Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/3A - Further Pure Mathematics 1

Thursday 16 June 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

An ellipse has equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$ and eccentricity e_1. A hyperbola has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and eccentricity e_2, where $e_1 e_2 = 1$.

(a) Show that $a^2 = 3b^2$. **(4)**

The foci of the ellipse and hyperbola have the same coordinates.

(b) Determine the equation of the hyperbola. **(3)**

Model solution

(a) For an ellipse, $b^2 = a^2(1 - e^2)$. *The given semi-axes are* 4 *and* 2*, so this formula first determines* e_1.

$$4 = 16(1 - e_1^2), \qquad e_1^2 = \frac{3}{4}, \qquad e_1 = \frac{\sqrt{3}}{2}.$$

For a hyperbola, $b^2 = a^2(e_2^2 - 1)$. *Since* $e_1 e_2 = 1$, $e_2^2 = 1/e_1^2 = 4/3$.

$$b^2 = a^2\left(\frac{4}{3} - 1\right) = \frac{1}{3}a^2,$$

$$\text{so } a^2 = 3b^2.$$

(4)

(b) Matching foci means matching their distances from the centre. For the ellipse this distance is ae_1*, while for the hyperbola it is* ae_2.

$$\textit{ellipse focus distance} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3},$$

$$\textit{hyperbola focus distance} = ae_2 = a\left(\frac{2}{\sqrt{3}}\right).$$

$$\frac{2a}{\sqrt{3}} = 2\sqrt{3} \quad \Rightarrow \quad a = 3, \qquad b^2 = \frac{a^2}{3} = 3.$$

$$\frac{x^2}{9} - \frac{y^2}{3} = 1.$$

(3)

Question 2

Question 2

During 2029, the hours of daylight per day in London are modelled by

$$H = 0.3\sin\left(\frac{x}{60}\right) - 4\cos\left(\frac{x}{60}\right) + 11.5, \qquad 0 \leqslant x < 365,$$

where x is the number of days after 1 January and angles are in radians.

(a) Show that on 31 January the model gives 8.13 hours, to 3 significant figures. **(1)**

(b) Using $t = \tan(x/120)$, show that $H = \dfrac{at^2 + bt + c}{1 + t^2}$, finding a, b, c. **(2)**

(c) Hence determine the date of the first day of 2029 for which there will be at least 12 hours of daylight. **(4)**

Model solution

(a) 31 January is 30 days after 1 January, so substitute $x = 30$ directly into the model.

$$H = 0.3\sin\left(\frac{30}{60}\right) - 4\cos\left(\frac{30}{60}\right) + 11.5 = 8.133\ldots$$

$$H = 8.13 \text{hours} (3\text{s.f.}).$$

(1)

(b) The chosen tangent substitution is useful because it converts both sine and cosine into rational expressions in t:

$$\sin\left(\frac{x}{60}\right) = \frac{2t}{1+t^2}, \qquad \cos\left(\frac{x}{60}\right) = \frac{1-t^2}{1+t^2}.$$

$$H = 0.3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right) + 11.5 = \frac{0.6t - 4 + 4t^2 + 11.5(1+t^2)}{1+t^2}.$$

$$H = \frac{15.5t^2 + 0.6t + 7.5}{1+t^2}, \text{so} \; a = 15.5, \; b = 0.6, \; c = 7.5.$$

(2)

(c) The first time that daylight reaches 12 hours occurs at the boundary $H = 12$. Solve this equation, then convert the appropriate value of t back to a day number.

$$\frac{15.5t^2 + 0.6t + 7.5}{1+t^2} = 12 \Rightarrow 3.5t^2 + 0.6t - 4.5 = 0.$$

$$t = \frac{-0.6 \pm \sqrt{0.6^2 - 4(3.5)(-4.5)}}{7} = 1.051\ldots \text{ or } -1.222\ldots$$

The positive value gives the first date in the year.

$$x = 120\tan^{-1}(1.051\ldots) = 97.254\ldots$$

The first such day is ***8 April 2029*** (approximately day 97).

(4)

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Question 3

Question 3

With origin O, points A and B have coordinates $(2, 2, -1)$ and $(4, 2p, 1)$ respectively.
Determine the possible values of p for which:

(a) $\overrightarrow{OB}$ makes an angle of $45°$ with the positive x-axis; **(3)**

(b) $\overrightarrow{OA} \times \overrightarrow{OB}$ is parallel to $\begin{pmatrix} 4 \\ -p \\ 2 \end{pmatrix}$; **(3)**

(c) the area of triangle OAB is $3\sqrt{2}$. **(3)**

Model solution

Write the position vectors first: $\overrightarrow{OA} = (2, 2, -1)$ *and* $\overrightarrow{OB} = (4, 2p, 1)$.

