \int_2^3 1 - \frac{2}{x+3} + \frac{1}{x-1} \, dx$$
$$= \Big[x - 2\ln|x+3| + \ln|x-1|\Big]_2^3$$
$$= (3 - 2\ln 6 + \ln 2) - (2 - 2\ln 5 + \ln 1)$$
$$= 1 - 2\ln 6 + \ln 2 + 2\ln 5$$
$$= 1 + \ln 2 + \ln 25 - \ln 36 = 1 + \ln\left(\tfrac{2\times 25}{36}\right) = 1 + \ln\left(\tfrac{25}{18}\right)$$

BY PARTIAL FRACTIONS
$$\frac{x^2+x+2}{x^2+2x-3} \equiv A + \frac{B}{x+3} + \frac{C}{x-1}$$
$$\boxed{x^2+x+2 \equiv A(x+3)(x-1) + B(x-1) + C(x+3)}$$
- If $x=1$, $4 = 4C \Rightarrow \boxed{C=1}$
- If $x=-3$, $9-3+2 = -4B \Rightarrow \boxed{B=-2}$
- If $x=0$, $2 = -3A - B + 3C$, $2 = -3A + 2 + 3 \Rightarrow \boxed{A=1}$

63. $$\int_0^1 \frac{9}{(2x+1)^2} \ dx = 3$$

$$\int_0^1 \frac{9}{(2x+1)^2} \, dx = \int_0^1 9(2x+1)^{-2} \, dx = \left[-\tfrac{9}{2}(2x+1)^{-1}\right]_0^1 = \tfrac{9}{2}\left[\tfrac{1}{2x+1}\right]_1^0$$
$$= \tfrac{9}{2}\left[1 - \tfrac{1}{3}\right] = \tfrac{9}{2} \times \tfrac{2}{3} = 3$$

64. $$\int_0^{\frac{\pi}{6}} \sin x \sin 3x \ dx = \frac{\sqrt{3}}{16}$$

$$\int_0^{\frac{\pi}{6}} \sin x \sin 3x \, dx$$
$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x \, dx$$
$$= \left[\frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x\right]_0^{\frac{\pi}{6}}$$
$$= \left(\frac{1}{4}\frac{\sqrt{3}}{2} - \frac{1}{8}\frac{\sqrt{3}}{2}\right) - (0) = \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{16} = \frac{\sqrt{3}}{16}$$

$\cos(3x+x) = \cos 3x \cos x - \sin 3x \sin x$
$\cos(3x-x) = \cos 3x \cos x + \sin 3x \sin x$
SUBTRACT "UPWARDS"
$\cos 2x - \cos 4x = 2\sin 3x \sin x$
$\sin 3x \sin x = \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x$

65. $$\int_0^{\sqrt{2}} x^3 \ln\left(x^2+2\right) \ dx = \frac{1}{2} + \ln 2$$

$$\int_0^{\sqrt{2}} x^3 \ln(x^2+2) \, dx = \dots \text{ BY SUBSTITUTION FIRST} \dots$$
$$= \int_2^4 x^3 \ln u \, \frac{du}{2x} = \int_2^4 \frac{1}{2}x^2 \ln u \, du$$
$$= \int_2^4 \frac{1}{2}(u-2)\ln u \, du \ \dots \text{ BY PARTS, IGNORING LIMITS}$$
$$= \frac{1}{4}(u-2)^2 \ln u - \int \frac{(u-2)^2}{4u} \, du$$
$$= \frac{1}{4}(u-2)^2 \ln u - \int \frac{u^2-4u+4}{4u} \, du$$
$$= \frac{1}{4}(u-2)^2 \ln u - \int \frac{1}{4}u - 1 + \frac{1}{u} \, du$$
$$= \left[\frac{1}{4}(u-2)^2 \ln u - \frac{1}{8}u^2 + u - \ln|u|\right]_2^4$$
$$= (\cancel{\ln 4} - 2 + 4 - \cancel{\ln 4}) - \left(0 - \frac{1}{2} + 2 - \ln 2\right)$$
$$= 2 - \frac{3}{2} + \ln 2 = \frac{1}{2} + \ln 2$$

$u = x^2+2$
$\frac{du}{dx} = 2x$
$dx = \frac{du}{2x}$
$x = 0 \mapsto u = 2$
$x = \sqrt{2} \mapsto u = 4$

$\ln u$	$\frac{1}{u}$
$\frac{1}{4}(u-2)^2$	$\frac{1}{2}(u-2)$

66. $$\int_0^{\frac{1}{4}} \frac{4}{(1+2x)(1-2x)} \ dx = \ln 3$$

$$\int_0^{\frac{1}{4}} \frac{4}{(1+2x)(1-2x)} \, dx$$
$$= \int_0^{\frac{1}{4}} \frac{2}{1+2x} + \frac{2}{1-2x} \, dx$$
$$= \Big[\ln|1+2x| - \ln|1-2x|\Big]_0^{\frac{1}{4}}$$
$$= \left(\ln\frac{3}{2} - \ln\frac{1}{2}\right) - (\cancel{\ln 1} - \cancel{\ln 1})$$
$$= \ln\left(\frac{3/2}{1/2}\right) = \ln 3$$

BY PARTIAL FRACTIONS
$\frac{4}{(1+2x)(1-2x)} \equiv \frac{A}{1+2x} + \frac{B}{1-2x}$
$\boxed{4 \equiv A(1-2x) + B(1+2x)}$
- If $x = \frac{1}{2}$, $4 = 2B \Rightarrow \boxed{B=2}$
- If $x = -\frac{1}{2}$, $4 = 2A \Rightarrow \boxed{A=2}$

67. $$\int_0^{\frac{\pi}{2}} \cos^3 x \ dx = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \cos^3 x \, dx = \int_0^{\frac{\pi}{2}} \cos^2 x \cos x \, dx = \int_0^{\frac{\pi}{2}} (1-\sin^2 x)\cos x \, dx = \int_0^{\frac{\pi}{2}} \cos x - \sin^2 x \cos x \, dx$$

$$= \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\frac{\pi}{2}} = \left(1-\frac{1}{3}\right) - (0-0) = \frac{2}{3}$$

ALTERNATIVE BY SUBSTITUTION

$$\int_0^{\frac{\pi}{2}} \cos^3 x \, dx = \int_0^1 \cos^3 x \frac{du}{\cos x} = \int_0^1 \cos^2 x \, du$$

$$= \int_0^1 1 - \sin^2 x \, du = \int_0^1 1 - u^2 \, du$$

$$= \left[u - \frac{1}{3}u^3\right]_0^1 = \left(1 - \frac{1}{3}\right) - (0-0) = \frac{2}{3}$$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$x = 0 \mapsto u = 0$

$x = \frac{\pi}{2} \mapsto u = 1$

AS ABOVE

68. $$\int_0^3 \frac{4}{2x+3} \ dx = \ln 9$$

$$\int_0^3 \frac{4}{2x+3} \, dx = \Big[2\ln|2x+3|\Big]_0^3 = 2\ln 9 - 2\ln 3 = 2(\ln 9 - \ln 3)$$

$$= 2\ln 3 = \ln 9$$

69. $$\int_0^2 \frac{6x^3}{\sqrt{x^2+1}} \ dx = 4\left(1+\sqrt{5}\right)$$

$$\int_0^2 \frac{6x^3}{\sqrt{x^2+1}} \, dx = \int_1^{\sqrt{5}} \frac{6x^3}{\cancel{u}} \frac{\cancel{u}\,du}{x} = \int_1^{\sqrt{5}} 6x^2 \, du$$

$$= \int_1^{\sqrt{5}} 6(u^2-1) \, du = \Big[2u^3 - 6u\Big]_1^{\sqrt{5}}$$

$$= \left(2 \times 5\sqrt{5} - 6\sqrt{5}\right) - (2-6) = 4\sqrt{5} - (-4)$$

$$= 4 + 4\sqrt{5} = 4\left(1+\sqrt{5}\right)$$

$u = \sqrt{x^2+1}$

$u^2 = x^2 + 1$

$2u\frac{du}{dx} = 2x$

$dx = \frac{u\,du}{x}$

$x = 0 \mapsto u = 1$

$x = 2 \mapsto u = \sqrt{5}$

$x^2 = u^2 - 1$

70. $$\int_5^8 \frac{2x^2}{x^2-16} \ dx = 6 + 4\ln 3$$

$$\int_5^8 \frac{2x^2}{x^2-16} \, dx = \int_5^8 2 - \frac{4}{x+4} + \frac{4}{x-4} \, dx$$

$$= \Big[2x - 4\ln|x+4| + 4\ln|x-4|\Big]_5^8$$

$$= (16 - 4\ln 12 + 4\ln 4) - (10 - 4\ln 9 + 4\ln 1)$$

$$= 16 - 4\ln 12 + 4\ln 4 - 10 + 4\ln 9$$

$$= 6 + 4\ln\left(\frac{9 \times 4}{12}\right) = 6 + 4\ln 3$$

THESE ARE IMPROPER PARTIAL FRACTIONS!

$$\frac{2x^2}{x^2-16} \equiv A + \frac{B}{x+4} + \frac{C}{x-4}$$

$$\boxed{2x^2 \equiv A(x+4)(x-4) + B(x-4) + C(x+4)}$$

- If $x = 4$, $32 = 8C \Rightarrow \boxed{C = 4}$
- If $x = -4$ $32 = -8B \Rightarrow \boxed{B = -4}$
- If $x = 0$ $0 = -16A - 4B + 4C$, $0 = -16A + 16 + 16$, $16A = 32$ $\boxed{A = 2}$

71. $$\int_1^2 \frac{\ln x}{x}\, dx = \frac{1}{2}(\ln 2)^2$$

$\int_1^2 \frac{\ln x}{x}\, dx$ RECOGNITION OR SUBSTITUTION

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\, du$

$x = 1 \mapsto u = 0$

$x = 2 \mapsto u = \ln 2$

$= \int_0^{\ln 2} \frac{u}{\cancel{x}}\, \cancel{x}\, du = \int_0^{\ln 2} u\, du = \left[\frac{1}{2}u^2\right]_0^{\ln 2}$

$= \frac{1}{2}\left[(\ln 2)^2 - 0\right] = \frac{1}{2}(\ln 2)^2$

BY PARTS

$\ln x \to \frac{1}{x}$

$\ln x \to \frac{1}{x}$

$\int_1^2 \ln x \times \frac{1}{x}\, dx = \left[(\ln x)^2\right]_1^2 - \int_1^2 \ln x \times \frac{1}{x}\, dx$

$2\int_1^2 \frac{\ln x}{x}\, dx = \left[(\ln x)^2\right]_1^2$

$\int \frac{\ln x}{x}\, dx = \frac{1}{2}\left[(\ln 2)^2 - 0\right] = \frac{1}{2}(\ln 2)^2$

72. $$\int_0^1 \frac{17-5x}{(3+2x)(2-x)^2}\, dx = \frac{1}{2} + \ln\left(\frac{10}{3}\right)$$

$\int_0^1 \frac{17-5x}{(3+2x)(2-x)^2}\, dx$ BY PARTIAL FRACTIONS

$\frac{17-5x}{(3+2x)(2-x)^2} \equiv \frac{A}{3+2x} + \frac{B}{(2-x)^2} + \frac{C}{2-x}$

