Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/02 - Core Pure Mathematics 2

Thursday 9 June 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

For $z_1 = 3 - 3\mathrm{i}$ and $z_2 = \sqrt{3} + \mathrm{i}$, a student's work for $\arg(z_1/z_2)$ has errors in its argument for z_1 and in its quotient rule. Correct both errors and find $\arg(z_1/z_2)$.

Model solution

(a)(i) The sign of the imaginary part tells us that z_1 is in the fourth quadrant, so its principal argument must be negative.

$$\arg z_1 = \tan^{-1}\left(\frac{-3}{3}\right) = -\frac{\pi}{4}.$$

The correction to line 1 is $\arg z_1 = -\pi/4$.

(1) (a)(ii) Arguments are subtracted when complex numbers are divided; they are not divided.

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2.$$

The correction to line 3 is $\arg(z_1/z_2) = \arg z_1 - \arg z_2$.

(1) (b) The first-quadrant number $z_2 = \sqrt{3} + \mathrm{i}$ has argument $\pi/6$. Subtract it from the corrected argument of z_1.

$$\arg\left(\frac{z_1}{z_2}\right) = -\frac{\pi}{4} - \frac{\pi}{6} = -\frac{5\pi}{12}.$$

$$\arg(z_1/z_2) = -\frac{5\pi}{12}.$$

(1)

Question 2

Question 2

In 2019 a college had 1110 students on Construction, Design and Hospitality. Construction had 370 more students than Hospitality. In 2020 the respective changes were 1.25%, 2.5% and -2%, and the total increased by 0.27% to 2 significant figures. Form and solve a matrix model for the 2019 enrolments.

Model solution

(a) Let C, D, H *be the 2019 numbers on Construction, Design and Hospitality respectively. The percentage increase in the total is* $0.0027 \times 1110 = 2.997$*, which rounds to 3 students, so the 2020 total is* 1113*. Each percentage change is applied to its own 2019 number.*

$$C + D + H = 1110, \qquad C - H = 370,$$

$$1.0125C + 1.025D + 0.98H = 1113.$$

These three equations model the data.

(4) (b) The three simultaneous linear equations are suited to a matrix equation.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix} \begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix}.$$

$$\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}^{-1} \begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix} = \begin{pmatrix} 720 \\ 40 \\ 350 \end{pmatrix}.$$

In 2019 there were 720 *Construction students,* 40 *Design students and* 350 *Hospitality students.*

(4)

Question 3

Question 3

$M = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$. Prove by induction that $M^n = \begin{pmatrix} 3^n & \frac{a}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}$, then use an area transformation and $(2, -2) \mapsto (123, -2)$ to find n and a.

Model solution

(a) Induction is appropriate because the claim is for every positive integer n. First check the base case, then multiply the assumed k-case by M.

$$M^1 = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3^1 & \frac{a}{2}(3^1 - 1) \\ 0 & 1 \end{pmatrix}.$$

So the statement holds for $n = 1$. Assume it holds for $n = k$, so

$$M^k = \begin{pmatrix} 3^k & \frac{a}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} M^{k+1} &= M^k M \\ &= \begin{pmatrix} 3^k & \frac{a}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{k+1} & 3^k a + \frac{a}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{k+1} & \frac{a}{2}(3^{k+1} - 1) \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Thus the result for k implies the result for $k + 1$. Since the base case is true, it holds for all $n \in \mathbb{N}$.

$$M^n = \begin{pmatrix} 3^n & \frac{a}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix} \text{ for all } n \in \mathbb{N}.$$

(6) (b)(i) A linear transformation multiplies area by the absolute value of its determinant. Here $\det(M^n) = 3^n$.

$$5 \times 3^n = 1215 \quad \Rightarrow \quad 3^n = 243 = 3^5.$$

$$n = 5.$$

(3) (b)(ii) Substitute $n = 5$ into the formula for M^n, then use the known image of A.

$$\begin{pmatrix} 243 & \frac{a}{2}(243 - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 123 \\ -2 \end{pmatrix}.$$

$$486 - a(242) = 123 \quad \Rightarrow \quad 242a = 363 \quad \Rightarrow \quad a = \frac{3}{2}.$$

$$a = \frac{3}{2}.$$

(2)

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Question 4

Question 4

Given $z_1 = 6e^{i\pi/3}$ and $z_2 = 6\sqrt{3}e^{5i\pi/6}$, show that their sum is $12e^{2i\pi/3}$. If $\arg(z-5) = 2\pi/3$, find the least possible $|z|$.

