Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/02 - Core Pure Mathematics 2

Monday 5 June 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

For $r = 2\sqrt{\sinh\theta + \cosh\theta}$, $0 \leqslant \theta \leqslant \pi$, find the exact area bounded by the initial line, curve and $\theta = \pi$, in the form $pe^q - r$.

Model solution

The stated boundaries give $0 \leqslant \theta \leqslant \pi$. Use the polar-area formula; squaring r removes the square root.

$$A = \frac{1}{2}\int_0^{\pi} \left(2\sqrt{\sinh\theta + \cosh\theta}\right)^2 d\theta = 2\int_0^{\pi} (\sinh\theta + \cosh\theta)\,d\theta.$$

$$A = 2[\cosh\theta + \sinh\theta]_0^{\pi} = 2(e^{\pi} - 1).$$

$$A = 2e^{\pi} - 2.$$

(4)

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Question 5

Question 5

$z_1 = 35 - 25i$ and $z_2 = -29 + 39i$ are opposite vertices of a regular hexagon with centre α. Given $\beta = (1+i)/64$: (a) show $\alpha = 3 + 7i$; (b) show $\beta(z_1 - \alpha) = 1$; (c) find the other four vertices.

Model solution

(a) The centre is the midpoint of opposite vertices.

$$\alpha = \frac{z_1 + z_2}{2} = \frac{(35 - 25i) + (-29 + 39i)}{2} = 3 + 7i.$$

$$\alpha = 3 + 7i.$$

(2) (b) Substitute the values and multiply, using $i^2 = -1$.

$$\beta(z_1 - \alpha) = \frac{1+i}{64}(32 - 32i) = \frac{32 - 32i + 32i - 32i^2}{64} = 1.$$

$$\beta(z_1 - \alpha) = 1.$$

(2) (c)(i) The sixth roots of unity are equally spaced by $\pi/3$.

$$w = e^{ik\pi/3}, \qquad k = 0, 1, 2, 3, 4, 5.$$

$$w = e^{ik\pi/3} \ (k = 0, \ldots, 5).$$

(1) (c)(ii) Since $\beta(z - \alpha) = w$, $z = \alpha + w/\beta$. *Also* $1/\beta = 32(1 - i)$. *Use* $k = 1, 2, 4, 5$, *excluding the two supplied vertices.*

$$z = 3 + 7i + 32(1 - i)e^{ik\pi/3}.$$

$$k = 1 : (19 + 16\sqrt{3}) + (-9 + 16\sqrt{3})i, \quad k = 2 : (-13 + 16\sqrt{3}) + (23 + 16\sqrt{3})i,$$
$$k = 4 : (-13 - 16\sqrt{3}) + (23 - 16\sqrt{3})i, \quad k = 5 : (19 - 16\sqrt{3}) - (9 + 16\sqrt{3})i.$$

The four vertices are the four values displayed above.

(4)

Question 6

Question 6

Given $y = e^{2x}\sinh x$, prove by induction that $\dfrac{d^n y}{dx^n} = e^{2x}\left(\dfrac{3^n+1}{2}\sinh x + \dfrac{3^n-1}{2}\cosh x\right)$.

Model solution

For induction, first verify $n = 1$, then differentiate the assumed $n = k$ formula using the product rule.

$$\frac{dy}{dx} = e^{2x}(2\sinh x + \cosh x) = e^{2x}\left(\frac{3+1}{2}\sinh x + \frac{3-1}{2}\cosh x\right).$$

Thus the formula is true for $n = 1$. Assume it is true for $n = k$. Then

$$\frac{d^{k+1}y}{dx^{k+1}} = e^{2x}\left[(3^k+1)\sinh x + (3^k-1)\cosh x + \frac{3^k+1}{2}\cosh x + \frac{3^k-1}{2}\sinh x\right].$$

$$= e^{2x}\left(\frac{3^{k+1}+1}{2}\sinh x + \frac{3^{k+1}-1}{2}\cosh x\right).$$

This is exactly the stated formula with $n = k+1$; therefore, as the base case holds, it is true for all $n \in \mathbb{N}$. ***(6)***

Question 7

Question 7

A berry is modelled by rotating $16x^2 + 3y^2 - y\cos(5y/2) = 6$, $x \geqslant 0$, about the y-axis. Its intercepts are $y = -1.545, 1.257$. (a) Find its volume. (b) Decide whether 100 berries are likely to fill 200 cm^3.