$\boxed{17-5x \equiv A(2-x)^2 + B(3+2x) + C(2-x)(3+2x)}$

- If $x = 2 \Rightarrow 7 = 7B \Rightarrow \boxed{B = 1}$
- If $x = -\frac{3}{2} \Rightarrow \frac{49}{2} = \frac{49}{4}A \Rightarrow \boxed{A = 2}$
- If $x = 0 \Rightarrow 17 = 4A + 3B + 6C$

$17 = 8 + 3 + 6C$

$6 = 6C$

$\boxed{C = 1}$

$= \int_0^1 \frac{2}{3+2x} + (2-x)^{-2} + \frac{1}{2-x}\, dx$

$= \left[\ln|3+2x| + (2-x)^{-1} - \ln|2-x|\right]_0^1$

$= \left[\frac{1}{2-x} + \ln\left|\frac{2x+3}{2-x}\right|\right]_0^1$

$= (1 + \ln 5) - \left(\frac{1}{2} + \ln\frac{3}{2}\right)$

$= 1 + \ln 5 - \frac{1}{2} - \ln\frac{3}{2} = \frac{1}{2} + \ln\left(\frac{5}{\frac{3}{2}}\right) = \frac{1}{2} + \ln\left(\frac{10}{3}\right)$

73. $$\int_0^{\pi} x\cos\left(\tfrac{1}{4}x\right)\, dx = 2\sqrt{2}(\pi + 4) - 16$$

$\int_0^{\pi} x\cos\left(\frac{1}{4}x\right)\, dx =$ BY PARTS & IGNORING LIMITS

$x \to 1$

$4\sin\left(\frac{1}{4}x\right) \to \cos\left(\frac{1}{4}x\right)$

$= 4x\sin\frac{1}{4}x - \int 4\sin\left(\frac{1}{4}x\right) dx$

$= \left[4x\sin\frac{1}{4}x + 16\cos\left(\frac{1}{4}x\right)\right]_0^{\pi}$

$= \left(4\pi\sin\frac{\pi}{4} + 16\cos\frac{\pi}{4}\right) - (0 + 16\cos 0)$

$= 2\pi\sqrt{2} + 8\sqrt{2} - 16$

$= 2\sqrt{2}(\pi + 4) - 16$

74. $$\int_0^4 e^{\frac{1}{2}x}\, dx = 2\left(e^2 - 1\right)$$

$\int_0^4 e^{\frac{1}{2}x}\, dx = \left[2e^{\frac{1}{2}x}\right]_0^4 = 2e^2 - 2 = 2(e^2 - 1)$

75. $$\int_4^9 \frac{5x^2-8x+1}{2x(x-1)^2}\,dx = \ln\left(\frac{32}{3}\right)-\frac{5}{24}$$

$$\int_4^9 \frac{5x^2-8x+1}{2x(x-1)^2}\,dx \xrightarrow{\text{PARTIAL FRACTION}} \left\{ \frac{5x^2-8x+1}{2x(x-1)^2} \equiv \frac{A}{2x}+\frac{B}{(x-1)^2}+\frac{C}{x-1} \right\}$$

$$\boxed{5x^2-8x+1 \equiv A(x-1)^2+B(2x)+C(2x)(x-1)}$$

- If $x=1$, $-2=2B \Rightarrow \boxed{B=-1}$
- If $x=0$, $1=A \Rightarrow \boxed{A=1}$
- If $x=2$, $20-16+1=A+4B+4C$

$$5=1-4+4C$$
$$8=4C$$
$$\boxed{C=2}$$

$$= \int_4^9 \frac{1}{2x}-(x-1)^{-2}+\frac{2}{x-1}\,dx$$

$$= \left[\frac{1}{2}\ln|x|+(x-1)^{-1}+2\ln|x-1|\right]_4^9$$

$$= \left(\frac{1}{2}\ln 9+\frac{1}{8}+2\ln 8\right)-\left(\frac{1}{2}\ln 4+\frac{1}{3}+2\ln 3\right)$$

$$= \ln 3+\frac{1}{8}+\ln 64-\ln 2-\frac{1}{3}-2\ln 3$$

$$= \frac{-5}{24}-\ln 3+\ln 32 = \ln\left(\frac{32}{3}\right)-\frac{5}{24}$$

76. $$\int_0^1 \frac{3}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+1\right)}\,dx = -\ln 4$$

$$\int_0^1 \frac{3}{(\sqrt{x}-2)(\sqrt{x}+1)}\,dx = \int_0^1 \frac{3}{(u-2)(u+1)}\,2u\,du = \int_0^1 \frac{6u}{(u-2)(u+1)}\,du$$

$$\left\{ u=\sqrt{x},\quad u^2=x,\quad 2u\frac{du}{dx}=1,\quad 2u\,du=dx,\quad x=0 \mapsto u=0,\quad x=1 \mapsto u=1 \right\}$$

BY PARTIAL FRACTIONS

$$\frac{6u}{(u-2)(u+1)} \equiv \frac{A}{u-2}+\frac{B}{u+1}$$

$$\boxed{6u \equiv A(u+1)+B(u-2)}$$

- If $u=2 \Rightarrow 12=3A$, $A=4$
- If $u=-1 \Rightarrow -6=-3B$, $B=2$

$$= \int_0^1 \frac{4}{u-2}+\frac{2}{u+1}\,du = \Big[4\ln|u-2|+2\ln|u+1|\Big]_0^1$$

$$= \left(\cancel{4\ln|-1|}+2\ln 2\right)-\left(4\ln|-2|+\cancel{2\ln 1}\right) = 2\ln 2-4\ln 2 = -2\ln 2$$

$$= -\ln 4$$

77. $$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\,dx = 1+\sqrt{3}$$

$$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\,dx = \int_0^{\frac{\pi}{3}} \frac{1+\sin x}{(1-\sin x)(1+\sin x)}\,dx = \int_0^{\frac{\pi}{3}} \frac{1+\sin x}{1-\sin^2 x}\,dx$$

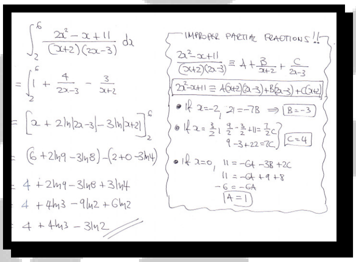
$$= \int_0^{\frac{\pi}{3}} \frac{1+\sin x}{\cos^2 x}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\,dx$$

$$= \int_0^{\frac{\pi}{3}} \sec^2 x+\frac{\sin x}{\cos x}\times\frac{1}{\cos x}\,dx = \int_0^{\frac{\pi}{3}} \sec^2 x+\tan x\sec x\,dx$$

$$= \Big[\tan x+\sec x\Big]_0^{\frac{\pi}{3}} = \left(\sqrt{3}+2\right)-(0+1)$$

$$= 1+\sqrt{3}$$

78. $$\int_{2}^{6} \frac{2x^2 - x + 11}{(x+2)(2x-3)}\, dx = 4 + 4\ln 3 - 3\ln 2$$


$\int_{2}^{6} \frac{2x^2-x+11}{(x+2)(2x-3)}\,dx$

$= \int_{2}^{6} 1 + \frac{4}{2x-3} - \frac{3}{x+2}$

$= \left[x + 2\ln|2x-3| - 3\ln|x+2|\right]_{2}^{6}$

$= (6 + 2\ln 9 - 3\ln 8) - (2 + 0 - 3\ln 4)$

$= 4 + 2\ln 9 - 3\ln 8 + 3\ln 4$

$= 4 + 4\ln 3 - 9\ln 2 + 6\ln 2$

$= 4 + 4\ln 3 - 3\ln 2$

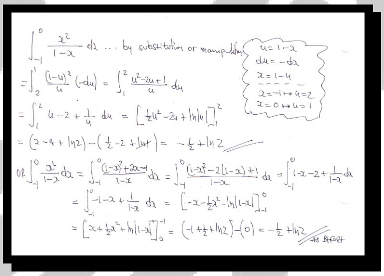
IMPROPER PARTIAL FRACTIONS!!

$\frac{2x^2-x+11}{(x+2)(2x-3)} \equiv A + \frac{B}{x+2} + \frac{C}{2x-3}$

$\boxed{2x^2-x+11 \equiv A(x+2)(2x-3) + B(2x-3) + C(x+2)}$

- If $x=-2$, $21 = -7B \Rightarrow \boxed{B=-3}$
- If $x=\frac{3}{2}$, $\frac{9}{2} - \frac{3}{2} + 11 = \frac{7}{2}C$; $9 - 3 + 22 = 7C$ $\boxed{C=4}$
- If $x=0$, $11 = -6A - 3B + 2C$; $11 = -6A + 9 + 8$; $-6 = -6A$; $\boxed{A=1}$

79. $$\int_{-1}^{0} \frac{x^2}{1-x}\, dx = -\frac{1}{2} + \ln 2$$


$\int_{-1}^{0} \frac{x^2}{1-x}\,dx$ ... by substitution or manipulation

$u = 1-x$, $du = -dx$, $x = 1-u$; $x=-1 \mapsto u=2$, $x=0 \mapsto u=1$

$= \int_{2}^{1} \frac{(1-u)^2}{u}(-du) = \int_{1}^{2} \frac{u^2-2u+1}{u}\,du$

$= \int_{1}^{2} u - 2 + \frac{1}{u}\,du = \left[\frac{1}{2}u^2 - 2u + \ln|u|\right]_{1}^{2}$

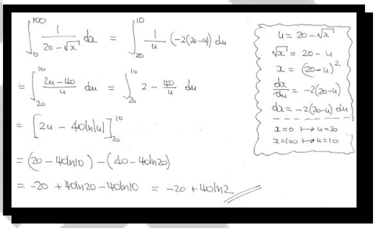
$= (2 - 4 + \ln 2) - (\frac{1}{2} - 2 + \cancel{\ln 1}) = -\frac{1}{2} + \ln 2$

OR $\int_{-1}^{0} \frac{x^2}{1-x}dx = \int_{-1}^{0} \frac{(1-x)^2 + 2x - 1}{1-x}dx = \int_{-1}^{0} \frac{(1-x)^2 - 2(1-x) + 1}{1-x}dx = \int_{-1}^{0} 1 - x - 2 + \frac{1}{1-x}dx$

$= \int_{-1}^{0} -1 - x + \frac{1}{1-x}\,dx = \left[-x - \frac{1}{2}x^2 - \ln|1-x|\right]_{-1}^{0}$

$= \left[x + \frac{1}{2}x^2 + \ln|1-x|\right]_{0}^{-1} = (-1 + \frac{1}{2} + \ln 2) - (0) = -\frac{1}{2} + \ln 2$ AS BEFORE

80. $$\int_{0}^{100} \frac{1}{20 - \sqrt{x}}\, dx = 40\ln 2 - 20$$


$\int_{0}^{100} \frac{1}{20-\sqrt{x}}\,dx = \int_{20}^{10} \frac{1}{u}(-2(20-u))\,du$