Model solution

(i) To add complex numbers, convert from modulus--argument form to Cartesian form.

$$z_1 = 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 3 + 3\sqrt{3}i, \qquad z_2 = 6\sqrt{3}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -9 + 3\sqrt{3}i.$$

$$z_1 + z_2 = -6 + 6\sqrt{3}i = 12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 12e^{2i\pi/3}.$$

$$z_1 + z_2 = 12e^{2i\pi/3}.$$

(3) (ii) The condition puts z on the ray from $(5,0)$ at angle $2\pi/3$. The least $|z|$ is the perpendicular distance from the origin to this ray, found by minimising squared distance.

$$z = 5 + t\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right), \qquad t \geqslant 0,$$

$$|z|^2 = \left(5 - \frac{t}{2}\right)^2 + \left(\frac{\sqrt{3}t}{2}\right)^2 = 25 - 5t + t^2.$$

$$\frac{d}{dt}|z|^2 = 2t - 5 = 0 \Rightarrow t = \frac{5}{2}, \qquad |z|^2 = 25 - 5\left(\frac{5}{2}\right) + \left(\frac{5}{2}\right)^2 = \frac{75}{4}.$$

$$\min|z| = \sqrt{\frac{75}{4}} = \frac{5\sqrt{3}}{2}.$$

(3)

Question 5

Question 5

Show that $d(\arcsin x)/dx = 1/\sqrt{1-x^2}$, and prove that $f(x) = \arcsin(e^x)$, $x \leqslant 0$, has no stationary points.

Model solution

(a) Use implicit differentiation because $y = \arcsin x$ *is defined by* $\sin y = x$. *The range of* $\arcsin$ *gives* $\cos y \geqslant 0$, *fixing the square-root sign.*

$$\sin y = x \Rightarrow \cos y \frac{dy}{dx} = 1, \qquad \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}.$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}.$$

(3) (b) Apply the chain rule to the outer $\arcsin$ *function. Since* $x \leqslant 0$, $0 < e^x \leqslant 1$, *so the function is defined.*

$$f'(x) = \frac{e^x}{\sqrt{1-(e^x)^2}} = \frac{e^x}{\sqrt{1-e^{2x}}} \qquad (x < 0).$$

For every $x < 0$, *numerator and denominator are positive, so* $f'(x) > 0$, *never zero. At* $x = 0$ *the derivative is not finite, so this endpoint is not stationary.*

$f(x)$ *has no stationary points.*

(3)

Question 6

Question 6

The roots of $4x^3 + px^2 - 14x + q = 0$, where $p, q > 0$, are α, β, γ. Given $\alpha^2 + \beta^2 + \gamma^2 = 16$ and $1/\alpha + 1/\beta + 1/\gamma = 14/3$, find p, q, and $(\alpha - 1)(\beta - 1)(\gamma - 1)$ without solving the cubic.

Model solution

(a) Vieta's relations connect the coefficients to symmetric combinations of roots, so there is no need to solve the cubic. Dividing by 4 gives

$$x^3 + \frac{p}{4}x^2 - \frac{7}{2}x + \frac{q}{4} = 0, \qquad \alpha + \beta + \gamma = -\frac{p}{4}, \quad \alpha\beta + \alpha\gamma + \beta\gamma = -\frac{7}{2}.$$

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma) = 16 - 7 = 9.$$

Because $p > 0$, the sum of roots is $-p/4 < 0$, so it is -3, not $+3$.

$$-\frac{p}{4} = -3 \Rightarrow p = 12.$$

$$p = 12.$$

(3) (b) The reciprocal sum is the pairwise-product sum divided by the product.

