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Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/01 - Core Pure Mathematics 1

Thursday 25 May 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

The equation $x^3 - 7x^2 - 12x + 6 = 0$ has roots α, β, γ. Without solving it, find the cubic equation with roots $\alpha + 2, \beta + 2, \gamma + 2$, in the form $w^3 + pw^2 + qw + r = 0$.

Model solution

A shift of every root by 2 is handled by setting $w = x + 2$, so $x = w - 2$. This transforms the equation without finding its roots.

$$(w-2)^3 - 7(w-2)^2 - 12(w-2) + 6 = 0$$

$$w^3 - 6w^2 + 12w - 8 - 7w^2 + 28w - 28 - 12w + 24 + 6 = 0,$$

$$w^3 - 13w^2 + 28w - 6 = 0.$$

$$w^3 - 13w^2 + 28w - 6 = 0.$$

(5)

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Question 2

Question 2

(a) Write x^2+4x-5 as $(x+p)^2+q$. (b) Hence find $\int \frac{1}{\sqrt{x^2+4x-5}}\,dx$. (c) Find the mean value of $f(x)=1/\sqrt{x^2+4x-5}$ on $3 \leqslant x \leqslant 13$.

Model solution

(a) Completing the square makes the quadratic match a standard integral form.

$$x^2+4x-5=(x+2)^2-4-5=(x+2)^2-9.$$

$$x^2+4x-5=(x+2)^2-9.$$

(1)

(b) This is now the standard $\sqrt{u^2-a^2}$ integral, with $u=x+2$ and $a=3$.

$$\int \frac{1}{\sqrt{x^2+4x-5}}\,dx=\ln\left|x+2+\sqrt{(x+2)^2-9}\right|+C.$$

$$\ln\left|x+2+\sqrt{x^2+4x-5}\right|+C.$$

(2)

(c) The mean value is $\frac{1}{b-a}\int_a^b f(x)\,dx$. The interval length is 10.

$$\bar{f}=\frac{1}{10}\left[\ln\left(x+2+\sqrt{(x+2)^2-9}\right)\right]_3^{13}=\frac{1}{10}\ln\left(\frac{15+6\sqrt{6}}{9}\right).$$

$$\frac{1}{10}\ln\left(\frac{5+2\sqrt{6}}{3}\right).$$

(3)

Question 3

Question 3

Let $z_1 = -4 + 4\mathrm{i}$ and $z_2 = 3(\cos\frac{17\pi}{12} + \mathrm{i}\sin\frac{17\pi}{12})$. (a) Express z_1 in modulus–argument form. (b) Find (i) z_1/z_2 and (ii) z_2^4, in Cartesian form. (c) On one Argand diagram show $z_1, z_2, z_1/z_2$, and the region $|z - z_1| \leqslant |z - z_2|$.

Model solution

(a) The point $(-4, 4)$ has modulus $4\sqrt{2}$ and lies in quadrant II, so its argument is $3\pi/4$.

$$z_1 = 4\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right).$$

$$z_1 = 4\sqrt{2}(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}).$$

(2)

(b)(i) Divide moduli and subtract arguments when dividing complex numbers in polar form.

$$\frac{z_1}{z_2} = \frac{4\sqrt{2}}{3}\left(\cos\left(\frac{3\pi}{4} - \frac{17\pi}{12}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4} - \frac{17\pi}{12}\right)\right) = \frac{4\sqrt{2}}{3}\left(\cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3}\right).$$

$$\frac{z_1}{z_2} = -\frac{2\sqrt{2}}{3} - \frac{2\sqrt{6}}{3}\mathrm{i}.$$

(2)

(b)(ii) De Moivre's theorem raises the modulus to the fourth power and multiplies the argument by four.

$$z_2^4 = 3^4\left(\cos\frac{17\pi}{3} + \mathrm{i}\sin\frac{17\pi}{3}\right) = 81\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right).$$

$$z_2^4 = \frac{81}{2} - \frac{81\sqrt{3}}{2}\mathrm{i}.$$

(2)

(c) The equality boundary consists of points equidistant from z_1 and z_2, so it is the perpendicular bisector of their joining segment. The required side is the one containing z_1. Plot $z_1 = (-4, 4)$, $z_2 = (-0.78, -2.90)$, and $z_1/z_2 = (-0.94, -1.63)$, then draw that bisector and shade its z_1-side, including the boundary. The construction is supplied in `assets/argand-question-3.png`.

An Argand diagram with those three labelled points and the stated closed half-plane.

(4)

Question 4

Question 4

Prove by induction that, for $n \in \mathbb{N}$, $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix}$.

Model solution

Induction is required because the identity is claimed for every natural number. First verify the base case.