$= \int_{20}^{10} \frac{2u-40}{u}\,du = \int_{20}^{10} 2 - \frac{40}{u}\,du$

$= \left[2u - 40\ln|u|\right]_{20}^{10}$

$= (20 - 40\ln 10) - (40 - 40\ln 20)$

$= -20 + 40\ln 20 - 40\ln 10 = -20 + 40\ln 2$

$u = 20 - \sqrt{x}$, $\sqrt{x} = 20 - u$, $x = (20-u)^2$, $\frac{dx}{du} = -2(20-u)$, $dx = -2(20-u)\,du$

$x = 0 \mapsto u = 20$, $x = 100 \mapsto u = 10$

81. $\displaystyle\int_0^1 \frac{x^2}{x^2-4}\,dx = 1-\ln 3$

$$\int_0^1 \frac{x^2}{x^2-4}\,dx \quad \xrightarrow[\text{PARTIAL FRACTIONS}]{\text{! IMPROPER !}}$$

$$\frac{x^2}{x^2-4} \equiv A + \frac{B}{x-2} + \frac{C}{x+2}$$

$$\boxed{x^2 \equiv A(x-2)(x+2) + B(x+2) + C(x-2)}$$

- if $x=2$, $4=4B \Rightarrow \boxed{B=1}$
- if $x=-2$, $4=-4C \Rightarrow \boxed{C=-1}$
- if $x=0$, $0=-4A+2B-2C$
  $0=-4A+2+2$
  $4A=4$
  $\boxed{A=1}$

$$= \int_0^1 1 + \frac{1}{x-2} - \frac{1}{x+2}\,dx$$

$$= \Big[x + \ln|x-2| - \ln|x+2|\Big]_0^1$$

$$= \big(1+\ln|-1| - \ln 3\big) - \big(0 + \ln|-2| - \ln 2\big)$$

$$= 1 + \cancel{\ln 1} - \ln 3 - \cancel{\ln 2} + \cancel{\ln 2}$$

$$= 1 - \ln 3$$

82. $\displaystyle\int_0^{\frac{\pi}{6}} \sin^3\theta \; d\theta = \frac{2}{3} - \frac{3}{8}\sqrt{3} = \frac{1}{24}\left(16 - 9\sqrt{3}\right)$

$$\int_0^{\frac{\pi}{6}} \sin^3\theta\,d\theta = \int_0^{\frac{\pi}{6}} \sin^2\theta\sin\theta\,d\theta = \int_0^{\frac{\pi}{6}} (1-\cos^2\theta)\sin\theta\,d\theta$$

$$= \int_0^{\pi/6} \sin\theta - \cos^2\theta\sin\theta\,d\theta = \Big[-\cos\theta + \tfrac{1}{3}\cos^3\theta\Big]_0^{\frac{\pi}{6}}$$

$$= \left(-\frac{\sqrt{3}}{2} + \frac{1}{3}\times\frac{3\sqrt{3}}{8}\right) - \left(-1+\frac{1}{3}\right) = \left(-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8}\right) - \left(-\frac{2}{3}\right)$$

$$= \frac{2}{3} - \frac{3}{8}\sqrt{3}$$

ALTERNATIVE BY SUBSTITUTION

Let $u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

$d\theta = -\frac{du}{\sin\theta}$

$\theta = 0 \mapsto u = 1$

$\theta = \frac{\pi}{6} \mapsto u = \frac{\sqrt{3}}{2}$

$$\int_0^{\frac{\pi}{6}} \sin^3\theta\,d\theta = \int_1^{\sqrt{3}/2} \sin^{\cancel{3}\,2}\theta \left(-\frac{du}{\cancel{\sin\theta}}\right)$$

$$= \int_1^{\sqrt{3}/2} -\sin^2\theta\,du = \int_{\sqrt{3}/2}^1 \sin^2\theta\,du = \int_{\sqrt{3}/2}^1 1-\cos^2\theta\,du$$

$$= \int_{\sqrt{3}/2}^1 1-u^2\,du = \Big[u - \tfrac{1}{3}u^3\Big]_{\sqrt{3}/2}^1$$

$$= \left(1-\frac{1}{3}\right) - \left(\frac{\sqrt{3}}{2} - \frac{1}{3}\left(\frac{3\sqrt{3}}{8}\right)\right) = \frac{2}{3} - \frac{3}{8}\sqrt{3}$$

AS BEFORE

83. $\displaystyle\int_0^{\ln 2} \frac{1}{1+e^x}\,dx = \ln\left(\frac{4}{3}\right)$

$u = 1+e^x$

$\frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

$x = 0 \mapsto u = 2$

$x = \ln 2 \mapsto u = 3$

$e^x = u-1$

$$\int_0^{\ln 2} \frac{1}{1+e^x}\,dx = \int_2^3 \frac{1}{u}\,\frac{du}{e^x} = \int_2^3 \frac{1}{u(u-1)}\,du$$

BY PARTIAL FRACTIONS

$$\frac{1}{u(u-1)} \equiv \frac{A}{u} + \frac{B}{u-1}$$

$$\boxed{1 \equiv A(u-1) + Bu}$$

- if $u=1$, $B=1$
- if $u=0$, $A=-1$

$$= \dots \int_2^3 \frac{1}{u-1} - \frac{1}{u}\,du = \Big[\ln|u-1| - \ln|u|\Big]_2^3$$

$$= (\ln 2 - \ln 3) - (\cancel{\ln 1} - \ln 2) = 2\ln 2 - \ln 3 = \ln 4/3$$

84. $\displaystyle\int_0^4 e^{\sqrt{2x+1}}\,dx = 2e^3$

$$\int_0^4 e^{\sqrt{2x+1}}\,dx = \int_1^3 e^u\,(u\,du) = \int_1^3 ue^u\,du \ldots$$

$u = (2x+1)^{\frac{1}{2}}$
$u^2 = 2x+1$
$2u\frac{du}{dx} = 2$
$dx = u\,du$
$x=0 \mapsto u=1$
$x=4 \mapsto u=3$

... BY PARTS & IGNORING LIMITS ... $\begin{array}{c|c} u & 1 \\ \hline e^u & e^u \end{array}$ ...

$$= ue^u - \int e^u\,du = \left[ue^u - e^u\right]_1^3$$

$$= (3e^3 - e^3) - (\cancel{e^1 - e^1})$$

$$= 2e^3$$

85. $\displaystyle\int_0^1 \frac{x^3}{x+1}\,dx = \frac{5}{6} - \ln 2$

$$\int_0^1 \frac{x^3}{x+1} = \text{BY SUBSTITUTION} = \int_1^2 \frac{(u-1)^3}{u}\,du = \ldots$$

$u = x+1$
$\frac{du}{dx} = 1$
$dx = du$
$x=0 \mapsto u=1$
$x=1 \mapsto u=2$
$x = u-1$

- NOW $(u-1)^3 = (u-1)(u-1)^2 = (u-1)(u^2-2u+1)$
$= u^3 - 2u^2 + u$
$\quad - u^2 + 2u - 1$
$= u^3 - 3u^2 + 3u - 1$

$$= \int_1^2 \frac{u^3-3u^2+3u-1}{u}\,du = \int_1^2 u^2 - 3u + 3 - \frac{1}{u}\,du$$

$$= \left[\tfrac{1}{3}u^3 - \tfrac{3}{2}u^2 + 3u - \ln|u|\right]_1^2 = \left(\tfrac{8}{3} \cancel{-6} \cancel{+6} - \ln 2\right) - \left(\tfrac{1}{3} - \tfrac{3}{2} + 3 - \cancel{\ln 1}\right)$$

$$= \tfrac{5}{6} - \ln 2$$

ALTERNATIVE BY MANIPULATING

$$\int_0^1 \frac{x^3}{x+1}\,dx = \int_0^1 \frac{x^2(x+1) - x(x+1) + (x+1) - 1}{(x+1)}\,dx = \int_0^1 x^2 - x + 1 - \frac{1}{x+1}\,dx$$

$$= \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + x - \ln|x+1|\right]_0^1 = \left(\tfrac{1}{3} - \tfrac{1}{2} + 1 - \ln 2\right) - (\cancel{-\ln 1})$$

$$= \tfrac{5}{6} - \ln 2$$

86. $\displaystyle\int_0^1 \frac{10}{(x+1)(x+3)(2x+1)}\,dx = 3\ln 3 - 3\ln 2$

$$\int_0^1 \frac{10}{(x+1)(x+3)(2x+1)}\,dx$$

$$= \int_0^1 \frac{1}{x+3} - \frac{5}{x+1} + \frac{8}{2x+1}\,dx$$

$$= \Big[\ln|x+3| - 5\ln|x+1| + 4\ln|2x+1|\Big]_0^1$$

$$= (\ln 4 - 5\ln 2 + 4\ln 3) - (\ln 3 - 0)$$

$$= \ln 4 - 5\ln 2 + 4\ln 3 - \ln 3$$

$$= 2\ln 2 - 5\ln 2 + 3\ln 3$$

$$= 3\ln 3 - 3\ln 2$$

PARTIAL FRACTIONS

$$\frac{10}{(x+1)(x+3)(2x+1)} \equiv \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{2x+1}$$

$$\boxed{10 \equiv A(x+3)(2x+1) + B(x+1)(2x+1) + C(x+1)(x+3)}$$

- If $x=-3$: $10 = B(-2)(-5) \Rightarrow \boxed{B=1}$
- If $x=-1$: $10 = A(2)(-1) \Rightarrow \boxed{A=-5}$
- If $x=0$: $10 = 3A + B + 3C$
$10 = -15 + 1 + 3C$
$24 = 3C$
$\boxed{C=8}$

87. $$\int_0^{\frac{\pi}{2}} \left[1+\tan\left(\tfrac{1}{2}x\right)\right]^2 \, dx = 2+\ln 4$$

$$\int_0^{\frac{\pi}{2}} \left(1+\tan\tfrac{x}{2}\right)^2 dx = \int_0^{\frac{\pi}{2}} 1+2\tan\tfrac{x}{2}+\tan^2\tfrac{x}{2}\, dx$$
$$= \int_0^{\pi/2} 2\tan\tfrac{x}{2}+\sec^2\tfrac{x}{2}\, dx$$
$$= \left[4\ln\left|\sec\tfrac{x}{2}\right| + 2\tan\tfrac{x}{2}\right]_0^{\pi/2}$$
$$= \left(4\ln\left|\sec\tfrac{\pi}{4}\right| + 2\tan\tfrac{\pi}{4}\right) - \left(4\ln|\sec 0| + \cancel{2\tan 0}\right)$$
$$= 4\ln(\sqrt{2}) + 2 - \cancel{4\ln 1}$$
$$= 2\ln 2 + 2$$
$$= 2 + \ln 4$$

88. $$\int_0^{\frac{\pi}{3}} \frac{\sin 2x}{1+\cos x} \, dx = 1+2\ln\left(\tfrac{3}{4}\right)$$

$$\int_0^{\pi/3} \frac{\sin 2x}{1+\cos x}\, dx = \int_0^{\pi/3} \frac{2\sin x\cos x}{1+\cos x}\, dx = \ldots \text{ by substitution}$$