Model solution

(a) Rotation about the y-axis calls for $V = \pi\int x^2 dy$. First make x^2 the subject.

$$x^2 = \frac{1}{16}\left(6 - 3y^2 + y\cos\frac{5y}{2}\right).$$

$$V = \frac{\pi}{16}\int_{-1.545}^{1.257}\left(6 - 3y^2 + y\cos\frac{5y}{2}\right)dy.$$

Integrate the product by parts: $\int y\cos(5y/2)dy = \frac{2y}{5}\sin(5y/2) + \frac{4}{25}\cos(5y/2)$.

$$V = \frac{\pi}{16}\left[6y - y^3 + \frac{2y}{5}\sin\frac{5y}{2} + \frac{4}{25}\cos\frac{5y}{2}\right]_{-1.545}^{1.257} = 2.259\ldots$$

$$V = 2.26\ cm^3 \text{ (3 s.f.).}$$

(6) (b) Even if every berry had the volume of the largest, the total would be only $100(2.26) = 226\ cm^3$. Actual berries are smaller and not all material becomes juice.

It is not likely that there is enough juice to fill $200\ cm^3$.

(2)

Question 2

Question 2

(a) Write e^x through x^3. (b) Hence find the Maclaurin series of $e^{(e^x-1)}$ through x^3.

Model solution

(a) Use the standard Maclaurin expansion.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots .$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots .}$$

(1) (b) Let $u = e^x - 1$. As u begins with x, retain terms only through u^3.

$$u = x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots , \quad u^2 = x^2 + x^3 + \cdots , \quad u^3 = x^3 + \cdots .$$

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \cdots = 1 + x + x^2 + \frac{5}{6}x^3 + \cdots .$$

$$\boxed{e^{(e^x-1)} = 1 + x + x^2 + \frac{5}{6}x^3 + \cdots .}$$

(5)

Question 3

Question 3

$M = \begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}$ and $M^2 + 11M = aI$. (a) Find a and show $k = -9$. (b) Find the invariant lines. (c) State which consist of fixed points.

Model solution

(a) Compare entries after expanding the given matrix equation.

$$M^2 + 11M = \begin{pmatrix} 34 & 5k-10 \\ 6k-12 & k^2+30 \end{pmatrix} + \begin{pmatrix} -22 & 55 \\ 66 & 11k \end{pmatrix} = \begin{pmatrix} 12 & 5k+45 \\ 6k+54 & k^2+11k+30 \end{pmatrix}.$$

$$a = 12, \qquad 5k + 45 = 0 \implies k = -9.$$

$$a = 12, \text{ and } k = -9.$$

(3) (b) Write a general line as $y = mx + c$. It is invariant when the image of every $(x, mx + c)$ again satisfies that equation.

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 6 & -9 \end{pmatrix} \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} -2x + 5mx + 5c \\ 6x - 9mx - 9c \end{pmatrix}.$$

$$Y = mX + c \implies 6x - 9mx - 9c = m(-2x + 5mx + 5c) + c.$$

$$5m^2 + 7m - 6 = 0 = (5m - 3)(m + 2).$$

For $m = 3/5$, the constant equation gives $c = 0$; for $m = -2$, it is true for every c.

$$y = \frac{3}{5}x \text{ and } y = -2x + c \ (c \in \mathbb{R}).$$

(6) (c) Fixed points satisfy $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. The first equation is $-2x + 5y = x$, hence $y = 3x/5$.

The line of fixed points is $y = \frac{3}{5}x$.

(1)

Question 4

Question 4

For the polar curve $r = 3 + \sqrt{5}\cos\theta$, $0 \leqslant \theta \leqslant 2\pi$: (a) sketch it, labelling pole, initial line and its initial-line intercept; (b) show that at the horizontal tangent with $0 < \theta < \pi/2$, $\cos\theta = 1/\sqrt{5}$; (c) find r.

Model solution

(a) Since $3 > \sqrt{5}$, this is a dimpled limacon, not an inner loop. On the positive initial line $\theta = 0$, $r = 3 + \sqrt{5}$.

Sketch: dimpled limacon with pole and initial line labelled; initial-line point has $r = 3 + \sqrt{5}$.

