Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/01 - Core Pure Mathematics 1

Monday 3 June 2019

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2019 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

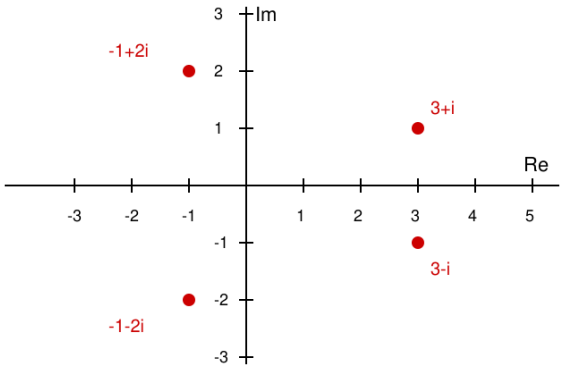
Let $f(z) = z^4 + az^3 + bz^2 + cz + d$, where a, b, c, d are real. Given that $-1 + 2i$ and $3 - i$ are roots of $f(z) = 0$: (a) show all roots on one Argand diagram; (b) find a, b, c, d. **(9)**

Model solution

(a) Since the polynomial has real coefficients, every non-real root has its complex conjugate as a root. This is the clue that supplies the two missing roots.

$$z = -1 + 2i, \quad -1 - 2i, \quad 3 - i, \quad 3 + i.$$

On an Argand diagram these are the points $(-1, 2), (-1, -2), (3, -1), (3, 1)$*, with real part horizontal and imaginary part vertical.*



The four points are $(-1, \pm 2)$ *and* $(3, \pm 1)$.

(4)

(b) Pair each conjugate pair first. This makes real quadratic factors and avoids expanding complex arithmetic unnecessarily.

$$f(z) = [z - (-1 + 2i)][z - (-1 - 2i)][z - (3 - i)][z - (3 + i)]$$

$$= [(z + 1)^2 + 2^2][(z - 3)^2 + 1^2] = (z^2 + 2z + 5)(z^2 - 6z + 10).$$

$$f(z) = z^4 - 4z^3 + 3z^2 - 10z + 50.$$

$$a = -4, \ b = 3, \ c = -10, \ d = 50.$$

(5)

Question 2

Question 2

Show that $\int_0^\infty \frac{8x-12}{(2x^2+3)(x+1)}\,dx = \ln k$, where k is rational. **(7)**

Model solution

The denominator has a quadratic factor and a linear factor, so use a linear numerator over the quadratic and a constant over the linear factor.

$$\frac{8x-12}{(2x^2+3)(x+1)} = \frac{Ax+B}{2x^2+3} + \frac{C}{x+1}.$$

$$8x-12 = (Ax+B)(x+1) + C(2x^2+3).$$

Comparing coefficients gives $A = 8$, $B = 0$, $C = -4$. *Hence*

$$\int \frac{8x-12}{(2x^2+3)(x+1)}\,dx = 2\ln(2x^2+3) - 4\ln(x+1) + C.$$

This is improper, so evaluate it as a limit. Keeping the logarithms together makes the limiting cancellation visible.

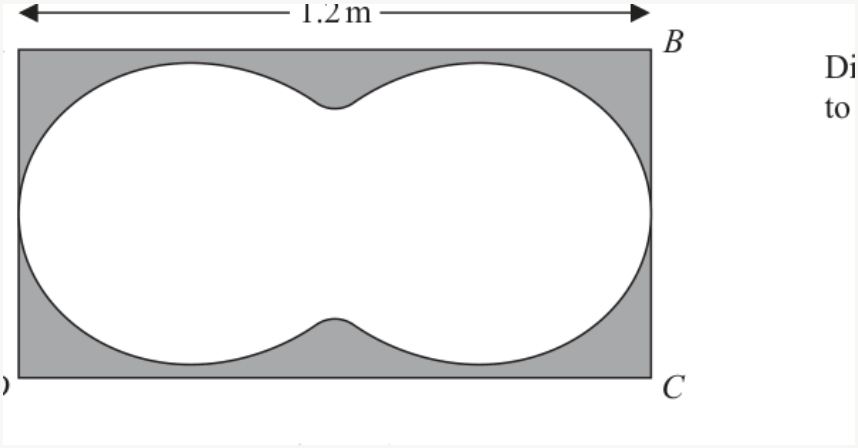
$$\begin{aligned} I &= \lim_{R\to\infty} \left[2\ln(2x^2+3) - 4\ln(x+1)\right]_0^R \\ &= \lim_{R\to\infty} \ln\left(\frac{(2R^2+3)^2}{(R+1)^4}\right) - \ln 9 \\ &= 2\ln 2 - 2\ln 3 = \ln\frac{4}{9}. \end{aligned}$$