$u = 1+\cos x$

$\frac{du}{dx} = -\sin x$

$dx = -\frac{du}{\sin x}$

$\cos x = u-1$

$x=0 \mapsto u=2$

$x=\frac{\pi}{3} \mapsto u=\frac{3}{2}$

$$= \int_2^{3/2} \frac{2\cancel{\sin x}\cos x}{u}\left(-\frac{du}{\cancel{\sin x}}\right) = \int_{\frac{3}{2}}^{2} \frac{2\cos x}{u}\, du = \int_{\frac{3}{2}}^{2} \frac{2(u-1)}{u}\, du$$
$$= \int_{3/2}^{2} 2-\frac{2}{u}\, du = \Big[2u - 2\ln|u|\Big]_{\frac{3}{2}}^{2} = (4-2\ln 2)-\left(3-2\ln\tfrac{3}{2}\right)$$
$$= 4-2\ln 2-3+2\ln\tfrac{3}{2} = 1+2\left(\ln\tfrac{3}{2}-\ln 2\right) = 1+2\ln\tfrac{3}{4}$$

89. $$\int_0^{1} 3x^2 \ln(x+1) \, dx = -\frac{5}{6}+\ln 4$$

$$\int_0^1 3x^2\ln(x+1)\, dx \ \ldots \text{ BY SUBSTITUTION FIRST}$$

$u = x+1$

$\frac{du}{dx} = 1$

$du = dx$

$x = u-1$

$x=0 \mapsto u=1$

$x=1 \mapsto u=2$

$$= \int_1^2 3(u-1)^2\ln u\, du \ \ldots \text{ THEN PARTS (IGNORE LIMITS)}$$

$\ln u$	$\frac{1}{u}$
$(u-1)^3$	$3(u-1)^2$

$(u-1)^3 = u^3-3u^2+3u-1$

$$= (u-1)^3\ln u - \int \frac{(u-1)^3}{u}\, du$$
$$= (u-1)^3\ln u - \int \frac{u^3-3u^2+3u-1}{u}\, du$$
$$= (u-1)^3\ln u - \int u^2-3u+3-\frac{1}{u}\, du$$
$$= (u-1)^3\ln u - \left(\tfrac{1}{3}u^3-\tfrac{3}{2}u^2+3u-\ln u\right)$$
$$= \left[(u-1)^3\ln u - \tfrac{1}{3}u^3+\tfrac{3}{2}u^2-3u+\ln u\right]_1^2$$
$$= \left(\ln 2-\tfrac{8}{3}\cancel{+6}\cancel{-6}+\ln 2\right)-\left(0-\tfrac{1}{3}+\tfrac{3}{2}-3\cancel{+\ln 1}\right)$$
$$= -\tfrac{5}{6}+2\ln 2 = -\tfrac{5}{6}+\ln 4$$

90. $$\int_{0}^{\frac{1}{4}} 2x\sqrt{1-4x}\ dx = \frac{1}{30}$$

$$\int_0^{\frac{1}{4}} 2x\sqrt{1-4x}\,dx = \int_1^0 2xu\left(-\tfrac{1}{2}u\,du\right) = \int_0^1 xu^2\,du$$
$$= \frac{1}{4}\int_0^1 4xu^2\,du \qquad 4x = 1-u^2$$
$$= \frac{1}{4}\int_0^1 (1-u^2)u^2\,du = \frac{1}{4}\int_0^1 u^2-u^4\,du$$
$$= \frac{1}{4}\left[\frac{1}{3}u^3-\frac{1}{5}u^5\right]_0^1$$
$$= \frac{1}{4}\left[\left(\frac{1}{3}-\frac{1}{5}\right)-(0)\right] = \frac{1}{30}$$

$u=\sqrt{1-4x}$
$u^2 = 1-4x$
$2u\frac{du}{dx} = -4$
$2u\,du = -4\,dx$
$dx = -\frac{1}{2}u\,du$
$x=0 \quad u=1$
$x=\frac{1}{4} \quad u=0$

91. $$\int_{1}^{e} x\left(1-\ln x\right)\ dx = \frac{1}{4}\left(e^2-3\right)$$

$$\int_1^e x(1-\ln x)\,dx = \text{BY PARTS \& IGNORING LIMITS} = \ldots$$
$$= \frac{1}{2}x^2(1-\ln x) - \int -\frac{1}{2}x\,dx$$
$$= \frac{1}{2}x^2(1-\ln x) + \int \frac{1}{2}x\,dx$$
$$= \frac{1}{2}x^2(1-\ln x) + \frac{1}{4}x^2 + C$$

$1-\ln x$	$-\frac{1}{x}$
$\frac{1}{2}x^2$	x

LIMITS NEXT ..

$$\int_1^e x(1-\ln x)\,dx = \left[\frac{1}{2}x^2(1-\ln x)+\frac{1}{4}x^2\right]_1^e$$
$$= \left(0+\frac{1}{4}e^2\right)-\left(\frac{1}{2}+\frac{1}{4}\right)$$
$$= \frac{1}{4}e^2-\frac{3}{4}$$
$$= \frac{1}{4}\left(e^2-3\right)$$

92. $$\int_{-\frac{1}{3}}^{0} \frac{1}{3-6x-9x^2}\ dx = \frac{1}{12}\ln 3$$

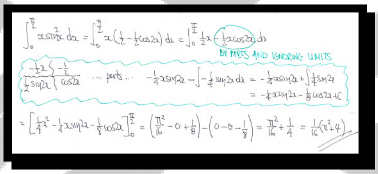
$$\int_{-\frac{1}{3}}^0 \frac{1}{3-6x-9x^2}\,dx = \int_0^{-\frac{1}{3}} \frac{1}{9x^2+6x-3}\,dx = \int_0^{-\frac{1}{3}} \frac{1}{3(3x^2+2x-1)}\,dx$$
$$= \frac{1}{3}\int_0^{-\frac{1}{3}} \frac{1}{(3x-1)(x+1)}\,dx$$
$$= \frac{1}{3}\int_0^{-\frac{1}{3}} \frac{\frac{3}{4}}{3x-1} - \frac{\frac{1}{4}}{x+1}\,dx$$
$$= \frac{1}{12}\int_0^{-\frac{1}{3}} \frac{3}{3x-1} - \frac{1}{x+1}\,dx$$
$$= \frac{1}{12}\Big[\ln|3x-1| - \ln|x+1|\Big]_0^{-\frac{1}{3}}$$
$$= \frac{1}{12}\left[\left(\ln|-2| - \ln\frac{2}{3}\right) - \left(\cancel{\ln|-1|} - \cancel{\ln 1}\right)\right] = \frac{1}{12}\left(\ln 2 - \ln\frac{2}{3}\right)$$
$$= \frac{1}{12}\left(\ln 2 + \ln\frac{3}{2}\right) = \frac{1}{12}\ln\left(2\times\frac{3}{2}\right) = \frac{1}{12}\ln 3$$

PARTIAL FRACTIONS

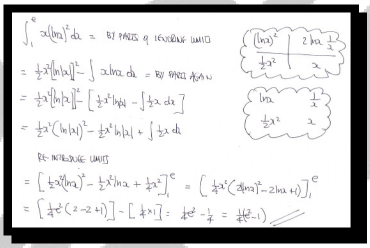
$$\frac{1}{(3x-1)(x+1)} \equiv \frac{A}{3x-1} + \frac{B}{x+1}$$
$$\boxed{1 \equiv A(x+1) + B(3x-1)}$$

- If $x=-1$, $1=-4B \Rightarrow \boxed{B=-\frac{1}{4}}$
- If $x=\frac{1}{3}$, $1=\frac{4}{3}A \Rightarrow \boxed{A=\frac{3}{4}}$

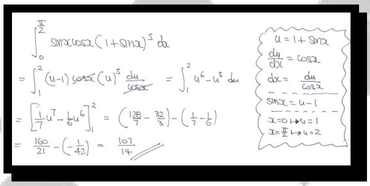
93. $$\int_0^{\frac{\pi}{2}} x\sin^2 x \ \ dx = \frac{1}{16}\left(\pi^2+4\right)$$



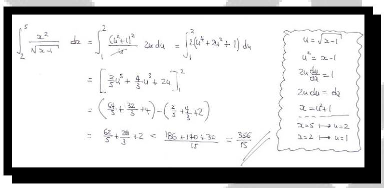
94. $$\int_1^{e} x(\ln x)^2 \ \ dx = \tfrac{1}{4}\left(e^2-1\right)$$



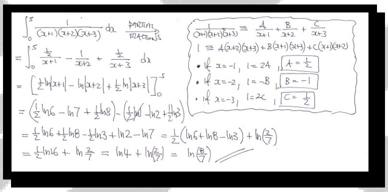
95. $$\int_0^{\frac{\pi}{2}} \sin x\cos x(1+\sin x)^5 \ \ dx = \frac{107}{14}$$



96. $$\int_2^{5} \frac{x^2}{\sqrt{x-1}} \ dx = \frac{356}{15}$$



97. $$\int_0^5 \frac{1}{(x+1)(x+2)(x+3)}\, dx = \ln\left(\frac{8}{7}\right)$$



$\int_0^5 \frac{1}{(x+1)(x+2)(x+3)}\, dx \xrightarrow[\text{FRACTIONS}]{\text{PARTIAL}}$

$\frac{1}{(x+1)(x+2)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$

$1 \equiv A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

• If $x=-1$, $1 = 2A$, $\boxed{A = \frac{1}{2}}$

• If $x=-2$, $1 = -B$, $\boxed{B = -1}$

• If $x=-3$, $1 = 2C$, $\boxed{C = \frac{1}{2}}$

$= \int_0^5 \frac{\frac{1}{2}}{x+1} - \frac{1}{x+2} + \frac{\frac{1}{2}}{x+3}\, dx$

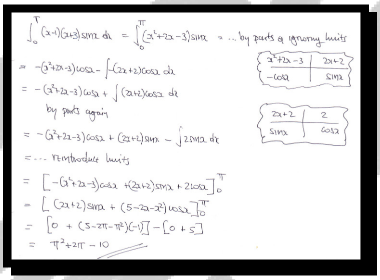
$= \left[\frac{1}{2}\ln|x+1| - \ln|x+2| + \frac{1}{2}\ln|x+3|\right]_0^5$

$= \left(\frac{1}{2}\ln 6 - \ln 7 + \frac{1}{2}\ln 8\right) - \left(\cancel{\frac{1}{2}\ln 1} - \ln 2 + \frac{1}{2}\ln 3\right)$

$= \frac{1}{2}\ln 6 + \frac{1}{2}\ln 8 - \frac{1}{2}\ln 3 + \ln 2 - \ln 7 = \frac{1}{2}(\ln 6 + \ln 8 - \ln 3) + \ln\left(\frac{2}{7}\right)$

$= \frac{1}{2}\ln 16 + \ln\frac{2}{7} = \ln 4 + \ln\left(\frac{2}{7}\right) = \ln\left(\frac{8}{7}\right)$