(2) (b) A tangent parallel to the initial line is horizontal, so differentiate $y = r\sin\theta$.

$$y = (3 + \sqrt{5}\cos\theta)\sin\theta, \quad \frac{dy}{d\theta} = 3\cos\theta + \sqrt{5}(\cos^2\theta - \sin^2\theta).$$

$$0 = 3\cos\theta + \sqrt{5}(2\cos^2\theta - 1) \implies 2\sqrt{5}\cos^2\theta + 3\cos\theta - \sqrt{5} = 0.$$

$$(\sqrt{5}\cos\theta - 1)(2\cos\theta + \sqrt{5}) = 0.$$

The given range has positive cosine, so $\cos\theta = 1/\sqrt{5}$.

$$\cos\theta = 1/\sqrt{5}.$$

(4) (c) Substitute this value in the polar equation.

$$r = 3 + \sqrt{5}\left(\frac{1}{\sqrt{5}}\right) = 4.$$

$$r = 4.$$

(1)

Question 8

Question 8

Cubics with real coefficients have three distinct collinear roots. (a) Describe possible lines. $f(z) = 8z^3 + bz^2 + cz + d$ has root $(3+3i)/2$. (b) Find its other roots. $g(z) = z^3 + Pz^2 + Qz + 12$ shares one root with f, has root -4, and its roots lie on a different such line. (c) Find its other roots. (d) Solve $f(z) = g(z)$.

Model solution

(a) Non-real roots of a real-coefficient polynomial occur in conjugate pairs.

The real axis $y = 0$, or a vertical line $x = k$ ($k \in \mathbb{R}$).

(2) (b) Use the conjugate pair and the vertical-line condition.

$$\frac{3}{2} - \frac{3}{2}i \text{ and } \frac{3}{2}.$$

(1) (c)(i) The common root must be real because the roots of g lie on a different line.

Common root $3/2$.

(1) (c)(ii) The product of g's roots is -12, so $(-4)(3/2)r = -12$.

$$r = 2.$$

Other root 2.

(3) (d) Form f from its roots and g from its roots, then equate.

$$f(z) = 8\left(z - \frac{3}{2}\right)\left(z^2 - 3z + \frac{9}{2}\right), \quad g(z) = (z+4)\left(z - \frac{3}{2}\right)(z-2).$$

$$f(z) = g(z) \implies (z - \tfrac{3}{2})(7z^2 - 26z + 44) = 0.$$

$$z = \frac{3}{2}, \ \frac{13 + i\sqrt{139}}{7}, \ \frac{13 - i\sqrt{139}}{7}.$$

(4)

Question 9

Question 9

$x' = 0.025y - 0.045x + 2$, $y' = 0.032x - 0.025y$. (a) Show $40000y'' + 2800y' + 13y = 2560$. (b) Find general y. (c) Find particular y. (d) Test its eventual level against 185–200 mg/ml.

Model solution

(a) Differentiate y' and eliminate x, x' using both model equations.

$$y'' = 0.032(0.025y - 0.045x + 2) - 0.025y', \quad x = \frac{y' + 0.025y}{0.032}.$$

$$y'' = -0.07y' - 0.000325y + 0.064.$$

$$40000y'' + 2800y' + 13y = 2560.$$

(3) (b) The auxiliary equation gives the complementary function; a constant forcing needs a constant particular integral.

$$40000m^2 + 2800m + 13 = 0 \implies m = -\frac{1}{200}, -\frac{13}{200}, \quad y_p = \frac{2560}{13}.$$

$$y = Ae^{-t/200} + Be^{-13t/200} + \frac{2560}{13}.$$

(5) (c) Initially $x = y = 0$, so $y(0) = 0$ and $y'(0) = 0$.

$$A + B = -\frac{2560}{13}, \quad -\frac{A}{200} - \frac{13B}{200} = 0 \implies A = -\frac{640}{3}, \ B = \frac{640}{39}.$$

$$y = -\frac{640}{3}e^{-t/200} + \frac{640}{39}e^{-13t/200} + \frac{2560}{13}.$$

(4) (d) The exponential terms tend to zero as $t \to \infty$.

$$\lim_{t\to\infty} y = \frac{2560}{13} = 196.9\ldots \ \textit{mg/ml}.$$

Yes: 196.9 mg/ml is in the required interval, so the rate is effective.

(2)