Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/3A - Further Pure Mathematics 1

Thursday 22 June 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) Use Simpson's rule with 4 intervals to estimate $\int_0^2 e^{\sin 2x}\,dx$, giving 3 significant figures.

(b) Given that the integral is 3.855 to 4 significant figures, comment on the accuracy of the estimate.

Model solution

(a) Four equal intervals over $[0,2]$ *give a step length* $h = 2/4 = 0.5$. *Simpson's rule needs the five ordinates at these equally spaced values.*

x	0	0.5	1	1.5	2
$y = e^{\sin 2x}$	1	$1.258\ldots$	$2.030\ldots$	$2.704\ldots$	$2.286\ldots$

$$\int_0^2 e^{\sin 2x}\,dx \approx \frac{h}{3}\left[y_0+4y_1+2y_2+4y_3+y_4\right] = \frac{0.5}{3}\left[1+4(1.258\ldots)+2(2.030\ldots)+4(2.704\ldots)+2.286\ldots\right] = 3.8$$

$$\int_0^2 e^{\sin 2x}\,dx \approx 3.87 (3\text{s.f.}).$$

(4)

(b) Comparing the estimate with 3.855, *both round to* 3.9 *at 2 significant figures, but not to the same value at 3 significant figures.*

The Simpson's rule estimate is accurate to 2 significant figures (equivalently, to 1 decimal place)

(1)

Question 5

Question 5

(a) Show that $t = \tan\frac{x}{2}$ transforms $\int \frac{dx}{2\sin x - \cos x + 5}$ into $\int \frac{dt}{3t^2+2t+2}$. (b) Hence evaluate the original integral.

Model solution

(a) The tangent half-angle substitution is appropriate because it converts both $\sin x$ *and* $\cos x$ *into rational expressions in* t.

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad \frac{dt}{dx} = \frac{1}{2}(1+t^2),$$

$$dx = \frac{2\,dt}{1+t^2}.$$

$$2\sin x - \cos x + 5 = \frac{4t - (1-t^2) + 5(1+t^2)}{1+t^2} = \frac{6t^2+4t+4}{1+t^2}.$$

$$\int \frac{dx}{2\sin x - \cos x + 5} = \int \frac{2\,dt}{6t^2+4t+4} = \int \frac{dt}{3t^2+2t+2}.$$

$$\int \frac{dx}{2\sin x - \cos x + 5} = \int \frac{dt}{3t^2+2t+2}.$$

(4)

(b) Complete the square to put the denominator into the arctangent form $u^2 + a^2$.

$$3t^2+2t+2 = 3\left(t+\frac{1}{3}\right)^2 + \frac{5}{3} = 3\left[\left(t+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2\right].$$

$$\int \frac{dt}{3t^2+2t+2} = \frac{1}{3}\cdot\frac{3}{\sqrt{5}}\tan^{-1}\left(\frac{t+1/3}{\sqrt{5}/3}\right) + C = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{3t+1}{\sqrt{5}}\right) + C.$$

Finally replace t *by* $\tan(x/2)$.

$$\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{3\tan(x/2)+1}{\sqrt{5}}\right) + C.$$

(4)

Question 6

Question 6

Let $y = \ln(e^{2x}\cos 3x)$, $-\frac{1}{2} < x < \frac{1}{2}$. (a) Show $y' = 2 - 3\tan 3x$. (b) Find $y^{(4)}$. (c) Find the first three non-zero terms of the Maclaurin expansion of $\ln(e^{2x}\cos 3x)$. (d) Write the first four non-zero terms of $\ln(1+kx)$. (e) Find k for which $\lim_{x\to 0} \dfrac{\ln(e^{2x}\cos 3x) - \ln(1+kx)}{x^2}$ exists.

Model solution

(a) Using the logarithm laws first makes the differentiation especially direct.

$$y = \ln(e^{2x}) + \ln(\cos 3x) = 2x + \ln(\cos 3x),$$

$$\frac{dy}{dx} = 2 + \frac{-3\sin 3x}{\cos 3x} = 2 - 3\tan 3x.$$

$$\boxed{\frac{dy}{dx} = 2 - 3\tan 3x.}$$

(2)

(b) Differentiate repeatedly, retaining factor 3 each time the chain rule is used.

$$y'' = -9\sec^2(3x),$$

$$y''' = -54\sec^2(3x)\tan(3x),$$

$$y^{(4)} = -162\sec^4(3x) - 324\sec^2(3x)\tan^2(3x).$$

$$\boxed{y^{(4)} = -162\sec^4(3x) - 324\sec^2(3x)\tan^2(3x).}$$

(3)

(c) Evaluate the derivatives at zero, then use the Maclaurin formula. The constant term is $y(0) = \ln 1 = 0$.

$$y'(0) = 2, \qquad y''(0) = -9, \qquad y'''(0) = 0, \qquad y^{(4)}(0) = -162.$$

$$y = 0 + 2x - \frac{9}{2!}x^2 + 0x^3 - \frac{162}{4!}x^4 + \cdots = 2x - \frac{9}{2}x^2 - \frac{27}{4}x^4 + \cdots.$$

$$\boxed{\ln(e^{2x}\cos 3x) = 2x - \frac{9}{2}x^2 - \frac{27}{4}x^4 + \cdots.}$$

(3)

(d) Substitute kx ***for*** x ***in*** $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$.

$$\ln(1+kx) = kx - \frac{k^2x^2}{2} + \frac{k^3x^3}{3} - \frac{k^4x^4}{4} + \cdots .$$