98. $$\int_0^{\pi} (x-1)(x+3)\sin x \ \ dx = \pi^2 + 2\pi - 10$$



$\int_0^{\pi} (x-1)(x+3)\sin x\, dx = \int_0^{\pi} (x^2+2x-3)\sin x = \dots$ by parts & ignoring limits

x^2+2x-3	$2x+2$
$-\cos x$	$\sin x$

$= -(x^2+2x-3)\cos x - \int -(2x+2)\cos x\, dx$

$= -(x^2+2x-3)\cos x + \int (2x+2)\cos x\, dx$

by parts again

$2x+2$	2
$\sin x$	$\cos x$

$= -(x^2+2x-3)\cos x + (2x+2)\sin x - \int 2\sin x\, dx$

$= \dots$ reintroduce limits

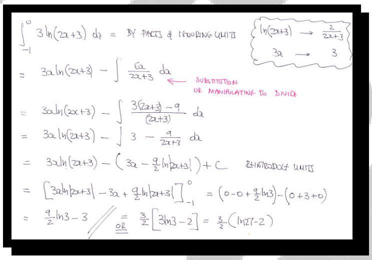
$= \left[-(x^2+2x-3)\cos x + (2x+2)\sin x + 2\cos x\right]_0^{\pi}$

$= \left[(2x+2)\sin x + (5-2x-x^2)\cos x\right]_0^{\pi}$

$= \left[0 + (5-2\pi-\pi^2)(-1)\right] - \left[0 + 5\right]$

$= \pi^2 + 2\pi - 10$

99. $$\int_{-1}^{0} 3\ln(2x+3)\ \ dx = \frac{3}{2}(\ln 27 - 2)$$



$\int_{-1}^{0} 3\ln(2x+3)\, dx =$ BY PARTS & IGNORING LIMITS

$\ln(2x+3) \rightarrow \frac{2}{2x+3}$

$3x \rightarrow 3$

$= 3x\ln(2x+3) - \int \frac{6x}{2x+3}\, dx \leftarrow$ SUBSTITUTION OR MANIPULATING TO DIVIDE

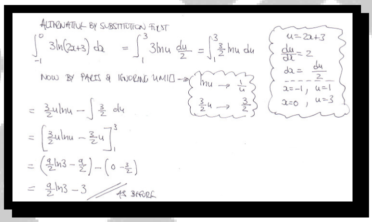
$= 3x\ln(2x+3) - \int \frac{3(2x+3) - 9}{(2x+3)}\, dx$

$= 3x\ln(2x+3) - \int 3 - \frac{9}{2x+3}\, dx$

$= 3x\ln(2x+3) - \left(3x - \frac{9}{2}\ln|2x+3|\right) + C$ REINTRODUCE LIMITS

$= \left[3x\ln|2x+3| - 3x + \frac{9}{2}\ln|2x+3|\right]_{-1}^{0} = \left(0 - 0 + \frac{9}{2}\ln 3\right) - (0 + 3 + 0)$

$= \frac{9}{2}\ln 3 - 3$ OR $= \frac{3}{2}\left[3\ln 3 - 2\right] = \frac{3}{2}(\ln 27 - 2)$



ALTERNATIVE BY SUBSTITUTION FIRST

$\int_{-1}^{0} 3\ln(2x+3)\, dx = \int_1^3 3\ln u\, \frac{du}{2} = \int_1^3 \frac{3}{2}\ln u\, du$

$u = 2x+3$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$x=-1$, $u=1$

$x=0$, $u=3$

NOW BY PARTS & IGNORING LIMITS $\rightarrow$

$\ln u \rightarrow \frac{1}{u}$

$\frac{3}{2}u \rightarrow \frac{3}{2}$

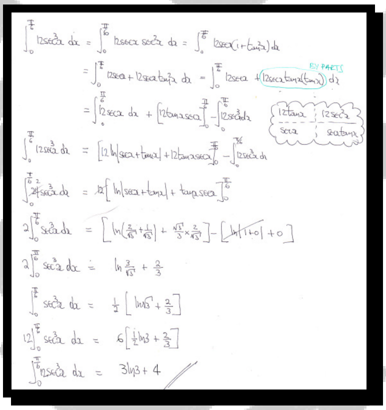
$= \frac{3}{2}u\ln u - \int \frac{3}{2}\, du$

$= \left[\frac{3}{2}u\ln u - \frac{3}{2}u\right]_1^3$

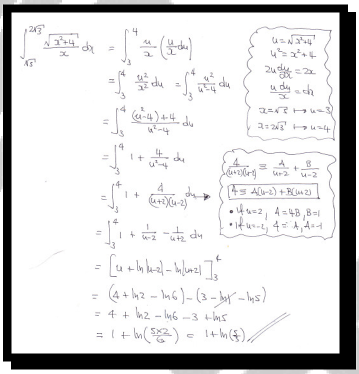
$= \left(\frac{9}{2}\ln 3 - \frac{9}{2}\right) - \left(0 - \frac{3}{2}\right)$

$= \frac{9}{2}\ln 3 - 3$ AS BEFORE

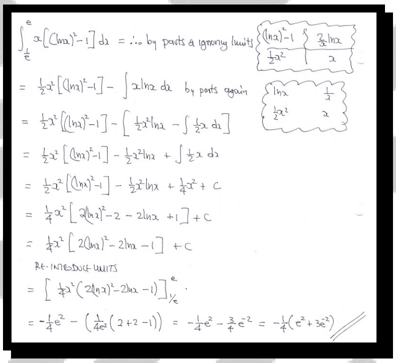
100. $$\int_0^{\frac{\pi}{6}} 12\sec^3 x \ dx = 4 + 3\ln 3$$



101. $$\int_{\sqrt{5}}^{2\sqrt{3}} \frac{\sqrt{x^2+4}}{x} \ dx = 1 + \ln\left(\frac{5}{3}\right)$$



102. $$\int_{e^{-1}}^{e} x\left[(\ln x)^2 - 1\right]\, dx = -\frac{1}{4}\left(e^2 + 3e^{-2}\right)$$



$\int_{\frac{1}{e}}^{e} x\left[(\ln x)^2-1\right]dx = \dots$ by parts & ignoring limits

$(\ln x)^2-1$	$\frac{2}{x}\ln x$
$\frac{1}{2}x^2$	x

$= \frac{1}{2}x^2\left[(\ln x)^2-1\right] - \int x\ln x\, dx$ by parts again

$\ln x$	$\frac{1}{x}$
$\frac{1}{2}x^2$	x

$= \frac{1}{2}x^2\left[(\ln x)^2-1\right] - \left[\frac{1}{2}x^2\ln x - \int \frac{1}{2}x\, dx\right]$

$= \frac{1}{2}x^2\left[(\ln x)^2-1\right] - \frac{1}{2}x^2\ln x + \int \frac{1}{2}x\, dx$

$= \frac{1}{2}x^2\left[(\ln x)^2-1\right] - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C$

$= \frac{1}{4}x^2\left[2(\ln x)^2 - 2 - 2\ln x + 1\right] + C$

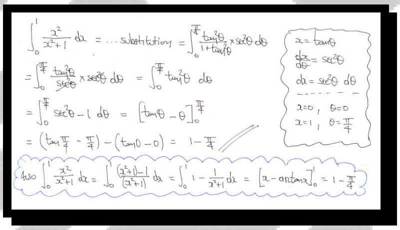
$= \frac{1}{4}x^2\left[2(\ln x)^2 - 2\ln x - 1\right] + C$

RE-INTRODUCE LIMITS

$= \left[\frac{1}{4}x^2\left(2(\ln x)^2 - 2\ln x - 1\right)\right]_{1/e}^{e}$

$= -\frac{1}{4}e^2 - \left(\frac{1}{4e^2}(2+2-1)\right) = -\frac{1}{4}e^2 - \frac{3}{4}e^{-2} = -\frac{1}{4}\left(e^2 + 3e^{-2}\right)$

103. $$\int_0^1 \frac{x^2}{x^2+1}\, dx = 1 - \frac{\pi}{4}$$



$\int_0^1 \frac{x^2}{x^2+1}\, dx = \dots$ substitution $= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{1+\tan^2\theta} \times \sec^2\theta\, d\theta$

$x = \tan\theta$

$\frac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta\, d\theta$

$x=0,\ \theta=0$

$x=1,\ \theta=\frac{\pi}{4}$

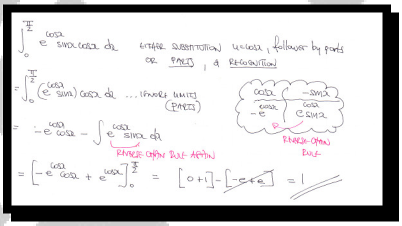
$= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\cancel{\sec^2\theta}} \times \cancel{\sec^2\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \tan^2\theta\, d\theta$

$= \int_0^{\frac{\pi}{4}} \sec^2\theta - 1\, d\theta = \left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}}$

$= \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - \left(\tan 0 - 0\right) = 1 - \frac{\pi}{4}$

Also $\int_0^1 \frac{x^2}{x^2+1}\, dx = \int_0^1 \frac{(x^2+1)-1}{(x^2+1)}\, dx = \int_0^1 1 - \frac{1}{x^2+1}\, dx = \left[x - \arctan x\right]_0^1 = 1 - \frac{\pi}{4}$

104. $$\int_0^{\frac{\pi}{2}} e^{\cos x}\sin x\cos x\, dx = 1$$



$\int_0^{\frac{\pi}{2}} e^{\cos x}\sin x\cos x\, dx$ EITHER SUBSTITUTION $u=\cos x$, followed by parts OR PARTS, & RECOGNITION

$= \int_0^{\frac{\pi}{2}} \left(e^{\cos x}\sin x\right)\cos x\, dx \dots$ IGNORE LIMITS (PARTS)

$\cos x$	$-\sin x$
$-e^{\cos x}$	$e^{\cos x}\sin x$

REVERSE CHAIN RULE

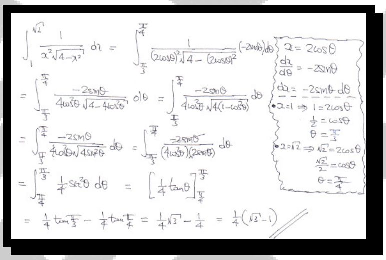
$= -e^{\cos x}\cos x - \int e^{\cos x}\sin x\, dx$

REVERSE CHAIN RULE AGAIN

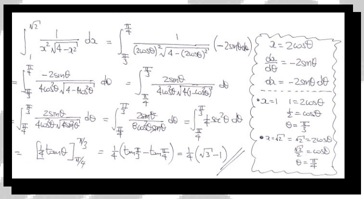
$= \left[-e^{\cos x}\cos x + e^{\cos x}\right]_0^{\frac{\pi}{2}} = \left[0+1\right] - \cancel{\left[-e+e\right]} = 1$

Part 2

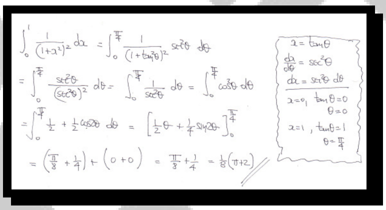
1. $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \frac{\pi}{2} - 1$, use $x = 2\sin\theta$



2. $\displaystyle\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,dx = \frac{1}{4}\left(\sqrt{3}-1\right)$, use $x = 2\cos\theta$



