Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/01 - Core Pure Mathematics 1

Thursday 26 May 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Let $f(z) = z^3 + az + 52$, where a is real. Given that $2 - 3i$ is a root of $f(z) = 0$: (a) write down the other complex root; (b) hence (i) solve $f(z) = 0$ completely and (ii) find a; (c) show the roots on an Argand diagram.

Model solution

(a) The coefficients are real, so non-real roots occur in conjugate pairs.

$$2 + 3i$$

$$\boxed{2 + 3i}$$

(1)

(b)(i) Use the conjugate pair first. Their sum is 4 and their product is 13, so their quadratic factor is $z^2 - 4z + 13$. The constant term of the cubic is 52, so the remaining root is -4, since $13(-4) = -52$.

$$f(z) = (z^2 - 4z + 13)(z + 4), \qquad z = 2 - 3i,\ 2 + 3i,\ -4.$$

$$\boxed{z = 2 \pm 3i \text{ or } z = -4}$$

(2)

(b)(ii) Expanding the factorisation lets us compare the coefficient of z with the given polynomial.

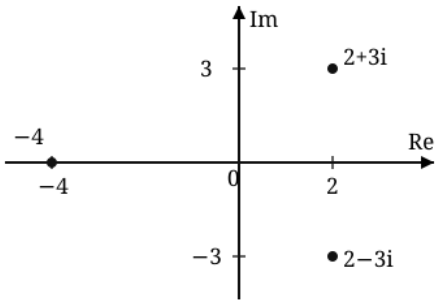
$$(z^2 - 4z + 13)(z + 4) = z^3 - 3z + 52.$$

Hence $a = -3$.

$$\boxed{a = -3}$$

(2)

(c) On an Argand diagram, plot real part horizontally and imaginary part vertically.



$$\boxed{\text{Points}(-4, 0), (2, 3) \text{and} (2, -3).}$$

(1)

Question 2

Question 2

Determine the values of x for which $64\cosh^4 x - 64\cosh^2 x - 9 = 0$, giving answers in the form $q \ln 2$, where q is rational.

Model solution

This is quadratic in $\cosh^2 x$*, so factor it before using the exponential definition of* $\cosh x$*.*

$$64\cosh^4 x - 64\cosh^2 x - 9 = (8\cosh^2 x - 9)(8\cosh^2 x + 1) = 0.$$

Since $\cosh^2 x \geqslant 1$*,* $\cosh^2 x = -\frac{1}{8}$ *is impossible. Thus*

$$\cosh x = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}.$$

Put $u = e^x > 0$*. This converts the hyperbolic equation into an ordinary quadratic.*

$$\frac{1}{2}\left(u + \frac{1}{u}\right) = \frac{3\sqrt{2}}{4} \quad \Longrightarrow \quad 2u^2 - 3\sqrt{2}u + 2 = 0,$$

$$u = \frac{3\sqrt{2} \pm \sqrt{18 - 16}}{4} = \frac{3\sqrt{2} \pm \sqrt{2}}{4} = \sqrt{2} \quad \textit{or} \quad \frac{1}{\sqrt{2}}.$$

Therefore $x = \ln\sqrt{2}$ *or* $x = \ln(1/\sqrt{2})$*.*

$$x = \pm\frac{1}{2}\ln 2$$

(4)

Question 3

Question 3

(a) Determine the general solution of $\cos x \dfrac{dy}{dx} + y \sin x = e^{2x} \cos^2 x$. (b) Given $y = 3$ when $x = 0$, determine the smallest positive x for which $y = 0$.

Model solution

(a) Divide by $\cos x$ *to obtain a linear differential equation. The coefficient of* y *is* $\tan x$*, so an integrating factor is* $e^{\int \tan x \, dx} = \sec x$.

$$\frac{dy}{dx} + y \tan x = e^{2x} \cos x, \qquad \sec x \frac{dy}{dx} + y \sec x \tan x = e^{2x}.$$

The left-hand side is $\dfrac{d}{dx}(y \sec x)$*, hence*

$$y \sec x = \frac{1}{2} e^{2x} + C, \qquad y = \left(\frac{1}{2} e^{2x} + C \right) \cos x.$$

$$y = \left(\frac{1}{2} e^{2x} + C \right) \cos x$$

(3)

