Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/01 - Core Pure Mathematics 1

Wednesday 22 May 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Let $f(z) = z^4 - 6z^3 + az^2 + bz + 145$, where a, b are real. Given that $2 + 5i$ is a root of $f(z) = 0$: (a) determine the other roots; (b) show all roots on one Argand diagram.

Model solution

(a) Real coefficients mean that a non-real root brings its complex conjugate. Hence $2-5i$ *is a root, and the conjugate pair gives*

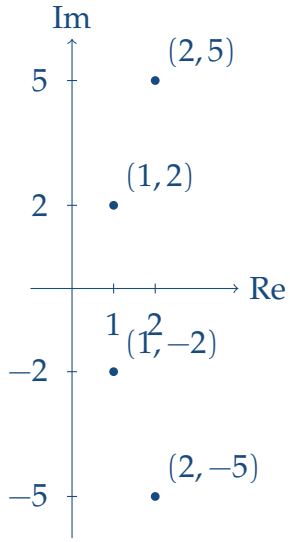
$$(z-(2+5i))(z-(2-5i)) = (z-2)^2 + 25 = z^2 - 4z + 29.$$

The remaining factor is monic quadratic. Comparing the z^3 *and constant coefficients in* $(z^2 - 4z + 29)(z^2 + cz + d)$ *with the given polynomial gives* $c = -2$, $d = 5$.

$$z^2 - 2z + 5 = 0 \quad \Rightarrow \quad z = 1 \pm 2i.$$

The other roots are $2 - 5i, 1 + 2i, 1 - 2i$.

((7)) (b) Plot each root as $(\Re z, \Im z)$*; conjugate pairs are reflections in the real axis.*



((2))

Question 5

Question 5

A raindrop falls from rest. Its downward velocity v m s^{-1}, after t seconds, satisfies $(t+2)\frac{dv}{dt}+3v = k(t+2)-3$, $t \geqslant 0$, where $k > 0$. (a) Show $v = \frac{k}{4}(t+2) - 1 + \frac{4(2-k)}{(t+2)^3}$. Given $v = 4$ when $t = 2$: (b) find v at $t = 5$; (c) comment for very large t.

Model solution

(a) Divide by $t+2$ *to obtain a linear equation. Its integrating factor is* $e^{\int 3/(t+2)\,dt} = (t+2)^3$.

$$\frac{dv}{dt} + \frac{3}{t+2}v = k - \frac{3}{t+2}, \qquad \frac{d}{dt}\big(v(t+2)^3\big) = k(t+2)^3 - 3(t+2)^2.$$

$$v(t+2)^3 = \frac{k}{4}(t+2)^4 - (t+2)^3 + C.$$

The initial condition "falls from rest" gives $v(0) = 0$, *hence* $C = 8 - 4k = 4(2-k)$.

$$v = \frac{k}{4}(t+2) - 1 + \frac{4(2-k)}{(t+2)^3}.$$

((5)) (b) Use the observation at $t = 2$ *to find the model parameter before predicting the later velocity.*

$$4 = k - 1 + \frac{2-k}{16} \quad \Rightarrow \quad k = 5.2.$$

$$v(5) = \frac{5.2}{4}(7) - 1 + \frac{4(2-5.2)}{7^3} = 8.062\ldots$$

8.06 m s^{-1} ***downwards.***

((3)) (c) As $t \to \infty$, $v \sim \frac{k}{4}(t+2) - 1$, *which increases without bound. A raindrop reaches the ground and has a finite terminal velocity, so the model is not valid for very large* t. *((1))*

Question 6

Question 6

Prove by induction, for all positive integers n, that $\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(4n^2-1)$.

Model solution

Start with a base case, assume the result for k, then add the next term to prove the $k+1$ case.

$$n=1: \quad \sum_{r=1}^{1}(2r-1)^2 = 1 = \frac{1}{3}(1)(4(1)^2-1).$$

So the statement is true for $n=1$. Assume that

$$\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(4k^2-1).$$

$$\sum_{r=1}^{k+1}(2r-1)^2 = \frac{1}{3}k(4k^2-1)+(2k+1)^2 = \frac{1}{3}(4k^3+12k^2+11k+3).$$

$$= \frac{1}{3}(k+1)(4k^2+8k+3) = \frac{1}{3}(k+1)\big(4(k+1)^2-1\big).$$

Thus true for $n=k$ implies true for $n=k+1$. Together with the base case, the result is true for all positive integers n. *((6))*

Question 7

Question 7

$l_1 : \mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$, $l_2 : \mathbf{r} = 5\mathbf{i} + p\mathbf{j} - 7\mathbf{k} + \mu(6\mathbf{i} + \mathbf{j} + 8\mathbf{k})$, lie in plane Π. (a) Show $3\mathbf{i} - 10\mathbf{j} - \mathbf{k} \perp \Pi$. (b) Find Π. (c) Find p. The lines meet at A, and $B = (12, -11, 6)$. (d) Find the acute angle between AB and Π.

