Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/02 - Core Pure Mathematics 2

Monday 3 June 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) Prove that $4\sinh^3 x + 3\sinh x \equiv \sinh 3x$.
(b) Hence solve $\sinh 3x = 19\sinh x$, giving simplified natural logarithms where appropriate.

Model solution

(a) The definition $\sinh x = (e^x - e^{-x})/2$ *is requested, so expand the left-hand side in exponentials and collect like terms.*

$$4\sinh^3 x + 3\sinh x = 4\left(\frac{e^x - e^{-x}}{2}\right)^3 + 3\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^{3x} - 3e^x + 3e^{-x} - e^{-3x}}{2} + \frac{3e^x - 3e^{-x}}{2} = \frac{e^{3x} - e^{-3x}}{2} = \sinh 3x.$$

$$4\sinh^3 x + 3\sinh x \equiv \sinh 3x.$$

(2)

(b) Use the identity just proved to turn the triple-angle equation into an equation in $\sinh x$.

$$4\sinh^3 x + 3\sinh x = 19\sinh x \implies 4\sinh x(\sinh^2 x - 4) = 0.$$

Thus $\sinh x = 0$, *giving* $x = 0$, *or* $\sinh x = \pm 2$. *Since* $\operatorname{arsinh} u = \ln(u + \sqrt{u^2+1})$,

$$x = \ln(2 + \sqrt{5}) \quad \textit{or} \quad x = \ln(-2 + \sqrt{5}).$$

$$x = 0,\ \ln(2 + \sqrt{5}),\ \ln(\sqrt{5} - 2).$$

(5)

Question 2

Question 2

$f(x) = \tanh^{-1}\left(\frac{3-x}{6+x}\right)$, $x < \frac{3}{2}$. (a) Show $f'(x) = -\frac{1}{2x+3}$. (b) Find $f''(x)$. (c) Find the Maclaurin series through x^2.

Model solution

(a) Differentiate the inner quotient first, then use $\frac{d}{dx}\tanh^{-1} u = \frac{u'}{1-u^2}$.

$$u = \frac{3-x}{6+x}, \qquad u' = \frac{-(6+x)-(3-x)}{(6+x)^2} = \frac{-9}{(6+x)^2}.$$

$$f'(x) = \frac{-9/(6+x)^2}{1-((3-x)/(6+x))^2} = \frac{-9}{(6+x)^2-(3-x)^2} = \frac{-9}{18x+27} = -\frac{1}{2x+3}.$$

$$f'(x) = -\frac{1}{2x+3}.$$

(4)

(b) Differentiate the simplified first derivative; this is the most direct form to use.

$$f''(x) = -(-1)(2x+3)^{-2}\cdot 2 = \frac{2}{(2x+3)^2}.$$

$$f''(x) = \frac{2}{(2x+3)^2}.$$

(1)

(c) A Maclaurin expansion uses the value and derivatives at $x = 0$*:* $f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \cdots$.

$$f(0) = \tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\ln\frac{1+1/2}{1-1/2} = \ln 3, \quad f'(0) = -\frac{1}{3}, \quad f''(0) = \frac{2}{9}.$$

$$f(x) = \ln 3 - \frac{1}{3}x + \frac{x^2}{2}\left(\frac{2}{9}\right) + \cdots = \ln 3 - \frac{1}{3}x + \frac{1}{9}x^2 + \cdots.$$

$$p = 3,\ q = -\tfrac{1}{3},\ r = \tfrac{1}{9}.$$

(3)

Question 3

Question 3

(a) Explain why $\int_{4/3}^{\infty} \frac{1}{9x^2+16}\,dx$ is improper. (b) Show that it is $k\pi$, finding k.

Model solution

(a) An integral is improper when a limit of integration is infinite (or the integrand is unbounded). Here the upper limit is infinite.

It is improper because its upper limit is ∞.