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2(1) \\ 0 & 1 \end{pmatrix}.$$

Assume it is true for $n = k$, *so* $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix}$. *Then*

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2k-2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2(k+1) \\ 0 & 1 \end{pmatrix}.$$

Thus truth for k *implies truth for* $k+1$*; with the base case, the result follows by induction.*

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix} \text{ for all } n \in \mathbb{N}.$$

(5)

Question 5

Question 5

$\ell_1 : (x+5)/1 = (y+4)/(-3) = (z-3)/5$, and $\Pi_1 : 2x+3y-2z = 6$. (a) Find their intersection. (b) Show that the reflection ℓ_2 is $\mathbf{r} = (-7,2,-7) + \mu(10,6,2)$. (c) Find $\Pi_1 \cap \Pi_2$, where Π_2 contains ℓ_1, ℓ_2. (d) Given $\Pi_3 : \mathbf{r} \cdot (1,1,a) = b$ and that the three planes form a sheaf, find a, b.

Model solution

(a) Substitute the parametric coordinates of the line into the plane equation.

$$2(-5+\lambda) + 3(-4-3\lambda) - 2(3+5\lambda) = 6 \implies -28 - 17\lambda = 6 \implies \lambda = -2.$$

$$(-7, 2, -7).$$

(2)

(b) Reflection in a plane reverses the component in the normal direction. Use $P = (-5,-4,3)$ *and normal* $\mathbf{n} = (2,3,-2)$. *The perpendicular through* P *is* $\mathbf{r} = P + t\mathbf{n}$.

$$2(-5+2t) + 3(-4+3t) - 2(3-2t) = 6 \implies -28 + 17t = 6 \implies t = 2.$$

$$H = (-1,2,-1), \qquad P' = 2H - P = (3,8,-5).$$

The reflected line contains the fixed point $(-7,2,-7)$ *and* $P' = (3,8,-5)$, *so its direction is*

$$(3,8,-5) - (-7,2,-7) = (10,6,2).$$

Hence

$$\mathbf{r} = (-7,2,-7) + \mu(10,6,2).$$

$$\mathbf{r} = (-7,2,-7) + \mu(10,6,2).$$

(5)

(c) First find Π_2. *Its normal is perpendicular to both line directions, so take their cross product. Then solve the two plane equations together.*

$$(1,-3,5) \times (10,6,2) = (-36,48,36) \parallel (-3,4,3),$$

$$\Pi_2 : \ -3x + 4y + 3z = 8.$$

$$2x + 3y - 2z = 6, \quad -3x + 4y + 3z = 8 \implies y = 2, \quad x = z.$$

$$\mathbf{r} = (-7,2,-7) + s(1,0,1).$$

$$\mathbf{r} = (-7, 2, -7) + s(1, 0, 1).$$

(2)

(d) A line common to all three planes must have direction perpendicular to the normal $(1, 1, a)$ ***of*** Π_3.

$$(1, 0, 1) \cdot (1, 1, a) = 1 + a = 0 \implies a = -1.$$

$$(-7, 2, -7) \cdot (1, 1, -1) = -7 + 2 + 7 = 2,$$

so $b = 2$.

$$a = -1, \ b = 2.$$

(3)

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Question 6

Question 6

Initially a tank contains 10 litres. Water enters at $3-\frac{4}{1+e^{0.8t}}$ litres per minute and leaves at a rate proportional to volume. Initially the volume decreases at 3 litres per minute. (a) Show $\frac{dV}{dt} = 3-\frac{4}{1+e^{0.8t}}-0.4V$. (b) Find $\frac{d}{dt}(\arctan e^{0.4t})$. (c) Solve for V. (d) Evaluate the model given $V(10) = 8$.

Model solution

(a) Rate of change is inflow minus outflow. Since the outflow is proportional to volume, call it kV.

$$\frac{dV}{dt} = 3 - \frac{4}{1+e^{0.8t}} - kV.$$

$$-3 = 3 - \frac{4}{2} - 10k = 1 - 10k \implies k = 0.4.$$

$$\frac{dV}{dt} = 3 - \frac{4}{1+e^{0.8t}} - 0.4V.$$

(3) (b) Apply the chain rule to the arctangent.

$$\frac{d}{dt}(\arctan e^{0.4t}) = \frac{0.4e^{0.4t}}{1+e^{0.8t}}.$$

$$\frac{0.4e^{0.4t}}{1+e^{0.8t}}.$$

(2) (c) This linear differential equation needs integrating factor $e^{0.4t}$*, which makes the left side one product derivative.*

$$\frac{d}{dt}(Ve^{0.4t}) = 3e^{0.4t} - \frac{4e^{0.4t}}{1+e^{0.8t}}.$$

$$Ve^{0.4t} = 7.5e^{0.4t} - 10\arctan(e^{0.4t}) + C,$$

$$V = 7.5 - 10e^{-0.4t}\arctan(e^{0.4t}) + Ce^{-0.4t}.$$

$$10 = 7.5 - \frac{10\pi}{4} + C \implies C = 2.5 + \frac{5\pi}{2}.$$

$$V = 7.5 - 10e^{-0.4t}\arctan(e^{0.4t}) + \left(2.5 + \frac{5\pi}{2}\right)e^{-0.4t}.$$

(6) (d) The model gives $V(10) \approx 7.4$ *litres, rather than the observed* 8 *litres.*

It underestimates by about 0.6 litres (about 7.5%), so it is not very accurate.