Model solution

(a) A vector orthogonal to both direction vectors in the plane is a normal.

$$(3, -10, -1) \cdot (2, 1, -4) = 0, \qquad (3, -10, -1) \cdot (6, 1, 8) = 0.$$

Thus $(3, -10, -1)$ *is perpendicular to* Π. *((2)) (b) Use this normal and the point* $(1, -2, 3)$ *on* l_1.

$$(3, -10, -1) \cdot (x - 1, y + 2, z - 3) = 0 \quad \Rightarrow \quad 3x - 10y - z = 20.$$

$$3x - 10y - z = 20.$$

((2)) (c) The point $(5, p, -7)$ *on* l_2 *lies in the plane.*

$$15 - 10p + 7 = 20 \quad \Rightarrow \quad p = 0.2.$$

$$p = 0.2.$$

((2)) (d) Equating line coordinates gives $\lambda = 2.3$, $\mu = 0.1$, *hence* $A = (5.6, 0.3, -6.2)$.

$$\mathbf{AB} = (6.4, -11.3, 12.2), \qquad \mathbf{AB} \cdot (3, -10, -1) = 120.$$

For line--plane angle α, *use* $\sin\alpha = |\mathbf{AB} \cdot \mathbf{n}|/(|\mathbf{AB}|\,|\mathbf{n}|)$.

$$\sin\alpha = \frac{120}{\sqrt{6.4^2 + 11.3^2 + 12.2^2}\sqrt{3^2 + 10^2 + 1^2}} = 0.642\ldots \quad \Rightarrow \quad \alpha = 39.9\ldots^\circ.$$

40°, *to the nearest degree.*

((4))

Question 2

Question 2

The roots of $2x^3 - 3x^2 + 12x + 7 = 0$ are α, β, γ. Without solving: (a) find $\alpha + \beta + \gamma$, $\alpha\beta + \alpha\gamma + \beta\gamma$, $\alpha\beta\gamma$; (b) find (i) $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma}$, (ii) $(\alpha - 1)(\beta - 1)(\gamma - 1)$, (iii) $\alpha^2 + \beta^2 + \gamma^2$.

Model solution

(a) Divide by two and compare with $x^3 - S_1x^2 + S_2x - S_3 = 0$, so Vieta's formulae give

$$\alpha + \beta + \gamma = \frac{3}{2}, \qquad \alpha\beta + \alpha\gamma + \beta\gamma = 6, \qquad \alpha\beta\gamma = -\frac{7}{2}.$$

((1)) (b)(i) Combine the fractions, then use the known pairwise sum and product.

$$\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} = \frac{2(\alpha\beta + \alpha\gamma + \beta\gamma)}{\alpha\beta\gamma} = \frac{2(6)}{-\frac{7}{2}}.$$

$$-\frac{24}{7}.$$

((2)) (b)(ii) Expand the product, then substitute the three symmetric sums.

$$(\alpha - 1)(\beta - 1)(\gamma - 1) = \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) - 1 = -9.$$

$$-9.$$

((3)) (b)(iii) Squaring the root sum introduces the known pairwise sum.

$$\alpha^2 + \beta^2 + \gamma^2 = \left(\frac{3}{2}\right)^2 - 2(6) = -\frac{39}{4}.$$

$$-\frac{39}{4}.$$

((2))

Question 3

Question 3

A pool edge is $r = 4 - a\sin 3\theta$, $0 \leqslant \theta \leqslant 2\pi$, inside a circle of radius 6 m with the same centre. Its shortest distance to the circle is 0.5 m. (a) Find a. (b) Find the exact tiled area between them.