(1)

(b) Use a limiting process at the infinite endpoint. The denominator has the form $a^2+(bx)^2$, which gives an arctangent antiderivative.

$$\int \frac{1}{9x^2+16}\,dx = \frac{1}{12}\arctan\left(\frac{3x}{4}\right).$$

$$\int_{4/3}^{\infty} \frac{1}{9x^2+16}\,dx = \lim_{t\to\infty}\left[\frac{1}{12}\arctan\left(\frac{3x}{4}\right)\right]_{4/3}^{t} = \frac{1}{12}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{48}.$$

$$k = \frac{1}{48}.$$

(4)

Question 4

Question 4

Use the method of differences to show that $\sum_{r=1}^{n} \frac{2}{(r+4)(r+6)} = \frac{n(11n+61)}{30(n+5)(n+6)}$.

Model solution

The factors in the denominator differ by 2, so partial fractions produce terms which telescope.

$$\frac{2}{(r+4)(r+6)} = \frac{A}{r+4} + \frac{B}{r+6}.$$

Putting $r = -4$ *and* $r = -6$ *gives* $A = 1$ *and* $B = -1$, *hence*

$$\sum_{r=1}^{n} \frac{2}{(r+4)(r+6)} = \sum_{r=1}^{n} \left(\frac{1}{r+4} - \frac{1}{r+6} \right).$$

Writing the first and last terms shows the cancellation:

$$\left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{n+3} - \frac{1}{n+5}\right) + \left(\frac{1}{n+4} - \frac{1}{n+6}\right) = \frac{1}{5} + \frac{1}{6} - \frac{1}{n+5} - \frac{1}{n+6}.$$

$$= \frac{11(n+5)(n+6) - 30(n+6) - 30(n+5)}{30(n+5)(n+6)} = \frac{n(11n+61)}{30(n+5)(n+6)}.$$

$$\sum_{r=1}^{n} \frac{2}{(r+4)(r+6)} = \frac{n(11n+61)}{30(n+5)(n+6)}.$$

(6)

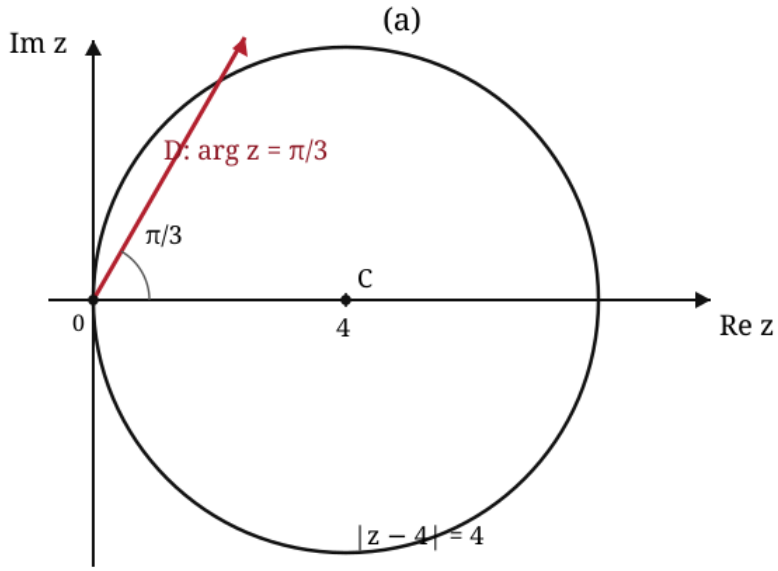
Question 5

Question 5

C : $|z-4|=4$, D : $\arg z = \pi/3$. The set A is the intersection of $|z-4|<4$ and $0<\arg z<\pi/3$. Sketch the loci and find the area of A.