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} = \frac{-7/2}{\alpha\beta\gamma} = \frac{14}{3}.$$

$$\alpha\beta\gamma = -\frac{3}{4} = -\frac{q}{4} \Rightarrow q = 3.$$

$$q = 3.$$

(3) (c) Evaluating the polynomial at $x = 1$ gives the product $(1 - \alpha)(1 - \beta)(1 - \gamma)$, then the requested product has the opposite sign.

$$4(1 - \alpha)(1 - \beta)(1 - \gamma) = 4 + 12 - 14 + 3 = 5.$$

$$(\alpha - 1)(\beta - 1)(\gamma - 1) = -\frac{5}{4}.$$

$$(\alpha - 1)(\beta - 1)(\gamma - 1) = -\frac{5}{4}.$$

(4)

Question 7

Question 7

The polar curve C is $r = 1 + \tan\theta$, $0 \leqslant \theta < \pi/3$. Its tangent at A is perpendicular to the initial line. Prove that $A = (2, \pi/4)$, then show that the shaded region between C, the tangent and the initial line has area $\frac{1}{2}(1 - \ln 2)$.

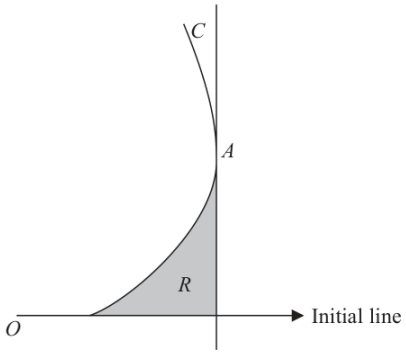


Figure 1

Figure 1 shows a sketch of the curve C with equation

Model solution

(a) A tangent perpendicular to the initial line is vertical in Cartesian coordinates. Write $x = r\cos\theta$*; a vertical tangent occurs where* $dx/d\theta = 0$.

$$x = (1 + \tan\theta)\cos\theta = \cos\theta + \sin\theta,$$

$$\frac{dx}{d\theta} = -\sin\theta + \cos\theta.$$

$$-\sin\theta + \cos\theta = 0 \quad \Rightarrow \quad \tan\theta = 1.$$

The permitted range gives $\theta = \pi/4$*. Substitution into the polar equation now gives the radius of* A.

$$r = 1 + \tan\frac{\pi}{4} = 2.$$

$$A = \left(2, \frac{\pi}{4}\right).$$

(4)

(b) The polar area formula is $\frac{1}{2}\int r^2\, d\theta$*. First find the area under the curve from the initial*

line to A, *that is, for* $0 \leqslant \theta \leqslant \pi/4$.

$$\begin{aligned}\textit{Area under } C &= \frac{1}{2}\int_0^{\pi/4} (1+\tan\theta)^2 \, d\theta \\ &= \frac{1}{2}\int_0^{\pi/4} (1+2\tan\theta+\tan^2\theta) \, d\theta \\ &= \frac{1}{2}\int_0^{\pi/4} (2\tan\theta+\sec^2\theta) \, d\theta \\ &= \left[\ln|\sec\theta| + \frac{1}{2}\tan\theta\right]_0^{\pi/4} \\ &= \ln\sqrt{2} + \frac{1}{2}.\end{aligned}$$

The tangent meets the initial line at $x = \sqrt{2}$, *because* A *has Cartesian coordinates* $(\sqrt{2}, \sqrt{2})$. *Thus the triangle enclosed by the tangent, the initial line and* OA *has area*

$$\frac{1}{2}(\sqrt{2})(\sqrt{2}) = 1.$$

The required shaded region is this triangle minus the area under the curve.

$$\textit{Area of } R = 1 - \left(\ln\sqrt{2} + \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{2}(1-\ln 2).$$

$$\textit{Area of } R = \frac{1}{2}(1-\ln 2).$$

(6)

Question 8

Question 8

Two bird flight paths are $\mathbf{r}_1 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. Their angle is $120°$.
Find a, their common point, its shortest distance from $2x - 3y + z = 2$, and comment on reliability.