(b) Apply the initial condition to fix the constant, then use the fact that the bracket is always positive. Thus zero can only occur when $\cos x = 0$.

$$3 = \frac{1}{2} + C \quad \Longrightarrow \quad C = \frac{5}{2},$$

$$y = \left(\frac{1}{2} e^{2x} + \frac{5}{2} \right) \cos x.$$

As $\frac{1}{2} e^{2x} + \frac{5}{2} > 0$, $y = 0 \iff \cos x = 0$. *The smallest positive solution is*

$$x = \frac{\pi}{2}$$

(3)

Question 4

Question 4

(a) Use differences to prove that, for $n > 2$, $\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) \equiv \ln\left(\frac{n(n+1)}{2}\right)$. (b) Hence find $\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)^{35}$.

Model solution

(a) The quotient inside each logarithm is the clue to use $\ln(A/B) = \ln A - \ln B$. *This makes the sum telescope.*

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) = \sum_{r=2}^{n} [\ln(r+1) - \ln(r-1)]$$

$$= (\ln 3 - \ln 1) + (\ln 4 - \ln 2) + (\ln 5 - \ln 3) + \cdots + (\ln(n+1) - \ln(n-1)).$$

All middle terms cancel, leaving

$$\ln n + \ln(n+1) - \ln 2 = \ln\left(\frac{n(n+1)}{2}\right).$$

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) = \ln\left(\frac{n(n+1)}{2}\right)$$

(4)

(b) First use $\ln^{35} u = 35 \ln u$, *then express the required range as the difference of two sums beginning at* $r = 2$.

$$\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)^{35} = 35\left[\ln\left(\frac{100 \cdot 101}{2}\right) - \ln\left(\frac{50 \cdot 51}{2}\right)\right]$$

$$= 35 \ln\left(\frac{5050}{1275}\right) = 35 \ln\left(\frac{202}{51}\right).$$

$$35 \ln\left(\frac{202}{51}\right)$$

(3)

Question 5

Question 5

$M = \begin{pmatrix} a & 2 & -3 \\ 2 & 3 & 0 \\ 4 & a & 2 \end{pmatrix}$, where a is a constant. (a) Show that M is non-singular for all a. (b) Determine M^{-1}.

Model solution

(a) A square matrix is non-singular precisely when its determinant is non-zero. Expand along the first row, which has a convenient minor.

$$\det M = a\begin{vmatrix} 3 & 0 \\ a & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 4 & a \end{vmatrix} = 6a - 8 - 3(2a - 12) = 28.$$

Since $28 \neq 0$, independently of a, M is non-singular for every value of a.

$$\det M = 28 \neq 0, \text{so} M \text{is non} - \text{singular for all} a.$$

(2)

(b) Use $M^{-1} = \frac{1}{\det M}\operatorname{adj}(M)$. Form the matrix of minors, apply the cofactor signs, and transpose.

Matrix of minors $= \begin{pmatrix} 6 & 4 & 2a - 12 \\ 4 + 3a & 2a + 12 & a^2 - 8 \\ 9 & 6 & 3a - 4 \end{pmatrix}$, *cofactor matrix* $= \begin{pmatrix} 6 & -4 & 2a - 1 \\ -4 - 3a & 2a + 12 & 8 - a^2 \\ 9 & -6 & 3a - 4 \end{pmatrix}$

$$\operatorname{adj}(M) = \begin{pmatrix} 6 & -4 - 3a & 9 \\ -4 & 2a + 12 & -6 \\ 2a - 12 & 8 - a^2 & 3a - 4 \end{pmatrix},$$

$$M^{-1} = \frac{1}{28}\begin{pmatrix} 6 & -4 - 3a & 9 \\ -4 & 2a + 12 & -6 \\ 2a - 12 & 8 - a^2 & 3a - 4 \end{pmatrix}$$

(4)

Question 6

Question 6

(a) Express $\dfrac{2x^2+3x+6}{(x+1)(x^2+4)}$ as partial fractions. (b) Hence show that $\displaystyle\int_0^2 \frac{2x^2+3x+6}{(x+1)(x^2+4)}\,dx = \ln a + \frac{b\pi}{4}$, finding a and b.