Model solution

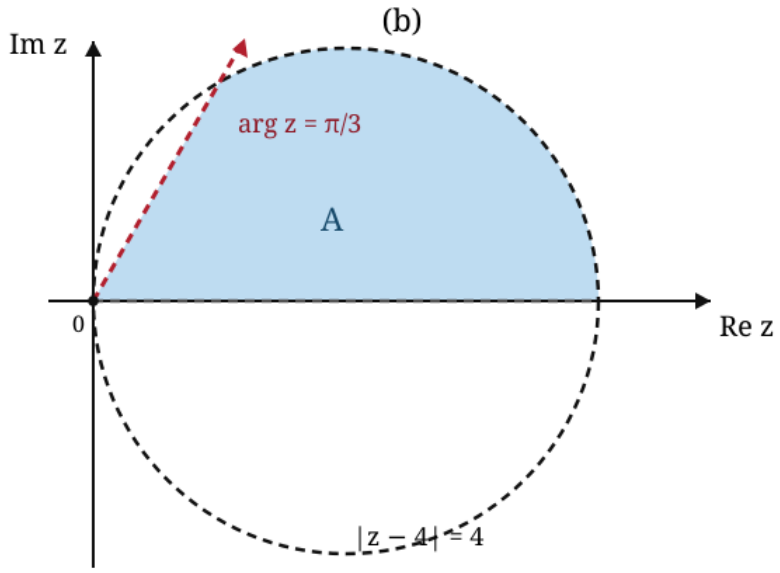
(a) $|z-4|=4$ *is a circle: its centre is the point* $4+0\mathrm{i}$ *and its radius is* 4. $\arg z = \pi/3$ *is the ray from the origin making an angle* $\pi/3$ *with the positive real axis.*



Sketch the circle centred at $(4,0)$ *of radius* 4*, together with the ray* $\theta = \pi/3$.

(4)

(b) The strict inequalities mean that neither boundary is included. Shade the part inside the circle lying between the positive real axis and the ray $\theta = \pi/3$.



Dashed boundaries are excluded.

The required region is $0 < \theta < \pi/3$ inside $|z-4| = 4$, with boundaries excluded.

(1)

(c) In polar coordinates, $(x-4)^2 + y^2 = 16$ becomes $r^2 - 8r\cos\theta = 0$, so the circle has boundary $r = 8\cos\theta$. This gives the radial limit needed for the shaded sector.

$$\text{Area}(A) = \frac{1}{2}\int_0^{\pi/3} (8\cos\theta)^2 \, d\theta = 32\int_0^{\pi/3} \cos^2\theta \, d\theta = 16\int_0^{\pi/3} (1+\cos 2\theta)\, d\theta$$

$$= 16\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/3} = \frac{16\pi}{3} + 4\sqrt{3}.$$

$$\frac{16\pi}{3} + 4\sqrt{3}.$$

(4)

Question 6

Question 6

For $t \geqslant 0$, $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 2x = 4t + 12$. (a) Find the general solution. (b) Find the solution with $x = 3$, $dx/dt = -2$ at $t = 0$. (c) Find and justify the minimum distance from the origin. (d) Comment on suitability if the particle is expected eventually to have constant speed.

Model solution

(a) Solve the homogeneous equation by its auxiliary equation. Since the forcing term is linear in t, try a linear particular integral $pt + q$.

$$2m^2 + 5m + 2 = 0 \implies (2m+1)(m+2) = 0,$$

so $x_{CF} = Ae^{-t/2} + Be^{-2t}$. *Let* $x_{PI} = pt + q$. *Then*

$$0 + 5p + 2(pt + q) = 4t + 12,$$

so $2p = 4$ *and* $5p + 2q = 12$, *giving* $p = 2,\ q = 1$.

$$x = Ae^{-t/2} + Be^{-2t} + 2t + 1.$$

(6)

(b) Apply both initial conditions to the general solution.

$$x(0) = 3 \implies A + B + 1 = 3 \implies A + B = 2,$$

$$\frac{dx}{dt} = -\frac{1}{2}Ae^{-t/2} - 2Be^{-2t} + 2, \quad \frac{dx}{dt}(0) = -2 \implies -\frac{1}{2}A - 2B = -4.$$

Solving gives $A = 0,\ B = 2$.

$$x = 2e^{-2t} + 2t + 1.$$

(3)

(c)(i) A closest point occurs at a stationary value of x. The expression found below is positive, so its value is also the distance from O.

$$\frac{dx}{dt} = -4e^{-2t} + 2 = 0 \implies e^{-2t} = \frac{1}{2} \implies t = \frac{1}{2}\ln 2.$$

$$x = 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\ln 2\right) + 1 = 2 + \ln 2.$$

$$\textit{Minimum distance} = 2 + \ln 2 \textit{ m.}$$

(3)

(c)(ii) Test the nature of the stationary value using the second derivative.

$$\frac{d^2x}{dt^2} = 8e^{-2t} > 0 \qquad (t \geqslant 0).$$

The stationary distance is a minimum.