3. $\displaystyle\int_0^{1} \frac{1}{\left(1+x^2\right)^2}\,dx = \frac{1}{8}(\pi+2)$, use $x = \tan\theta$



4. $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^2\sqrt{x^2-1}}\,dx = \frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$, use $x = \sec\theta$

$$\int_{\sqrt{2}}^{2} \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^2\theta\sqrt{\sec^2\theta-1}}(\sec\theta\tan\theta)\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\tan^2\theta}}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cancel{\sec\theta\tan\theta}}{\sec^2\theta\,\cancel{\tan\theta}}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos\theta\,d\theta = \Big[\sin\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \sin\frac{\pi}{3} - \sin\frac{\pi}{4} = \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} = \frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$$

$x = \sec\theta$

$\frac{dx}{d\theta} = \sec\theta\tan\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$x = 2 \quad 2 = \sec\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \pi/3$

$x = \sqrt{2} \quad \sqrt{2} = \sec\theta$

$\frac{1}{\sqrt{2}} = \cos\theta$

$\theta = \pi/4$

5. $\displaystyle\int_{0}^{\frac{3}{4}} \frac{1}{\sqrt{3-4x^2}}\,dx = \frac{\pi}{6}$, use $x = \frac{\sqrt{3}}{2}\sin\theta$

$$\int_{0}^{\frac{3}{4}} \frac{1}{\sqrt{3-4x^2}}\,dx = \int_{0}^{\frac{\pi}{3}} \frac{1}{\sqrt{3-4\left(\frac{\sqrt{3}}{2}\sin\theta\right)^2}}\,\frac{\sqrt{3}}{2}\cos\theta\,d\theta$$

$$= \int_{0}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2}\cos\theta}{\sqrt{3-\cancel{4}\times\frac{3}{\cancel{4}}\sin^2\theta}}\,d\theta = \int_{0}^{\frac{\pi}{3}} \frac{\frac{\sqrt{3}}{2}\cos\theta}{\sqrt{3(1-\sin^2\theta)}}\,d\theta$$

$$= \int_{0}^{\frac{\pi}{3}} \frac{\sqrt{3}/2\,\cancel{\cos\theta}}{\sqrt{3}\,\cancel{\cos\theta}}\,d\theta = \int_{0}^{\frac{\pi}{3}} \frac{1}{2}\,d\theta = \left[\frac{1}{2}\theta\right]_{0}^{\frac{\pi}{3}}$$

$$= \left(\frac{1}{2}\times\frac{\pi}{3}\right) - (0) = \frac{\pi}{6}$$

$x = \frac{\sqrt{3}}{2}\sin\theta$

$\frac{dx}{d\theta} = \frac{\sqrt{3}}{2}\cos\theta$

$dx = \frac{\sqrt{3}}{2}\cos\theta\,d\theta$

$x = 0, \quad \sin\theta = 0$

$\theta = 0$

$x = \frac{3}{4}, \quad \frac{3}{4} = \frac{\sqrt{3}}{2}\sin\theta$

$\frac{\sqrt{3}}{2} = \sin\theta$

$\theta = \frac{\pi}{3}$

6. $\displaystyle\int_{0}^{1} \frac{1}{\left(1+3x^2\right)^{\frac{3}{2}}}\,dx = \frac{1}{2}$, use $x = \frac{1}{\sqrt{3}}\tan\theta$

$$\int_{0}^{1} \frac{1}{(1+3x^2)^{\frac{3}{2}}}\,dx = \int_{0}^{\frac{\pi}{3}} \frac{1}{\left[1+3\left(\frac{1}{\sqrt{3}}\tan\theta\right)^2\right]^{3/2}} \times \frac{1}{\sqrt{3}}\sec^2\theta\,d\theta$$

$$= \int_{0}^{\frac{\pi}{3}} \frac{}{(1+\tan^2\theta)^{3/2}} \times \frac{1}{\sqrt{3}}\sec^2\theta\,d\theta = \frac{1}{\sqrt{3}}\int_{0}^{\frac{\pi}{3}} \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\,d\theta$$

$$= \frac{1}{\sqrt{3}}\int_{0}^{\pi/3} \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{\sqrt{3}}\int_{0}^{\frac{\pi}{3}} \frac{1}{\sec\theta}\,d\theta = \frac{1}{\sqrt{3}}\int_{0}^{\frac{\pi}{3}} \cos\theta\,d\theta$$

$$= \frac{1}{\sqrt{3}}\Big[\sin\theta\Big]_{0}^{\frac{\pi}{3}} = \frac{1}{\sqrt{3}}\left[\sin\frac{\pi}{3} - 0\right] = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{1}{2}$$

$x = \frac{1}{\sqrt{3}}\tan\theta$

$\frac{dx}{d\theta} = \frac{1}{\sqrt{3}}\sec^2\theta$

$dx = \frac{1}{\sqrt{3}}\sec^2\theta\,d\theta$

• $x = 0 \quad \tan\theta = 0$

$\theta = 0$

• $x = 1 \quad \tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

7. $\displaystyle\int_0^1 \frac{1}{\sqrt{2-x^2}}\,dx = \frac{\pi}{4}$, use $x = \sqrt{2}\sin\theta$

$$\int_0^1 \frac{1}{\sqrt{2-x^2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{2-(\sqrt{2}\sin\theta)^2}}\sqrt{2}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{\sqrt{2}\cos\theta}{\sqrt{2-2\sin^2\theta}}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\sqrt{2}\cos\theta}{\sqrt{2(1-\sin^2\theta)}}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\sqrt{2}\cos\theta}{\sqrt{2\cos^2\theta}}\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{\cancel{\sqrt{2}\cos\theta}}{\cancel{\sqrt{2}\cos\theta}}\,d\theta = \int_0^{\frac{\pi}{4}} 1\,d\theta = \Big[\theta\Big]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$x = \sqrt{2}\sin\theta$
$\frac{dx}{d\theta} = \sqrt{2}\cos\theta$
$dx = \sqrt{2}\cos\theta\,d\theta$
$x=1 \quad \sin\theta = \frac{1}{\sqrt{2}}, \quad \theta = \pi/4$
$x=0 \quad \sin\theta = 0, \quad \theta = 0$

8. $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{4x^2+3}\,dx = \frac{\pi\sqrt{3}}{36}$, use $x = \frac{\sqrt{3}}{2}\tan\theta$

$$\int_0^{\frac{1}{2}} \frac{1}{4x^2+3}\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{4\left(\frac{\sqrt{3}}{2}\tan\theta\right)^2+3}\,\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\cancel{4}\left(\frac{3}{\cancel{4}}\tan^2\theta\right)+3}\,d\theta = \int_0^{\frac{\pi}{6}} \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{3(1+\tan^2\theta)}\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{\frac{\sqrt{3}}{2}\cancel{\sec^2\theta}}{3\cancel{\sec^2\theta}}\,d\theta = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}}{6}\,d\theta = \left[\frac{\sqrt{3}}{6}\theta\right]_0^{\frac{\pi}{6}}$$
$$= \frac{\sqrt{3}}{6}\times\frac{\pi}{6} - 0 = \frac{\pi\sqrt{3}}{36}$$

$x = \frac{\sqrt{3}}{2}\tan\theta$
$\frac{dx}{d\theta} = \frac{\sqrt{3}}{2}\sec^2\theta$
$dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$
$x=0, \quad \tan\theta = 0, \quad \theta = 0$
$x=\frac{1}{2}, \quad \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta, \quad \tan\theta = \frac{1}{\sqrt{3}}, \quad \theta = \frac{\pi}{6}$

9. $\displaystyle\int_0^1 \frac{1}{\left(4-x^2\right)^{\frac{3}{2}}}\,dx = \frac{\sqrt{3}}{12}$, use $x = 2\sin\theta$

$$\int_0^1 \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{(4-4\sin^2\theta)^{\frac{3}{2}}}(2\cos\theta\,d\theta)$$
$$= \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{(4(1-\sin^2\theta))^{\frac{3}{2}}}\,d\theta = \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{(4\cos^2\theta)^{\frac{3}{2}}}\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{8\cos^3\theta}\,d\theta = \int_0^{\frac{\pi}{6}} \frac{1}{4}\sec^2\theta\,d\theta$$
$$= \left[\frac{1}{4}\tan\theta\right]_0^{\frac{\pi}{6}} = \frac{1}{4}\tan\frac{\pi}{6} - \cancel{\frac{1}{4}\tan 0} = \frac{\sqrt{3}}{12}$$

$x = 2\sin\theta$
$\frac{dx}{d\theta} = 2\cos\theta$
$dx = 2\cos\theta\,d\theta$
$x=0 \quad \sin\theta = 0, \quad \theta = 0$
$x=1 \quad \sin\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{6}$

10. $\displaystyle\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x}\,dx = \sqrt{3}-1-\frac{\pi}{12}$, use $x = \operatorname{cosec}\theta$

$$\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\cancel{\operatorname{cosec}\theta}}\left(-\cancel{\operatorname{cosec}\theta}\cot\theta\right)d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sqrt{\cot^2\theta}\,(\cot\theta)\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^2\theta\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^2\theta - 1\,d\theta$$

$$= \left[-\cot\theta-\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \left[\cot\theta+\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{6}} = \left(\cot\frac{\pi}{6}+\frac{\pi}{6}\right)-\left(\cot\frac{\pi}{4}+\frac{\pi}{4}\right)$$

$$= \sqrt{3}+\frac{\pi}{6}-1-\frac{\pi}{4} = \sqrt{3}-1-\frac{\pi}{12}$$

$x = \operatorname{cosec}\theta$

$\frac{dx}{d\theta} = -\operatorname{cosec}\theta\cot\theta$

$dx = -\operatorname{cosec}\theta\cot\theta\,d\theta$

$x=2 \quad 2 = \operatorname{cosec}\theta$, $\frac{1}{2} = \sin\theta$, $\theta = \pi/6$

$x=\sqrt{2} \quad \sqrt{2} = \operatorname{cosec}\theta$, $\frac{1}{\sqrt{2}} = \sin\theta$, $\theta = \frac{\pi}{4}$

11. $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{4-3x^2}}\,dx = \frac{\pi\sqrt{3}}{9}$, use $x = \frac{2}{\sqrt{3}}\sin\theta$

$$\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4-3\left(\frac{2}{\sqrt{3}}\sin\theta\right)^2}} \times \frac{2}{\sqrt{3}}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4-\cancel{3}\left(\frac{4}{\cancel{3}}\sin^2\theta\right)}} \times \frac{2}{\sqrt{3}}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4-4\sin^2\theta}} \times \frac{2}{\sqrt{3}}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4(1-\sin^2\theta)}} \times \frac{2}{\sqrt{3}}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{4\cos^2\theta}} \times \frac{2}{\sqrt{3}}\cos\theta\,d\theta$$

$$= \int_0^{\pi/3} \frac{1}{\cancel{2\cos\theta}} \times \frac{\cancel{2}}{\sqrt{3}}\cancel{\cos\theta}\,d\theta = \int_0^{\pi/3} \frac{1}{\sqrt{3}}\,d\theta$$

$$= \left[\frac{1}{\sqrt{3}}\theta\right]_0^{\pi/3} = \frac{\pi}{3\sqrt{3}} - 0 = \frac{\sqrt{3}}{9}\pi$$