Model solution

(a) The denominator has one linear and one irreducible quadratic factor, so use a constant numerator over the linear factor and a linear numerator over the quadratic factor.

$$\frac{2x^2+3x+6}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}.$$

Multiplying through gives

$$2x^2+3x+6 = A(x^2+4) + (Bx+C)(x+1).$$

Comparing coefficients gives $2 = A+B$, $3 = B+C$, $6 = 4A+C$, *and hence* $A = 1$, $B = 1$, $C = 2$.

$$\frac{1}{x+1} + \frac{x+2}{x^2+4}$$

(3)

(b) Integrate the decomposition term by term. The $x/(x^2+4)$ *term uses the derivative of the denominator, while the constant-over-quadratic term gives an arctangent.*

$$\int_0^2 \left(\frac{1}{x+1} + \frac{x}{x^2+4} + \frac{2}{x^2+4}\right) dx = \left[\ln(x+1) + \frac{1}{2}\ln(x^2+4) + \arctan\left(\frac{x}{2}\right)\right]_0^2$$

$$= \ln 3 + \frac{1}{2}\ln 8 + \frac{\pi}{4} - \frac{1}{2}\ln 4 = \ln(3\sqrt{2}) + \frac{\pi}{4}.$$

$$\ln(3\sqrt{2}) + \frac{\pi}{4}, \text{so } a = 3\sqrt{2},\ b = 1.$$

(4)

Question 7

Question 7

Given $z = a + bi$, where a, b are real: (a) show zz^* is real. Given $zz^* = 18$ and $\frac{z}{z^*} = \frac{7}{9} + \frac{4\sqrt{2}}{9}i$, (b) determine the possible values of z.

Model solution

(a) Multiply a complex number by its conjugate; the cross terms cancel.

$$zz^* = (a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2.$$

As $a^2 + b^2$ is real, zz^* is real.

$$zz^* = a^2 + b^2 \in \mathbb{R}$$

(2)

(b) Multiplying the quotient by $zz^* = 18$ removes the conjugate denominator and produces an equation for z^2.

$$z^2 = 18\left(\frac{7}{9} + \frac{4\sqrt{2}}{9}i\right) = 14 + 8\sqrt{2}i.$$

Let $z = a + bi$. Equating real and imaginary parts of $z^2 = (a^2 - b^2) + 2abi$ gives

$$a^2 - b^2 = 14, \qquad 2ab = 8\sqrt{2}.$$

Also $a^2 + b^2 = 18$. Adding and subtracting the first and this equation gives

$$2a^2 = 32 \Rightarrow a^2 = 16, \qquad 2b^2 = 4 \Rightarrow b^2 = 2.$$

The condition $ab = 4\sqrt{2} > 0$ means a and b have the same sign.

$$z = 4 + \sqrt{2}i \quad \textit{or} \quad z = -4 - \sqrt{2}i$$

(5)

Question 8

Question 8

(a) Given $z^n + z^{-n} = 2\cos n\theta$, show $32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$. A paperweight is modelled by rotating through $180°$ about the x-axis the region under $y = H\cos^3(x/4)$, $-4 \leqslant x \leqslant 4$. Its maximum height is 2 cm. (b) Find H. (c) Find its volume. (d) State a limitation.

Model solution

(a) Raise the given result for $n = 1$ to the sixth power. Pairing reciprocal powers then allows the supplied identity to replace each pair by a multiple-angle cosine.

$$64\cos^6\theta = \left(z + \frac{1}{z}\right)^6 = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$$

$$= \left(z^6 + z^{-6}\right) + 6\left(z^4 + z^{-4}\right) + 15\left(z^2 + z^{-2}\right) + 20$$

$$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20.$$

Dividing by 2 proves the required identity.

$$32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

(5)

(b) At $x = 0$, $\cos^3(x/4) = 1$, so the maximum height equals H.

$$H = 2$$

(1)

(c) A $180°$ rotation gives semicircular cross-sections, of area $\frac{1}{2}\pi y^2$. The curve is even, so symmetry changes the integral from $[-4, 4]$ to twice the integral from $[0, 4]$.

$$V = \frac{1}{2}\pi\int_{-4}^{4}\left(2\cos^3\frac{x}{4}\right)^2 dx = 4\pi\int_0^4 \cos^6\frac{x}{4}\,dx.$$

Using part (a) with $\theta = x/4$,

$$V = \frac{4\pi}{32}\int_0^4\left(\cos\frac{3x}{2} + 6\cos x + 15\cos\frac{x}{2} + 10\right)dx$$

$$= \frac{\pi}{8}\left[\frac{2}{3}\sin\frac{3x}{2} + 6\sin x + 30\sin\frac{x}{2} + 10x\right]_0^4 = 24.56\ldots$$

$$V = 24.56\ \textit{cm}^3 (\text{to 2 d.p.})$$

(5)

(d) A mathematical curve cannot capture every physical feature of a manufactured object.