(1)

(d) As t *becomes large, the decaying exponential tends to zero, so the velocity tends to a constant. This agrees with the stated physical expectation.*

$$\frac{dx}{dt} = -4e^{-2t} + 2 \longrightarrow 2 \text{ m s}^{-1}.$$

The model is suitable.

(1)

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Question 7

Question 7

(a) Find the roots of $z^6 = 1$. (b) Show them on an Argand diagram. (c) Show $(\sqrt{3}+i)^6 = -64$. (d) Solve $z^6 + 64 = 0$.

Model solution

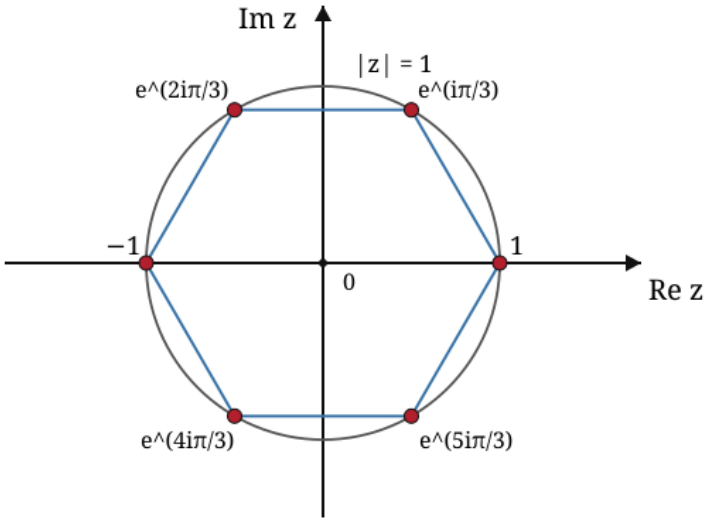
(a) Write 1 in all its polar forms, $e^{2\pi ik}$, then divide the arguments by 6.

$$z = e^{ik\pi/3}, \qquad k = 0, 1, 2, 3, 4, 5.$$

$$1, e^{i\pi/3}, e^{2i\pi/3}, e^{i\pi}, e^{4i\pi/3}, e^{5i\pi/3}.$$

(2)

(b) Each root has modulus 1, and consecutive arguments differ by $\pi/3$.



The six equally spaced roots are shown as the vertices of a regular hexagon on the unit circle.

(2)

(c) Convert $\sqrt{3}+i$ to polar form. Its modulus is 2, and its argument is $\pi/6$, so De Moivre's theorem applies directly.

$$(\sqrt{3}+i)^6 = \left(2e^{i\pi/6}\right)^6 = 64e^{i\pi} = -64.$$

$$(\sqrt{3}+i)^6 = -64.$$

(2)

(d) Here $z^6 = -64 = 64e^{i(\pi+2\pi k)}$. *Taking sixth roots gives modulus* 2 *and six equally spaced arguments.*

$$z = 2e^{i(\pi+2\pi k)/6}, \qquad k = 0, 1, 2, 3, 4, 5.$$

$$2e^{i\pi/6}, 2e^{i\pi/2}, 2e^{5i\pi/6}, 2e^{7i\pi/6}, 2e^{3i\pi/2}, 2e^{11i\pi/6}.$$

(3)

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Question 8

Question 8

$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ k & 3 & 6 \end{pmatrix}$, $k \neq 0$. (a) Find A^{-1}. (b) Find the intersection of $3x + y - z = 3$, $x + y + z = 1$, $kx + 3y + 6z = 6$.