$x = \frac{2}{\sqrt{3}}\sin\theta$

$\frac{dx}{d\theta} = \frac{2}{\sqrt{3}}\cos\theta$

$dx = \frac{2}{\sqrt{3}}\cos\theta\,d\theta$

$x=0, \ \sin\theta = 0$, $\theta = 0$

$x=1 \quad \sin\theta = \frac{\sqrt{3}}{2}$, $\theta = \frac{\pi}{3}$

12. $\displaystyle\int_{1}^{\sqrt{3}} \frac{x^2}{x^2+1}\,dx = \sqrt{3}-1-\frac{\pi}{12}$, use $x = \tan\theta$

$$\int_1^{\sqrt{3}} \frac{x^2}{x^2+1}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan^2\theta}{1+\tan^2\theta}\left(\sec^2\theta\,d\theta\right)$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan^2\theta}{\cancel{\sec^2\theta}} \times \cancel{\sec^2\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2\theta\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2\theta - 1\,d\theta = \left[\tan\theta-\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \left(\tan\frac{\pi}{3}-\frac{\pi}{3}\right)-\left(\tan\frac{\pi}{4}-\frac{\pi}{4}\right)$$

$$= \left(\sqrt{3}-\frac{\pi}{3}\right)-\left(1-\frac{\pi}{4}\right) = \sqrt{3}-1-\frac{\pi}{12}$$

$x = \tan\theta$

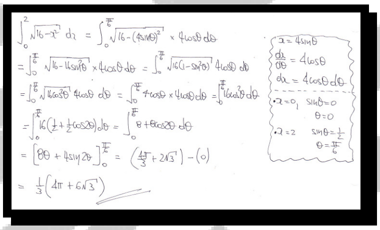
$\frac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta\,d\theta$

$x=\sqrt{3} \quad \tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$

$x=1 \quad \tan\theta = 1$, $\theta = \frac{\pi}{4}$

13. $\displaystyle\int_0^2 \sqrt{16-x^2}\ dx = \frac{1}{3}\left(4\pi+6\sqrt{3}\right)$, use $x = 4\sin\theta$



$$\int_0^2 \sqrt{16-x^2}\,dx = \int_0^{\frac{\pi}{6}} \sqrt{16-(4\sin\theta)^2} \times 4\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \sqrt{16-16\sin^2\theta} \times 4\cos\theta\,d\theta = \int_0^{\frac{\pi}{6}} \sqrt{16(1-\sin^2\theta)}\,4\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \sqrt{16\cos^2\theta}\,4\cos\theta\,d\theta = \int_0^{\frac{\pi}{6}} 4\cos\theta \times 4\cos\theta\,d\theta = \int_0^{\frac{\pi}{6}} 16\cos^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} 16\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta = \int_0^{\frac{\pi}{6}} 8+8\cos 2\theta\,d\theta$$
$$= \left[8\theta + 4\sin 2\theta\right]_0^{\frac{\pi}{6}} = \left(\tfrac{4\pi}{3}+2\sqrt{3}\right) - (0)$$
$$= \tfrac{1}{3}\left(4\pi+6\sqrt{3}\right)$$

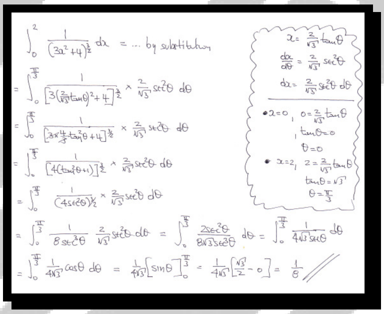
$x = 4\sin\theta$

$\frac{dx}{d\theta} = 4\cos\theta$

$dx = 4\cos\theta\,d\theta$

- $x=0$, $\sin\theta = 0$, $\theta = 0$
- $x=2$, $\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$

14. $\displaystyle\int_0^2 \frac{1}{\left(3x^2+4\right)^{\frac{3}{2}}}\ dx = \frac{1}{8}$, use $x = \frac{2}{\sqrt{3}}\tan\theta$



$$\int_0^2 \frac{1}{(3x^2+4)^{\frac{3}{2}}}\,dx = \dots \text{by substitution}$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{\left[3\left(\frac{2}{\sqrt{3}}\tan\theta\right)^2+4\right]^{\frac{3}{2}}} \times \frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{\left[3\times\frac{4}{3}\tan^2\theta+4\right]^{\frac{3}{2}}} \times \frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{\left[4(\tan^2\theta+1)\right]^{\frac{3}{2}}} \times \frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{(4\sec^2\theta)^{\frac{3}{2}}} \times \frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{8\sec^3\theta}\,\frac{2}{\sqrt{3}}\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{3}} \frac{2\sec^2\theta}{8\sqrt{3}\sec^3\theta}\,d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{4\sqrt{3}\sec\theta}\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{4\sqrt{3}}\cos\theta\,d\theta = \frac{1}{4\sqrt{3}}\left[\sin\theta\right]_0^{\frac{\pi}{3}} = \frac{1}{4\sqrt{3}}\left[\frac{\sqrt{3}}{2}-0\right] = \frac{1}{8}$$

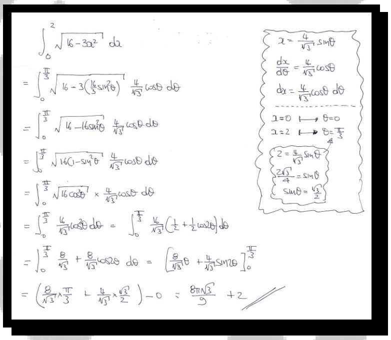
$x = \frac{2}{\sqrt{3}}\tan\theta$

$\frac{dx}{d\theta} = \frac{2}{\sqrt{3}}\sec^2\theta$

$dx = \frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$

- $x=0$, $0 = \frac{2}{\sqrt{3}}\tan\theta$, $\tan\theta = 0$, $\theta = 0$
- $x=2$, $2 = \frac{2}{\sqrt{3}}\tan\theta$, $\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$

15. $\displaystyle\int_0^2 \sqrt{16-3x^2}\ dx = \frac{8\pi\sqrt{3}}{9}+2$, use $x = \frac{4}{\sqrt{3}}\sin\theta$



$$\int_0^2 \sqrt{16-3x^2}\,dx$$
$$= \int_0^{\frac{\pi}{3}} \sqrt{16-3\left(\tfrac{16}{3}\sin^2\theta\right)}\,\frac{4}{\sqrt{3}}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \sqrt{16-16\sin^2\theta}\,\frac{4}{\sqrt{3}}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \sqrt{16(1-\sin^2\theta)}\,\frac{4}{\sqrt{3}}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \sqrt{16\cos^2\theta} \times \frac{4}{\sqrt{3}}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{16}{\sqrt{3}}\cos^2\theta\,d\theta = \int_0^{\frac{\pi}{3}} \frac{16}{\sqrt{3}}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{8}{\sqrt{3}} + \frac{8}{\sqrt{3}}\cos 2\theta\,d\theta = \left[\frac{8}{\sqrt{3}}\theta + \frac{4}{\sqrt{3}}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$
$$= \left(\frac{8}{\sqrt{3}}\times\frac{\pi}{3} + \frac{4}{\sqrt{3}}\times\frac{\sqrt{3}}{2}\right) - 0 = \frac{8\pi\sqrt{3}}{9} + 2$$

$x = \frac{4}{\sqrt{3}}\sin\theta$

$\frac{dx}{d\theta} = \frac{4}{\sqrt{3}}\cos\theta$

$dx = \frac{4}{\sqrt{3}}\cos\theta\,d\theta$

$x=0 \longmapsto \theta = 0$

$x=2 \longmapsto \theta = \frac{\pi}{3}$

$2 = \frac{4}{\sqrt{3}}\sin\theta$

$\frac{2\sqrt{3}}{4} = \sin\theta$

$\sin\theta = \frac{\sqrt{3}}{2}$

16. $\displaystyle\int_0^3 \frac{27}{\left(9+x^2\right)^2}\,dx = \frac{\pi}{8}+\frac{1}{4}$, use $x = 3\tan\theta$

$$\int_0^3 \frac{27}{(9+x^2)^2}\,dx = \ldots = \int_0^{\frac{\pi}{4}} \frac{27}{(9+9\tan^2\theta)^2}\left(3\sec^2\theta\,d\theta\right)$$

$$= \int_0^{\frac{\pi}{4}} \frac{81\sec^2\theta}{\left[9(1+\tan^2\theta)\right]^2}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{81\sec^2\theta}{81(\sec^2\theta)^2}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta = \int_0^{\pi/4} \frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{8}+\frac{1}{4}\right) - \cancel{(0+0)}$$

$$= \frac{\pi}{8}+\frac{1}{4}$$

$x = 3\tan\theta$

$\frac{dx}{d\theta} = 3\sec^2\theta$

$dx = 3\sec^2\theta\,d\theta$

$x=0,\ \theta=0$

$x=3,\ \theta=\frac{\pi}{4}$

Part 3

1. $\int_{4}^{8} \sqrt{x^2-16}\ dx = 16\sqrt{3} - 8\ln\left(2+\sqrt{3}\right)$

2. $\int_{0}^{1} \frac{1}{\sqrt{x}\,(x+1)}\ dx = \frac{\pi}{2}$

3. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x\ dx = \ln\left|\frac{2}{3}\sqrt{3}+1\right|$

4. $\int_{0}^{1} x^3\sqrt{x^2+1}\ dx = \frac{2}{15}\left(\sqrt{2}+1\right)$

5. $\int_{\ln 2}^{\ln 3} \frac{\cosh x+1}{\sinh x(\cosh x-1)}\ dx = \frac{5}{2}$

6. $\int_{1}^{\sqrt{3}} x\arctan x\ dx = \frac{5\pi}{12} + \frac{1}{2}\left(1-\sqrt{3}\right)$

7. $\int_{\frac{4}{3}}^{\frac{5}{3}} \frac{x+1}{\sqrt{9x^2-16}}\ dx = \frac{1}{3}(1+\ln 2)$

8. $\int_{\frac{3}{2}}^{\frac{5}{2}} \sqrt{4x^2-9}\ dx = 5 - \frac{9}{4}\ln 3$

9. $\int_{0}^{4} \operatorname{arsinh}\sqrt{x}\ dx = \frac{9}{2}\ln\left(2+\sqrt{5}\right) - \sqrt{5}$

10. $\int_{0}^{1} \frac{x^2}{\sqrt{x^2+1}}\ dx = \frac{1}{2}\left(\sqrt{2} - \ln\left(1+\sqrt{2}\right)\right)$

11. $\int_{5}^{7} \frac{1}{x^2-10x+29}\ dx = \frac{\pi}{8}$

12. $\int_{5}^{7} \frac{1}{\sqrt{x^2-10x+29}}\ dx = \ln\left(1+\sqrt{2}\right)$

13. $$\int_{2.5}^{7.5} \frac{180}{4x^2+75}\,dx = \sqrt{3}\pi$$

14. $$\int_{2}^{3} \frac{1}{\sqrt{3+2x-x^2}}\,dx = \frac{\pi}{3}$$

15. $$\int_{5}^{7} \frac{x+1}{x^2+9}\,dx = \frac{1}{2}\ln 2 + \frac{\pi}{12}$$

16. $$\int_{0}^{\frac{\pi}{3}} \frac{1}{9\cos^2 x + \sin^2 x}\,dx = \frac{\pi}{18}$$