$$k = \frac{4}{9}, \text{ so } I = \ln\frac{4}{9}.$$

(7)

Question 3

Question 3



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The glass perimeter is $r = 0.4 + a\cos 2\theta$, $0 \leqslant \theta < 2\pi$. The rectangular top has AB $= 1.2$ m. (a) Show $a = 0.2$. Hence, given AD $= 60$ cm, (b) find the wooden area in m^2, to 3 significant figures. **(10)**

Model solution

(a) The greatest radius occurs when $\cos 2\theta = 1$. *The diagram shows that this is half the 1.2 m length, namely* 0.6 *m.*

$$0.4 + a = 0.6 \quad \Longrightarrow \quad a = 0.2.$$

$$\boxed{a = 0.2.}$$

(2)

(b) The wooden area is rectangle area minus the polar area of the glass. First convert 60 *cm to* 0.6 *m.*

$$A_{\text{rectangle}} = 1.2(0.6) = 0.72.$$

$$\begin{aligned} A_{\text{glass}} &= \frac{1}{2}\int_0^{2\pi} (0.4 + 0.2\cos 2\theta)^2 \, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left(0.16 + 0.16\cos 2\theta + 0.04\cos^2 2\theta\right) d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (0.18 + 0.16\cos 2\theta + 0.02\cos 4\theta) \, d\theta \\ &= [0.09\theta + 0.04\sin 2\theta + 0.0025\sin 4\theta]_0^{2\pi} \\ &= 0.18\pi. \end{aligned}$$

The sine terms vanish at both limits, so the required surrounding area is

$$A_{\text{wood}} = 0.72 - 0.18\pi = 0.154513\ldots$$

The wooden area is $\boxed{0.155\ m^2}$ *(3 s.f.).*

(8)

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Question 4

Question 4

Prove, for $n \in \mathbb{Z},\ n \geqslant 0$, that $\displaystyle\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$, where a, b, c are integers to be found. **(5)**

Model solution

The product of three consecutive factors suggests partial fractions. The resulting shifted fractions will telescope when summed.

$$\frac{1}{(r+1)(r+2)(r+3)} = \frac{A}{r+1} + \frac{B}{r+2} + \frac{C}{r+3}.$$

$$1 = A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2),$$

so, by using $r = -1, -2, -3$,

$$A = \frac{1}{2}, \qquad B = -1, \qquad C = \frac{1}{2}.$$

$$\begin{aligned}\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} &= \frac{1}{2}\sum_{r=0}^{n}\left(\frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}\right)\\ &= \frac{1}{4} - \frac{1}{2(n+2)} + \frac{1}{2(n+3)}\\ &= \frac{(n+1)(n+4)}{4(n+2)(n+3)}.\end{aligned}$$

The middle terms cancel in the telescoping sum; putting the remaining terms over a common denominator gives the required form.

$$a = 1,\ b = 4,\ c = 4.$$

(5)

Question 5

Question 5

A 250 litre tank initially holds 100 litres of pure water. Salt water enters at 3 L min^{-1}, containing 1 g of salt per litre; well-mixed solution leaves at 2 L min^{-1}. If S grams of salt are present after t minutes: (a) show $\frac{dS}{dt} = 3 - \frac{2S}{100+t}$; (b) find S after 10 minutes; (c) find when concentration reaches 0.9 g L^{-1}; (d) evaluate the model. **(13)**

Model solution

(a) Use rate of change = rate in − rate out. The volume increases by $3 - 2 = 1$ L each minute, so it is $100 + t$ litres. Instantaneous mixing means the outgoing concentration equals the tank concentration.