Model solution

(a) Use the adjugate formula $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$. *The condition* $k \neq 0$ *anticipates a determinant containing* k.

$$\det A = 3(6-3) - 1(6-k) - 1(3-k) = 2k.$$

The cofactor matrix is

$$\begin{pmatrix} 3 & k-6 & 3-k \\ -9 & 18+k & k-9 \\ 2 & -4 & 2 \end{pmatrix},$$

so transposing it gives the adjugate. Therefore

$$A^{-1} = \frac{1}{2k} \begin{pmatrix} 3 & -9 & 2 \\ k-6 & 18+k & -4 \\ 3-k & k-9 & 2 \end{pmatrix}.$$

$$A^{-1} = \frac{1}{2k} \begin{pmatrix} 3 & -9 & 2 \\ k-6 & 18+k & -4 \\ 3-k & k-9 & 2 \end{pmatrix}.$$

(4)

(b) The three plane equations have coefficient matrix A, *hence* $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$. *Multiply by the inverse found above.*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2k} \begin{pmatrix} 3 & -9 & 2 \\ k-6 & 18+k & -4 \\ 3-k & k-9 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 6/k \\ (2k-12)/k \\ (6-k)/k \end{pmatrix}.$$

$$\left(\frac{6}{k},\ 2 - \frac{12}{k},\ \frac{6}{k} - 1 \right).$$

(3)

Question 9

Question 9

The vase curve is $x = a + 3\sin 2t$, $y = b\cos t$, $0 \leqslant t \leqslant \pi/2$. (a) Find a, b. (b) Find its volume to the nearest cm^3. (c) State a limitation of the model.

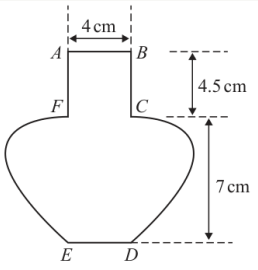


Figure 1

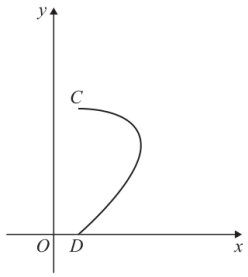


Figure 2

Figure 1 shows the central vertical cross-section $ABCDEFA$ of a vase together with measurements that have been taken from the vase.

The horizontal cross-section between AB and FC is a circle with diameter 4 cm

Model solution

(a) At C, $t = 0$, *so* $x = a = 2$ *cm because the diameter is* 4 *cm. At* D, $t = \pi/2$, *so the vertical change is* $y(0) - y(\pi/2) = b = 7$ *cm.*

$$a = 2,\ b = 7.$$

(2)

(b) The curved lower part is formed by rotating the curve about the y*-axis. With* y *decreasing as* t *increases, use* $V = \pi\int x^2(-dy/dt)\,dt$ *to keep the volume positive. Then add the cylindrical top of radius* 2 *cm and height* 4.5 *cm.*

$$\frac{dy}{dt} = -7\sin t, \qquad V_{\text{curve}} = 7\pi\int_0^{\pi/2}(2 + 3\sin 2t)^2\sin t\,dt.$$

Using $\sin 2t = 2\sin t\cos t$,

$$V_{\text{curve}} = 7\pi\int_0^{\pi/2}\left(4\sin t + 24\sin^2 t\cos t + 36\sin^3 t\cos^2 t\right)dt.$$

$$\int\left(4\sin t + 24\sin^2 t\cos t + 36\sin^3 t\cos^2 t\right)dt = -4\cos t + 8\sin^3 t - 12\cos^3 t + \frac{36}{5}\cos^5 t.$$

Thus

$$V_{\text{curve}} = 7\pi\left[-4\cos t + 8\sin^3 t - 12\cos^3 t + \frac{36}{5}\cos^5 t\right]_0^{\pi/2} = \frac{588\pi}{5}.$$

$$V_{\text{cylinder}} = \pi(2)^2(4.5) = 18\pi, \qquad V_{\text{total}} = \frac{588\pi}{5} + 18\pi = \frac{678\pi}{5} \approx 426.0.$$

$$426\ \text{cm}^3\ \textit{to the nearest}\ \text{cm}^3.$$

(7)

(c) A mathematical curve fitted to a few measurements need not reproduce the physical vase exactly.

For example, the vase may not be perfectly rotationally symmetric.

(1)

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