(1)

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Question 7

Question 7

(a) Explain why $\sum_{r=1}^{2n}(-1)^r f(r) = \sum_{r=1}^{n}(f(2r)-f(2r-1))$. (b) Show that $\sum_{r=1}^{2n} r(((-1)^r+2r)^2) = n(2n+1)(8n^2+4n+5)$. (c) Hence evaluate $\sum_{r=14}^{50} r(((-1)^r+2r)^2)$.

Model solution

(a) Even subscripts have positive sign and odd subscripts negative sign, so pair each even term with its preceding odd term.

$$-f(1)+f(2)-f(3)+f(4)+\cdots-f(2n-1)+f(2n) = \sum_{r=1}^{n}(f(2r)-f(2r-1)).$$

The sums agree by grouping odd--even pairs.

(2) (b) Expand first, then use standard formulas for $\sum r$, $\sum r^3$, and part (a) for the alternating term.

$$r(((-1)^r+2r)^2) = r+4(-1)^r r^2+4r^3.$$

$$S = n(2n+1)+4\sum_{r=1}^{n}((2r)^2-(2r-1)^2)+4[n(2n+1)]^2.$$

$$\sum_{r=1}^{n}((2r)^2-(2r-1)^2) = \sum_{r=1}^{n}(4r-1) = n(2n+1),$$

$$S = n(2n+1)+4n(2n+1)+4n^2(2n+1)^2 = n(2n+1)(8n^2+4n+5).$$

$$n(2n+1)(8n^2+4n+5).$$

(6) (c) Subtract the terms up to 13. To use part (b), split these as the first 12 terms plus term 13.

$$\sum_{r=14}^{50} = \sum_{r=1}^{50} - \sum_{r=1}^{12} -13(25)^2 = 25(51)(5105)-6(13)(317)-8125.$$

$$6476024.$$

(4)

Question 8

Question 8

A population is modelled by $\begin{pmatrix} N_{n+1} \\ J_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix} \begin{pmatrix} N_n \\ J_n \\ B_n \end{pmatrix}$. Initially there are breeders only. After two months $N = 48, J = 40$. Answer all parts (a)–(e).

Model solution

(a) A model with strict age categories makes simplifying assumptions.

For example, not every mammal over three months will necessarily be able to breed.

(1) (b)(i) Let the initial number of breeders be k. *Applying the matrix once gives* $(2k, 0, 0.96k)^T$, *so after a second month* $N_2 = 2(0.96k)$.

$$1.92k = 48 \implies k = 25.$$

Initially there were 25 *mammals.*

(3) (b)(ii) The second-month juvenile number is $J_2 = aN_1$.

$$40 = a(2 \times 25) = 50a \implies a = 0.8.$$

$$a = 0.8.$$

(1) (c) The determinant is non-zero, so an inverse exists. Solve the three transition equations backwards to form it.

$$\det M = 0.768, \qquad M^{-1} = \begin{pmatrix} \frac{5b}{4} & \frac{5}{4} & -\frac{125b}{48} \\ -1 & 0 & \frac{25}{12} \\ \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 0 & 2 \\ 0.8 & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{5b}{4} & \frac{5}{4} & -\frac{125b}{48} \\ -1 & 0 & \frac{25}{12} \\ \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

(3) (d) Work backwards from the known next-month vector, then use the total 1015. *The last and first transition equations give* B *and* J *directly.*

$$B = 596/2 = 298, \qquad J = \frac{437 - 0.96(298)}{0.48} = \frac{3773}{12}.$$

$$N = 1015 - 298 - \frac{3773}{12} = \frac{4831}{12}, \qquad 464 = 0.8N + bJ \implies b \approx 0.4515.$$

$b = 0.45$ *(to 2 d.p.).*

(3) *(e) Split newborns into male and female. The breeder contribution of* 2 *becomes* $0.42(2) = 0.84$ *male and* $0.58(2) = 1.16$ *female newborns.*

$$\begin{pmatrix} NM_{n+1} \\ NF_{n+1} \\ J_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0.84 \\ 0 & 0 & 0 & 1.16 \\ ? & ? & b & 0 \\ 0 & 0 & 0.48 & 0.96 \end{pmatrix} \begin{pmatrix} NM_n \\ NF_n \\ J_n \\ B_n \end{pmatrix}.$$

The refined matrix shown, with the two unknown sex-dependent juvenile transitions.

(2)

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