$$\textit{salt in} = 3 \times 1 = 3 \textit{ g min}^{-1}, \qquad \textit{salt out} = 2\left(\frac{S}{100+t}\right) \textit{ g min}^{-1}.$$

$$\frac{dS}{dt} = 3 - \frac{2S}{100+t}.$$

$$\boxed{\frac{dS}{dt} = 3 - \frac{2S}{100+t}.}$$

(4)

(b) This is a first-order linear differential equation. An integrating factor is appropriate because S and dS/dt occur linearly.

$$\frac{dS}{dt} + \frac{2}{100+t}S = 3, \qquad \text{IF} = e^{\int 2/(100+t)\,dt} = (100+t)^2.$$

$$\frac{d}{dt}\left[(100+t)^2 S\right] = 3(100+t)^2.$$

$$(100+t)^2 S = (100+t)^3 + C.$$

Initially the water is pure, so $S = 0$ when $t = 0$. Thus $C = -10^6$, and

$$S = (100+t) - \frac{10^6}{(100+t)^2}.$$

$$S(10) = 110 - \frac{10^6}{110^2} = \frac{3310}{121} = 27.355\ldots$$

$$\boxed{S = \frac{3310}{121} \textit{ g} \approx 27.4 \textit{ g after 10 minutes.}}$$

(5)

(c) Concentration is salt divided by volume, not the amount S itself. Set $S/(100+t) = 0.9$.

$$\frac{(100+t) - 10^6/(100+t)^2}{100+t} = 0.9.$$

$$0.1(100+t)^3 = 10^6 \quad \Longrightarrow \quad 100+t = 10\sqrt[3]{100}.$$

$$t = 10\sqrt[3]{100} - 100 = 115.443\ldots$$

The valve should be closed after $\boxed{115}$ *minutes (nearest minute).*

(3)

(d) A limitation is the assumption that mixing is instantaneous; a real tank need not have uniform concentration at every instant.

For example, instantaneous mixing is unrealistic.

(1)

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Question 6

Question 6

Prove by induction that $f(n) = 3^{2n+4} - 2^{2n}$ is divisible by 5 for all positive integers n. **(6)**

Model solution

Induction is requested, so establish a base case, assume the statement at k, then use that assumption to prove the next case.

$$f(1) = 3^6 - 2^2 = 729 - 4 = 725 = 5(145),$$

so the statement is true for $n = 1$. Assume that for $n = k$,

$$f(k) = 3^{2k+4} - 2^{2k} = 5M$$

for some integer M. For $n = k + 1$, factor powers so that $f(k)$ appears:

$$\begin{aligned} f(k+1) &= 3^{2k+6} - 2^{2k+2} \\ &= 9 \cdot 3^{2k+4} - 4 \cdot 2^{2k} \\ &= 9(3^{2k+4} - 2^{2k}) + 5 \cdot 2^{2k} \\ &= 9(5M) + 5 \cdot 2^{2k} \\ &= 5\left(9M + 2^{2k}\right). \end{aligned}$$

The bracket is an integer, so $f(k+1)$ is divisible by 5. Since the base case and inductive step are true, the result follows for every positive integer n.

$3^{2n+4} - 2^{2n}$ is divisible by 5 for all positive integers n.

(6)

Question 7

Question 7

$l_1: \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-4}{3}$, and $l_2: \mathbf{r} = \mathbf{i} + 3\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$. (a) Show they lie in the same plane. (b) Give a vector equation of that plane. (c) Find the acute angle between them, to the nearest degree. **(7)**

Model solution

(a) Convert the symmetric equation to a parametric vector equation, then test whether the two lines share a point. Intersecting lines necessarily lie in one plane.

$$l_1: \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \qquad l_2: \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

$$1 + 2\lambda = 1 + t, \quad -1 - \lambda = -t, \quad 4 + 3\lambda = 3 + 2t.$$

The first two equations give $\lambda = 1$, $t = 2$; the third then gives $7 = 7$. Thus both contain $(3, -2, 7)$.

The lines intersect at $(3, -2, 7)$, so they are coplanar.