Model solution

(a) The shortest radial gap is at the largest pool radius. Since $\sin 3\theta \geqslant -1$, *this maximum is* $4 + a$.

$$6 - (4 + a) = 0.5 \quad \Rightarrow \quad a = \frac{3}{2}.$$

$$a = \frac{3}{2}.$$

((2)) (b) For a polar curve use $\frac{1}{2}\int r^2\,d\theta$, *then subtract the pool area from the circle area.*

$$A_P = \frac{1}{2}\int_0^{2\pi}\left(4 - \frac{3}{2}\sin 3\theta\right)^2 d\theta = \frac{1}{2}\int_0^{2\pi}\left(16 - 12\sin 3\theta + \frac{9}{4}\sin^2 3\theta\right) d\theta.$$

Use $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$.

$$A_P = \frac{1}{2}\left[\frac{137}{8}\theta + 4\cos 3\theta - \frac{3}{16}\sin 6\theta\right]_0^{2\pi} = \frac{137\pi}{8}.$$

$$A_T = 36\pi - \frac{137\pi}{8} = \frac{151\pi}{8}\ \text{m}^2.$$

$$\frac{151\pi}{8}\ \text{m}^2.$$

((6))

Question 4

Question 4

Let $z = e^{i\theta}$, where θ is real. (a) Show $z^n + z^{-n} \equiv 2\cos n\theta$. (b) Show $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$. (c) Hence solve $\cos 5\theta + 5\cos 3\theta + 12\cos\theta = 0$, $0 \leqslant \theta < 2\pi$.

Model solution

(a) Substitute exponential form; the imaginary parts cancel.

$$z^n + z^{-n} = e^{in\theta} + e^{-in\theta} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) = 2\cos n\theta.$$

((2)) (b) Expand $(z + z^{-1})^5$, using $z + z^{-1} = 2\cos\theta$, then group powers as in part (a).

$$32\cos^5\theta = z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5} = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta.$$

$$\therefore \quad \cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta).$$

((5)) (c) Part (b) turns the multiple-angle expression into a factorised equation in $\cos\theta$.

$$16\cos^5\theta + 2\cos\theta = 0 \quad \Rightarrow \quad 2\cos\theta(8\cos^4\theta + 1) = 0.$$

Because $8\cos^4\theta + 1 > 0$, only $\cos\theta = 0$ is possible.

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

((3))

Question 8

Question 8

The bacteria populations satisfy $dx/dt = x + y$ and $dy/dt = 3y - 2x$, with t in days. (a) Show $d^2x/dt^2 - 4dx/dt + 5x = 0$. (b) Find a general solution for x. (c) Find one for y. Given $x = 100$, $y = 275$ at $t = 0$, the populations are equal after T hours: (d) find T; (e) state a limitation.

Model solution

(a) Differentiate the first equation, substitute for dy/dt, *then eliminate* y *using* $y = dx/dt - x$.

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + 3y - 2x = \frac{dx}{dt} + 3\left(\frac{dx}{dt} - x\right) - 2x = 4\frac{dx}{dt} - 5x.$$

$$\therefore \quad \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 5x = 0.$$

((3)) (b) The auxiliary equation has complex roots, giving an exponential-trigonometric complementary function.

$$m^2 - 4m + 5 = 0 \Rightarrow m = 2 \pm i, \qquad x = e^{2t}(A\cos t + B\sin t).$$

$$\boxed{x = e^{2t}(A\cos t + B\sin t).}$$

((4)) (c) Use $y = dx/dt - x$, *which keeps the constants linked to those in* x.

$$\frac{dx}{dt} = e^{2t}\big((2A + B)\cos t + (2B - A)\sin t\big),$$

$$y = e^{2t}\big((A + B)\cos t + (B - A)\sin t\big).$$

$$\boxed{y = e^{2t}\big((A + B)\cos t + (B - A)\sin t\big).}$$

((2)) (d) The initial values determine $A = 100$ *and* $A + B = 275$, *so* $B = 175$.

$$x = e^{2t}(100\cos t + 175\sin t), \quad y = e^{2t}(275\cos t + 75\sin t).$$

Set $x = y$; *the non-zero exponential factor cancels. The first positive time is required.*

$$100\cos t + 175\sin t = 275\cos t + 75\sin t \Rightarrow \tan t = 1.75.$$

$$t = \tan^{-1}(1.75) = 1.05165\ldots \text{ days}, \qquad T = 24t = 25.2396\ldots \text{ hours}.$$

$$\boxed{T = 25.24 \text{ hours}.}$$

((5)) (e) The trigonometric formulae produce negative population values at some times; a number of bacteria cannot be negative, so this limits the model. *((1))*