For example, the assumed curve may not accurately represent the paperweight's actual shape.

(1)

Question 9

Question 9

(i)(a) Explain why $\int_0^\infty \cosh x \, dx$ is improper. (b) Show it is divergent. (ii) Given $4\sinh x = p\cosh x$ has real solutions, find the range of real p.

Model solution

(i)(a) The interval of integration has an infinite upper endpoint.

Itisimproperbecauseitsupperlimitis∞(theintervalisunbounded).

(1)

(i)(b) Define the improper integral as a limit before integrating.

$$\int_0^\infty \cosh x \, dx = \lim_{t\to\infty} \int_0^t \cosh x \, dx = \lim_{t\to\infty} [\sinh x]_0^t = \lim_{t\to\infty} \sinh t.$$

Since $\sinh t \to \infty$ *as* $t \to \infty$*, the limit is not finite.*

The integral is divergent.

(3)

(ii) Divide by $\cosh x$*, which is always positive. The range* $-1 < \tanh x < 1$ *gives the condition for a real* x*.*

$$4\tanh x = p, \qquad -1 < \frac{p}{4} < 1.$$

$$-4 < p < 4$$

(2)

Question 10

Question 10

A pendulum angle θ, in radians, is modelled by $\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$. (a)(i) Show $\theta = \frac{1}{12}t\sin 3t$ is a particular solution. (ii) Find the general solution. Initially $\theta = \pi/3$ and the pendulum is at rest. At $t = 10$, $\theta = \alpha$. (b) Find α to 3 s.f. The true value is 0.62. (c) Evaluate the model. (d) Refine the equation for simple harmonic motion.

Model solution

(a)(i) The forcing term has the same frequency as the complementary solution, so the suggested particular integral contains the extra factor t. Differentiate using the product rule.

$$\theta_p = \frac{1}{12}t\sin 3t, \qquad \theta_p' = \frac{1}{12}\sin 3t + \frac{1}{4}t\cos 3t,$$

$$\theta_p'' = \frac{1}{2}\cos 3t - \frac{3}{4}t\sin 3t.$$

Therefore

$$\theta_p'' + 9\theta_p = \frac{1}{2}\cos 3t - \frac{3}{4}t\sin 3t + \frac{3}{4}t\sin 3t = \frac{1}{2}\cos 3t,$$

which verifies the particular solution.

$$\theta_p = \frac{1}{12}t\sin 3t$$

(4)

(a)(ii) Solve the associated homogeneous equation, then add the particular solution.

$$m^2 + 9 = 0 \quad \Longrightarrow \quad m = \pm 3i,$$

$$\theta = A\cos 3t + B\sin 3t + \frac{1}{12}t\sin 3t.$$

(4)

(b) "At rest" means $\theta'(0) = 0$. Apply the two initial conditions to determine A and B, then evaluate at $t = 10$.

$$\theta(0) = A = \frac{\pi}{3}.$$

$$\theta' = -3A\sin 3t + 3B\cos 3t + \frac{1}{12}\sin 3t + \frac{1}{4}t\cos 3t,$$

so $\theta'(0) = 3B = 0$, giving $B = 0$. Hence

$$\alpha = \frac{\pi}{3}\cos 30 + \frac{10}{12}\sin 30 = -0.661827\ldots$$

$$\alpha = -0.662 \textbf{ radians} (3\text{s.f.})$$

(4)

(c) Compare the size predicted by the model with the true value; a small difference supports the model at this time.

$$|-0.662| = 0.662 \quad \textit{is close to} \quad 0.62.$$

$$\text{The model is reasonably good at } t = 10\text{s, since it predicts an angle of magnitude } 0.662, \text{ close to } 0.62.$$

(1)

(d) Simple harmonic motion has no external forcing term, so remove the right-hand side.

$$\frac{d^2\theta}{dt^2} + 9\theta = 0.$$

(1)

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