(a) The cosine of the angle with the positive x*-axis is the* x*-component divided by the vector magnitude.*

$$|\overrightarrow{OB}| = \sqrt{4^2 + (2p)^2 + 1^2} = \sqrt{17 + 4p^2},$$

$$\cos 45° = \frac{4}{\sqrt{17 + 4p^2}}.$$

$$\frac{1}{\sqrt{2}} = \frac{4}{\sqrt{17 + 4p^2}} \Rightarrow 17 + 4p^2 = 32 \Rightarrow p^2 = \frac{15}{4}.$$

$$p = \pm\frac{\sqrt{15}}{2}.$$

(3)

(b) A cross product is perpendicular to both original vectors. For it to be parallel to the given vector, its components must be in a common ratio.

$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} 2 + 2p \\ -6 \\ 4p - 8 \end{pmatrix} = \lambda \begin{pmatrix} 4 \\ -p \\ 2 \end{pmatrix}.$$

$$-6 = -p\lambda, \qquad 4p - 8 = 2\lambda.$$

From the second equation $\lambda = 2p - 4$*; substituting into the first gives* $p(2p - 4) = 6$*, so* $p = 3$ *or* -1*. Checking the first component rejects* -1*: when* $p = -1$*,* $2 + 2p = 0$ *but* $4\lambda \neq 0$.

$$p = 3.$$

(3)

(c) The area of the triangle is half the magnitude of the cross product.

$$\frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = 3\sqrt{2} \Rightarrow (2 + 2p)^2 + (-6)^2 + (4p - 8)^2 = (6\sqrt{2})^2.$$

$$20p^2 - 56p + 32 = 0 \Rightarrow 5p^2 - 14p + 8 = 0 \Rightarrow (5p - 4)(p - 2) = 0.$$

$$p = 2 \text{ or } p = \frac{4}{5}.$$

(3)

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Question 4

Question 4

The velocity $v\,\mathrm{m\,s^{-1}}$ of a raindrop t seconds after falling is modelled by

$$\frac{dv}{dt} = -0.1v^2 + 10, \qquad t \geqslant 0.$$

Initially the raindrop is at rest.

(a) Use two iterations of $y_{n+1} = y_n + h\left(\frac{dy}{dx}\right)_n$ to estimate its velocity after 1 second. **(5)**

(b) The observed initial acceleration is smaller than that suggested by the model. Refine the model by changing one constant. **(1)**

Model solution

(a) Two iterations over one second require a step size $h = 0.5$. *The initial condition gives* $t_0 = 0$, $v_0 = 0$, *and hence* $(dv/dt)_0 = 10$.

$$v_1 = v_0 + 0.5\left(\frac{dv}{dt}\right)_0 = 0 + 0.5(10) = 5.$$

Use this first estimate to calculate the new acceleration before making the second step.

$$\left(\frac{dv}{dt}\right)_1 = -0.1(5)^2 + 10 = 7.5,$$

$$v_2 = v_1 + 0.5\left(\frac{dv}{dt}\right)_1 = 5 + 0.5(7.5) = 8.75.$$

$$v(1) \approx 8.75\ \mathrm{m\,s^{-1}}.$$

(5)

(b) At rest, $v = 0$, *so the constant term is the initial acceleration. It must therefore be reduced, while remaining positive.*

$$\frac{dv}{dt} = -0.1v^2 + A, \quad 0 < A < 10.$$

(1)

Question 5

Question 5

The rectangular hyperbola H has equation $xy = 36$.
(a) Use calculus to show that the tangent at $P(6t, 6/t)$ is $yt^2 + x = 12t$. **(3)**
The point $Q(12t, 3/t)$ also lies on H.
(b) Find the equation of the tangent at Q. **(2)**
The two tangents meet at R.
(c) Show that, as t varies, the locus of R is also a rectangular hyperbola. **(4)**

Model solution

(a) Differentiate implicitly, since x and y are linked by the curve equation.

$$y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}.$$

At P, this gradient is $-\frac{6/t}{6t} = -\frac{1}{t^2}$. Use point--gradient form for the tangent.