Model solution

(a) The angle between the paths is the angle between their direction vectors, so use the scalar-product formula. The sign of $\cos 120°$ is important because the angle is obtuse.

$$\frac{\begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{4 + a^2}\sqrt{2}} = \cos 120° = -\frac{1}{2},$$

$$\frac{a}{\sqrt{4 + a^2}\sqrt{2}} = -\frac{1}{2}.$$

$$2a = -\sqrt{2(4 + a^2)} \Rightarrow 4a^2 = 8 + 2a^2 \Rightarrow a^2 = 4.$$

The unsquared equation has a negative right-hand side, so a must be negative.

$$a = -2.$$

(4)

(b) At an intersection the corresponding i, j and k components must agree. Use two component equations to find the parameters, then check the third; this verifies that the lines really meet.

$$-1 + 2\lambda = 4, \qquad 2 = 3 - \mu.$$

$$\lambda = \frac{5}{2}, \qquad \mu = 1.$$

$$5 - 2\lambda = 5 - 2\left(\frac{5}{2}\right) = 0 = -1 + \mu,$$

so the remaining component agrees.

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}.$$

The paths have common point $(4, 0, 2)$.

(5)

(c) The shortest distance from a point (x_0, y_0, z_0) *to* $Ax + By + Cz = D$ *is* $|Ax_0 + By_0 + Cz_0 - D|/\sqrt{A^2 + B^2 + C^2}$. *Here the common point models the nest.*

$$\text{distance} = \frac{|2(4) - 3(0) + 2 - 2|}{\sqrt{2^2 + (-3)^2 + 1^2}} = \frac{8}{\sqrt{14}} = \frac{4\sqrt{14}}{7}.$$

$$\frac{8}{\sqrt{14}} \text{ units } \left(= \tfrac{4\sqrt{14}}{7} \text{ units}\right).$$

(3)

(d) The numerical result depends on idealised assumptions in the model. Birds do not generally travel on exact straight-line paths.

The answer is not reliable, because the birds' flight paths are modelled as straight lines.

(1)

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Question 9

Question 9

For $y = \cosh^n x$, $n \geqslant 5$, find the second and fourth derivatives and hence the first three non-zero terms of the Maclaurin series.

Model solution

(a)(i) Differentiate with the chain and product rules. Replacing $\sinh^2 x$ *by* $\cosh^2 x - 1$ *puts the result into powers of* $\cosh x$*, as required.*

$$\frac{dy}{dx} = n\cosh^{n-1} x \sinh x,$$

$$\frac{d^2y}{dx^2} = n(n-1)\cosh^{n-2} x \sinh^2 x + n\cosh^n x.$$

$$\frac{d^2y}{dx^2} = n(n-1)\cosh^{n-2} x(\cosh^2 x - 1) + n\cosh^n x,$$

$$\frac{d^2y}{dx^2} = n^2\cosh^n x - n(n-1)\cosh^{n-2} x.$$

$$\frac{d^2y}{dx^2} = n^2\cosh^n x - n(n-1)\cosh^{n-2} x.$$

(4)

(a)(ii) Apply the second-derivative result to each power of $\cosh x$ *in* y''*.*

$$\frac{d^4y}{dx^4} = n^2\left[n^2\cosh^n x - n(n-1)\cosh^{n-2} x\right] - n(n-1)\left[(n-2)^2\cosh^{n-2} x - (n-2)(n-3)\cosh^{n-4} x\right].$$

$$\frac{d^4y}{dx^4} = n^4\cosh^n x - n(n-1)\left[n^2 + (n-2)^2\right]\cosh^{n-2} x + n(n-1)(n-2)(n-3)\cosh^{n-4} x.$$

(2)

(b) Maclaurin's formula uses derivatives at $x = 0$*. Since* $\cosh 0 = 1$ *and* $\sinh 0 = 0$*, all odd derivatives at zero vanish.*

$$y(0) = 1, \qquad y''(0) = n^2 - n(n-1) = n,$$

$$y^{(4)}(0) = n(3n-2).$$

$$y = 1 + \frac{n}{2!}x^2 + \frac{n(3n-2)}{4!}x^4 + \cdots.$$

$$\cosh^n x = 1 + \frac{n}{2}x^2 + \frac{n(3n-2)}{24}x^4 + \cdots.$$

(2)

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