17. $$\int_{0}^{\frac{1}{2}\ln 3} \operatorname{sech} x\,dx = \frac{\pi}{6}$$

18. $$\int_{0}^{\frac{1}{\sqrt{3}}} \frac{4}{1-x^4}\,dx = \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) + \frac{\pi}{3}$$

19. $$\int_{0}^{\frac{3}{2}} \frac{8}{4x^2+9}\,dx = \frac{\pi}{3}$$

20. $$\int_{0}^{1} \frac{10}{(x+1)(x^2+4)}\,dx = \ln\left(\frac{16}{5}\right) + \arctan\left(\frac{1}{2}\right)$$

21. $$\int_{0}^{3} \frac{8x}{(x+2)(x^2+4)}\,dx = \ln\left(\frac{26}{25}\right) + 2\left(\arctan\left(\frac{1}{2}\right) - \arctan 1\right)$$

22. $$\int_{0}^{\frac{9}{4}} \frac{1}{\sqrt{x(9-x)}}\,dx = \frac{\pi}{3}$$

23. $$\int_{-1}^{0} \frac{(x+1)(x+2)}{(x-1)(x^2+1)}\,dx = \frac{\pi}{4} - 2\ln 2$$

24. $$\int_{0}^{\frac{\pi}{2}} \frac{1}{1+3\cos 3x}\,dx = \frac{\sqrt{2}}{6}\ln\left(\sqrt{2}-1\right)$$

25. $$\int_{-1}^{1} \frac{1}{\sqrt{x^2+2x+5}}\, dx = \ln\left(1+\sqrt{2}\right)$$

26. $$\int_{0}^{\sqrt{3}} \frac{x}{x^4+9}\, dx = \frac{\pi}{24}$$

27. $$\int_{-\frac{5}{3}}^{\frac{5}{6}} \frac{1}{\sqrt{25-9x^2}}\, dx = \frac{2\pi}{9}$$

28. $$\int_{1}^{2} \sqrt{x^2-2x+2}\, dx = \frac{1}{2}\ln\left(1+\sqrt{2}\right)-\frac{1}{2}\sqrt{2}$$

29. $$\int_{1}^{2} \frac{\sqrt{x}}{\sqrt{9-x^3}}\, dx = \frac{2}{3}\arccos\frac{1}{3}$$

30. $$\int_{0}^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{3-\sec^2 x}}\, dx = \frac{\pi}{4}$$

31. $$\int_{0}^{2} \frac{1}{\left(x^2+4\right)^{\frac{3}{2}}}\, dx = \frac{1}{8}\sqrt{2}$$

32. $$\int_{1}^{2} \frac{x^2}{\sqrt{x^2+1}}\, dx = \frac{1}{2}\left[\sqrt{2}-\ln\left(\sqrt{2}+1\right)\right]$$

33. $$\int_{0}^{2} \frac{36-3x}{4+3x^2}\, dx = 2\pi\sqrt{3}-\ln 2$$

$$\int_0^2 \frac{36-3x}{4+3x^2}\,dx = \int_0^2 \frac{36}{4+3x^2}\,dx - \int_0^2 \frac{3x}{4+3x^2}\,dx$$
$$= \int_0^2 \frac{12}{\frac{4}{3}+x^2}\,dx - \frac{1}{2}\int_0^2 \frac{6x}{4+3x^2}\,dx$$
$$= \int_0^2 \frac{12}{\left(\frac{2}{\sqrt{3}}\right)^2+x^2}\,dx - \frac{1}{2}\left[\ln\left[4+3x^2\right]\right]_0^2$$
$$= \frac{12}{2/\sqrt{3}}\left[\arctan\left(\frac{x}{2/\sqrt{3}}\right)\right]_0^2 - \frac{1}{2}\left[\ln 16-\ln 4\right]$$
$$= 6\sqrt{3}\left(\arctan\sqrt{3}-0\right) - \frac{1}{2}\ln 4$$
$$= 6\sqrt{3}\times\frac{\pi}{3} - \ln 2$$
$$= 2\pi\sqrt{3}-\ln 2$$

34. $\displaystyle\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\, dx = \frac{1}{4}[\pi - 2]$

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\, 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cancel{\cos\theta}} \times 2\sin\theta\cancel{\cos\theta}\, d\theta = \int_0^{\frac{\pi}{4}} 2\sin^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta = \int_0^{\frac{\pi}{4}} 1 - \cos 2\theta\, d\theta$$

$$= \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} - \frac{1}{2}\right) - (0) = \frac{\pi}{4} - \frac{1}{2} = \frac{1}{4}(\pi - 2)$$

$x = \sin^2\theta$

$\frac{dx}{d\theta} = 2\sin\theta\cos\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$x = \frac{1}{2} \quad \theta = \frac{\pi}{4}$

$x = 0 \quad \theta = 0$

35. $\displaystyle\int_1^4 \frac{1}{(x+9)\sqrt{x}}\, dx = \frac{2}{3}\arctan\left(\frac{3}{11}\right)$

$\displaystyle\int_1^4 \frac{1}{(9+x)\sqrt{x}}\, dx$ ... by substitution

$$= \int_1^2 \frac{1}{(9+u^2)u}\, 2u\, du = \int_1^2 \frac{2}{9+u^2}\, du$$

$$= \frac{2}{3}\left[\arctan\frac{u}{3}\right]_1^2 = \frac{2}{3}\left(\arctan\frac{2}{3} - \arctan\frac{1}{3}\right)$$

$$= \frac{2}{3}\arctan\frac{3}{11}$$

(≈ 0.178)

$\tan\left(\arctan\frac{2}{3} - \arctan\frac{1}{3}\right) = \dfrac{\frac{2}{3} - \frac{1}{3}}{1 + \frac{1}{3}\times\frac{2}{3}} = \frac{2}{11}$

$u = \sqrt{x}$

$u^2 = x$

$2u\, du = dx$

$x = 1 \mapsto u = 1$

$x = 4 \mapsto u = 2$

36. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\, dx = \frac{\sqrt{3}}{9}\pi$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x}\, dx = \int_0^1 \frac{1}{2 + \frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\, dt$$

MULTIPLY TOP | BOTTOM BY $(1+t^2)$

$$= \int_0^1 \frac{1+t^2}{2+2t^2+2t} \times \frac{2}{\cancel{1+t^2}}\, dt$$

$$= \int_0^1 \frac{1}{t^2+t+1}\, dt = \int_0^1 \frac{4}{4t^2+4t+4}\, dt$$

$$= \int_0^1 \frac{4}{(2t+1)^2+3}\, dt$$

$u = 2t+1$, $\frac{du}{dt} = 2$, $dt = \frac{du}{2}$; $t = 0, u = 1$; $t = 1, u = 3$

$$= \int_1^3 \frac{4}{u^2+(\sqrt{3})^2}\, \frac{du}{2} = \int_1^3 \frac{2}{u^2+(\sqrt{3})^2}\, du$$

$$= \frac{2}{\sqrt{3}}\left[\arctan\left(\frac{u}{\sqrt{3}}\right)\right]_1^3 = \frac{2}{3}\sqrt{3}\left(\arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{3}\sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{2}{3}\sqrt{3} \times \frac{\pi}{6} = \frac{\sqrt{3}}{9}\pi$$

$t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)$

$dt = \frac{1}{2}(1+t^2)\, dx$

$dx = \frac{2}{1+t^2}\, dt$

$x = 0 \mapsto t = 0$

$x = \frac{\pi}{2} \mapsto t = 1$

$\sin x = \frac{2t}{1+t^2}$

37. $\displaystyle\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\cosh x - 4\sinh x}\, dx = \frac{\pi}{18}$

$$\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\cosh x - 4\sinh x}\,dx = \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\left(\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\right)-4\left(\frac{1}{2}e^{x}-\frac{1}{2}e^{-x}\right)}\,dx = \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{\frac{1}{2}e^{x}+\frac{9}{2}e^{-x}}\,dx$$

$$= \int \frac{2}{e^{x}+9e^{-x}}\,dx$$

LET $u = e^{x}$, $\frac{du}{dx} = e^{x}$, $\frac{du}{dx} = u$, i.e. $dx = \frac{du}{u}$; $x = \ln 3 \mapsto u = 3$, $x = \frac{1}{2}\ln 3 \mapsto u = \sqrt{3}$

$$= \int_{\sqrt{3}}^{3} \frac{2}{u+9u^{-1}} \times \frac{du}{u} = \int_{\sqrt{3}}^{3} \frac{2}{u^2+9}\,du = \int_{\sqrt{3}}^{3} \frac{2}{u^2+3^2}\,du$$

$$= \frac{2}{3}\left[\arctan\frac{u}{3}\right]_{\sqrt{3}}^{3} = \frac{2}{3}\left[\arctan 1 - \arctan\frac{\sqrt{3}}{3}\right] = \frac{2}{3}\left[\frac{\pi}{4}-\frac{\pi}{6}\right] = \frac{\pi}{18}$$

38. $\displaystyle\int_{0}^{\sqrt{12}} \operatorname{arsinh}\left(\tfrac{1}{2}x\right)\, dx = 2\sqrt{3}\ln\left(2+\sqrt{3}\right) - 2$

$$\int_0^{\sqrt{12}} \operatorname{arsinh}\left(\tfrac{1}{2}x\right)dx = \int_0^{\sqrt{12}} 1 \times \operatorname{arsinh}\left(\tfrac{1}{2}x\right)dx = \dots \text{by parts}$$

$\operatorname{arsinh}\frac{1}{2}x$	$\frac{\frac{1}{2}}{\sqrt{1+\frac{1}{4}x^2}}$
x	1

$$= \left[x \operatorname{arsinh}\tfrac{1}{2}x\right]_0^{\sqrt{12}} - \int_0^{\sqrt{12}} \frac{\frac{1}{2}x}{\sqrt{1+\frac{1}{4}x^2}}\,dx$$

$$= \left[\sqrt{12}\operatorname{arsinh}\left(\tfrac{1}{2}\sqrt{12}\right) - 0\right] - \int_0^{\sqrt{12}} \frac{\cancel{\frac{1}{2}}x}{\cancel{\frac{1}{2}}\sqrt{4+x^2}}\,dx$$

$$= 2\sqrt{3}\operatorname{arsinh}(\sqrt{3}) - \int_0^{\sqrt{12}} x(4+x^2)^{-\frac{1}{2}}\,dx$$

$$= 2\sqrt{3}\operatorname{arsinh}(\sqrt{3}) - \left[(4+x^2)^{\frac{1}{2}}\right]_0^{\sqrt{12}}$$

$$= 2\sqrt{3}\ln\left(\sqrt{3}+\sqrt{3+1}\right) - (4-2)$$

$$= 2\sqrt{3}\ln\left(2+\sqrt{3}\right) - 2$$

recognition/substitution

partial fractions

parts

trig identities

mixed tech

antiderivatives/linear adjustment