$$y - \frac{6}{t} = -\frac{1}{t^2}(x - 6t).$$

$$yt^2 - 6t = -x + 6t \Rightarrow yt^2 + x = 12t.$$

$$\boxed{yt^2 + x = 12t.}$$

(3)

(b) The same derivative gives the gradient at Q: $-\frac{3/t}{12t} = -\frac{1}{4t^2}$.

$$y - \frac{3}{t} = -\frac{1}{4t^2}(x - 12t).$$

$$\boxed{4yt^2 + x = 24t.}$$

(2)

(c) At R, both tangent equations hold. Solve them simultaneously to obtain its coordinates in terms of t.

$$yt^2 + x = 12t,$$
$$4yt^2 + x = 24t.$$

$$3yt^2 = 12t \Rightarrow y = \frac{4}{t}, \qquad x = 12t - yt^2 = 8t.$$

Eliminating the parameter by multiplying the coordinates gives a constant product.

$$xy = (8t)\left(\frac{4}{t}\right) = 32.$$

The locus is $xy = 32$, which is a rectangular hyperbola.

(4)

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Question 6

Question 6

Points P, Q, R have position vectors $\begin{pmatrix}1\\-2\\4\end{pmatrix}$, $\begin{pmatrix}3\\1\\-5\end{pmatrix}$, and $\begin{pmatrix}2\\0\\3\end{pmatrix}$, respectively.

(a) Determine a vector equation of the plane through P, Q, R, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. **(2)**

(b) Determine the coordinates of its intersection with the x-axis. **(4)**

Model solution

(a) A plane is fixed by one point on it and two non-parallel directions in it. Subtracting position vectors gives suitable directions.

$$\overrightarrow{PQ} = \begin{pmatrix}2\\3\\-9\end{pmatrix}, \qquad \overrightarrow{PR} = \begin{pmatrix}1\\2\\-1\end{pmatrix}.$$

$$\mathbf{r} = \begin{pmatrix}1\\-2\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-9\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-1\end{pmatrix}.$$

(2)

(b) A point on the x-axis has $y = 0$ and $z = 0$. Apply these conditions to the plane equation before finding its x-coordinate.

$$-2 + 3\lambda + 2\mu = 0,$$
$$4 - 9\lambda - \mu = 0.$$

$$\lambda = \frac{2}{5}, \qquad \mu = \frac{2}{5}.$$

$$x = 1 + 2\lambda + \mu = 1 + 2\left(\frac{2}{5}\right) + \frac{2}{5} = \frac{11}{5}.$$

$$\left(\frac{11}{5}, 0, 0\right).$$

(4)

Question 7

Question 7

The curve $y = |x^2 - 8|$ and the line $y = mx + c$, where $m, c > 0$, meet in exactly three points.
(a) Show that $m^2 - 4c + 32 = 0$. **(2)**
Given $c = 3m$:
(b) determine m and c; **(3)**
(c) hence solve $|x^2 - 8| \geqslant mx + c$. **(3)**

Model solution

(a) Exactly three intersections means one branch touches the line. The lower branch is $8 - x^2$, so equality with the line gives a repeated root.

$$x^2 - 8 = -(mx + c) \Rightarrow x^2 + mx - 8 + c = 0.$$

Set its discriminant equal to zero for tangency.

$$m^2 - 4(-8 + c) = 0.$$

$$\boxed{m^2 - 4c + 32 = 0.}$$

(2)

(b) Substitute the given relationship, then retain only the values consistent with the stated three roots.

$$m^2 - 4(3m) + 32 = 0 \Rightarrow m^2 - 12m + 32 = 0 \Rightarrow m = 4 \text{ or } 8.$$

$$(m, c) = (4, 12) \text{ or } (8, 24).$$

The pair $(8, 24)$ does not produce the required three roots, so it is rejected.

$$\boxed{m = 4, \quad c = 12.}$$

(3)

(c) Find every boundary point by considering both branches of the modulus.

$$x^2 - 8 = 4x + 12 \Rightarrow x^2 - 4x - 20 = 0 \Rightarrow x = 2 \pm 2\sqrt{6},$$

$$x^2 - 8 = -(4x + 12) \Rightarrow x^2 + 4x + 4 = 0 \Rightarrow x = -2.$$

Testing the intervals separated by these boundary points shows where the curve is on or above the line.

$$\boxed{x \leqslant 2 - 2\sqrt{6}, \quad x = -2, \quad \text{or} \quad x \geqslant 2 + 2\sqrt{6}.}$$

(3)