(3)

(b) A plane containing both lines uses their common point and their two direction vectors.

$$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} + p \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + q \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

$$\mathbf{r} = (3, -2, 7) + p(2, -1, 3) + q(1, -1, 2).$$

(1)

(c) Use the scalar product of the direction vectors; taking the positive value gives the acute angle.

$$\cos\theta = \frac{(2, -1, 3) \cdot (1, -1, 2)}{\sqrt{2^2 + (-1)^2 + 3^2}\sqrt{1^2 + (-1)^2 + 2^2}} = \frac{9}{\sqrt{14}\sqrt{6}}.$$

$$\theta = 10.893\ldots^\circ.$$

The acute angle is $\boxed{11^\circ}$.

(3)

Question 8

Question 8

The populations w (white-clawed) and s (signal) at time t years satisfy $\frac{dw}{dt} = \frac{5}{2}(w - s)$, $\frac{ds}{dt} = \frac{2}{5}w - 90e^{-t}$. (a) Show $2w'' - 5w' + 2w = 450e^{-t}$. (b) Find a general solution for w. (c) Find one for s. Given $w = 65$, $s = 85$ at $t = 0$, (d) find extinction time T, 3 d.p. (e) give a limitation. **(18)**

Model solution

(a) Differentiate the equation for w, then substitute the supplied equation for s'. Eliminate s using the original equation for w'.

$$w'' = \frac{5}{2}(w' - s'), \qquad s' = \frac{2}{5}w - 90e^{-t}, \qquad s = w - \frac{2}{5}w'.$$

$$\begin{aligned} w'' &= \frac{5}{2}\left(w' - \frac{2}{5}w + 90e^{-t}\right) \\ &= \frac{5}{2}w' - w + 225e^{-t}. \end{aligned}$$

$$2w'' - 5w' + 2w = 450e^{-t}.$$

$$2\frac{d^2w}{dt^2} - 5\frac{dw}{dt} + 2w = 450e^{-t}.$$

(3)

(b) Solve the constant-coefficient equation. The auxiliary equation gives the complementary function; the forcing term e^{-t} suggests a trial Ke^{-t}.

$$2m^2 - 5m + 2 = 0 \quad \Longrightarrow \quad m = 2,\ \frac{1}{2},$$

$$w_{CF} = Ae^{t/2} + Be^{2t}.$$

$$w_{PI} = Ke^{-t} \quad \Longrightarrow \quad (2 + 5 + 2)Ke^{-t} = 450e^{-t},$$

so $K = 50$. Therefore

$$w = Ae^{t/2} + Be^{2t} + 50e^{-t}.$$

(6)

(c) Rearranging $w' = \frac{5}{2}(w - s)$ gives $s = w - \frac{2}{5}w'$. Substitute the general solution just found.

$$w' = \frac{1}{2}Ae^{t/2} + 2Be^{2t} - 50e^{-t},$$

$$s = \frac{4}{5}Ae^{t/2} + \frac{1}{5}Be^{2t} + 70e^{-t}.$$

$$s = \frac{4}{5}Ae^{t/2} + \frac{1}{5}Be^{2t} + 70e^{-t}.$$

(2)

(d) Apply both initial populations to determine the two constants.

$$A + B + 50 = 65, \qquad \frac{4}{5}A + \frac{1}{5}B + 70 = 85.$$

$$A = 20, \qquad B = -5.$$

$$w = 20e^{t/2} - 5e^{2t} + 50e^{-t}.$$

At extinction $w = 0$. Multiplying by e^t, then setting $u = e^{3t/2}$, produces a quadratic.

$$20e^{3t/2} - 5e^{3t} + 50 = 0 \quad \Longrightarrow \quad u^2 - 4u - 10 = 0.$$

Only the positive root can equal $e^{3t/2}$:

$$u = 2 + \sqrt{14}, \qquad \frac{3T}{2} = \ln(2 + \sqrt{14}).$$

$$T = \frac{2}{3}\ln(2 + \sqrt{14}) = \boxed{1.165 \text{ years}} \text{ (3 d.p.).}$$

(6) *(e) A population model is invalid once it predicts a negative population; it also cannot describe the system after extinction.*

It is not valid when a population becomes negative.

(1)