Question 8

Question 8

(i)(a) Use differentiation to determine the Taylor series expansion of $\ln x$, in ascending powers of $(x-1)$, up to and including $(x-1)^2$. **(4)**

(i)(b) Hence prove that $\lim_{x\to 1} \dfrac{\ln x}{x-1} = 1$. **(2)**

(ii) Use L'Hospital's rule to determine $\lim_{x\to 0} \dfrac{1}{(x+3)\tan(6x)\csc(2x)}$. **(4)**

Model solution

(i)(a) A Taylor expansion about $x = 1$ needs function values and derivatives evaluated at 1.

$$f(x) = \ln x, \quad f(1) = 0, \qquad f'(x) = \frac{1}{x}, \quad f'(1) = 1,$$

$$f''(x) = -\frac{1}{x^2}, \quad f''(1) = -1.$$

$$\ln x = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots$$

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots .$$

(4)

(i)(b) The word "hence" directs us to substitute the expansion just found into the limit. Cancelling $x-1$ exposes the limiting constant term.

$$\frac{\ln x}{x-1} = \frac{(x-1) - \frac{1}{2}(x-1)^2 + \cdots}{x-1} = 1 - \frac{1}{2}(x-1) + \cdots .$$

$$\lim_{x\to 1}\left(1 - \frac{1}{2}(x-1) + \cdots\right) = 1.$$

$$\lim_{x\to 1} \frac{\ln x}{x-1} = 1.$$

(2)

(ii) First rewrite the expression as a quotient. Both numerator and denominator tend to zero, so L'Hospital's rule applies.

$$\frac{1}{(x+3)\tan(6x)\csc(2x)} = \frac{\sin(2x)}{(x+3)\tan(6x)} \xrightarrow[x\to 0]{} \frac{0}{0}.$$

$$\lim_{x\to 0} \frac{\sin(2x)}{(x+3)\tan(6x)} = \lim_{x\to 0} \frac{2\cos(2x)}{\tan(6x) + 6(x+3)\sec^2(6x)}.$$

$$= \frac{2}{0+6(3)} = \frac{1}{9}.$$

$$\frac{1}{9}.$$

(4)

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Question 9

Question 9

A particle has displacement x metres from a fixed point O at time t minutes, where

$$t^2\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 2x + 16t^2x = 4t^3\sin 2t. \qquad \text{(I)}$$

(a) Show that $x = ty$ transforms (I) into $\dfrac{d^2y}{dt^2} + 16y = 4\sin 2t$. **(5)**

(b) Hence find a general solution for the displacement of P from O at time t. **(8)**

Model solution

(a) The substitution contains a product of t and y, so differentiate using the product rule twice.

$$x = ty, \qquad \frac{dx}{dt} = y + t\frac{dy}{dt}, \qquad \frac{d^2x}{dt^2} = 2\frac{dy}{dt} + t\frac{d^2y}{dt^2}.$$

Substitute these expressions into (I). The terms involving y and dy/dt cancel, which is the benefit of this transformation.

$$t^2\left(2\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right) - 2t\left(y + t\frac{dy}{dt}\right) + 2ty + 16t^2(ty) = 4t^3\sin 2t,$$

$$t^3\frac{d^2y}{dt^2} + 16t^3y = 4t^3\sin 2t.$$

$$\frac{d^2y}{dt^2} + 16y = 4\sin 2t.$$

(5)

(b) Solve the transformed constant-coefficient equation first. Its complementary solution comes from the auxiliary equation, and a sine forcing term suggests a sine--cosine trial particular integral.

$$m^2 + 16 = 0 \Rightarrow m = \pm 4i, \qquad y_c = A\cos 4t + B\sin 4t.$$

$$y_p = \lambda\sin 2t + \mu\cos 2t.$$

$$\frac{d^2y_p}{dt^2} = -4\lambda\sin 2t - 4\mu\cos 2t.$$

$$(-4\lambda + 16\lambda)\sin 2t + (-4\mu + 16\mu)\cos 2t = 4\sin 2t.$$

$$12\lambda = 4, \qquad 12\mu = 0 \Rightarrow \lambda = \frac{1}{3}, \quad \mu = 0.$$

$$y = A\cos 4t + B\sin 4t + \frac{1}{3}\sin 2t.$$

Finally use $x = ty$ *to return from the transformed variable to the particle's displacement.*

$$x = t\left(A\cos 4t + B\sin 4t + \frac{1}{3}\sin 2t\right), \text{where A and B are arbitrary constants.}$$

(8)

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