(1)

(e) For division by x^2 ***to give a finite limit, the numerator must have no term in*** x. ***Compare the linear terms of the two expansions.***

$$\ln(e^{2x}\cos 3x) - \ln(1+kx) = (2-k)x + \left(-\frac{9}{2} + \frac{k^2}{2}\right)x^2 + \cdots .$$

$$2 - k = 0.$$

$$k = 2.$$

(3)

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Question 7

Question 7

$A = (3,6,5)$ and $l : (\mathbf{r} - (12,30,39)) \times (7,13,24) = \mathbf{0}$. Points B, C lie on l with $AB = AC = 15$, and B has negative x-coordinate. (a) Find B, C. (b) Find the plane ABC. (c) For $D = (-2,1,\alpha)$, find α if the tetrahedron ABCD has volume 147. (d) Given $\alpha > 0$, find the shortest distance between l and AD, to 2 significant figures.

Model solution

(a) Write a general point on the line using its given point and direction vector. The distance condition then determines its parameter.

$$\mathbf{r} = (12,30,39) + \lambda(7,13,24),$$

$$\overrightarrow{A\mathbf{r}} = (9 + 7\lambda, 24 + 13\lambda, 34 + 24\lambda).$$

$$(9+7\lambda)^2 + (24+13\lambda)^2 + (34+24\lambda)^2 = 15^2 \Rightarrow 794\lambda^2 + 2382\lambda + 1588 = 0.$$

$$\lambda = -1 \quad \textit{or} \quad \lambda = -2.$$

The negative x-coordinate selects $\lambda = -2$ for B.

$$B = (12,30,39) - 2(7,13,24) = (-2,4,-9),$$

$$C = (12,30,39) - (7,13,24) = (5,17,15).$$

$$B = (-2,4,-9), \quad C = (5,17,15).$$

(4)

(b) A normal to plane ABC is the cross product of two non-parallel vectors in it.

$$\overrightarrow{AB} = (-5,-2,-14), \qquad \overrightarrow{AC} = (2,11,10),$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = (134,22,-51).$$

Use point A in the point--normal form of the plane.

$$134(x-3) + 22(y-6) - 51(z-5) = 0.$$

$$134x + 22y - 51z = 279.$$

(3)

(c) The volume is one sixth of the magnitude of a scalar triple product. Use the plane normal already found, which is $\overrightarrow{AB} \times \overrightarrow{AC}$.

$$\overrightarrow{AD} = (-5,-5,\alpha - 5),$$

$$147 = \frac{1}{6}|(134, 22, -51) \cdot (-5, -5, \alpha - 5)| = \frac{1}{6}|-525 - 51\alpha|.$$

$$|525 + 51\alpha| = 882 \Rightarrow \alpha = 7 \quad \textit{or} \quad \alpha = -\frac{469}{17}.$$

$$\alpha = 7 \textit{ or } -\frac{469}{17}.$$

(4)

(d) *Since* $\alpha > 0$, $D = (-2, 1, 7)$, *so a direction vector for* AD *is* $(-5, -5, 2)$. *For skew lines, their shortest separation is the component of any joining vector in the direction perpendicular to both lines.*

$$\mathbf{u} = (7, 13, 24), \qquad \mathbf{v} = (-5, -5, 2),$$

$$\mathbf{v} \times \mathbf{u} = (-146, 134, -30).$$

Take the vector from A *to* $(12, 30, 39)$ *on* l: $\mathbf{w} = (9, 24, 34)$.

$$d = \frac{|\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})|}{|\mathbf{v} \times \mathbf{u}|} = \frac{|(9, 24, 34) \cdot (-146, 134, -30)|}{\sqrt{146^2 + 134^2 + 30^2}} = \frac{882}{\sqrt{40172}} = 4.400\ldots$$

Theshortestdistanceis4.4(2s.f.).

(4)

Question 2

Question 2

The height h metres satisfies $t^2\frac{d^2h}{dt^2} - 2t\frac{dh}{dt} + 2h = t^3$, where $t \leqslant 5$. Given $t = e^x$: (a) show $\frac{dh}{dx} = t\frac{dh}{dt}$ and $\frac{d^2h}{dx^2} = t^2\frac{d^2h}{dt^2} + t\frac{dh}{dt}$; (b) transform the equation; (c) show that $h = At + Bt^2 + \frac{1}{2}t^3$; (d) use $h = 2.5$ at $t = 1$ and $\frac{dh}{dt} = -1$ at $t = 2$ to find the height at $t = 5$.

Model solution

(a) The substitution changes the independent variable, so the chain rule links derivatives with respect to x and t. Since $t = e^x$, $dt/dx = t$.

$$\frac{dh}{dx} = \frac{dh}{dt}\frac{dt}{dx} = t\frac{dh}{dt}.$$

$$\boxed{\frac{dh}{dx} = t\frac{dh}{dt}.}$$

(2) Differentiate this result once more, using the product rule.

$$\frac{d^2h}{dx^2} = \frac{d}{dx}\left(t\frac{dh}{dt}\right) = t\frac{dh}{dt} + t\frac{d}{dx}\left(\frac{dh}{dt}\right) = t\frac{dh}{dt} + t^2\frac{d^2h}{dt^2}.$$

$$\boxed{\frac{d^2h}{dx^2} = t^2\frac{d^2h}{dt^2} + t\frac{dh}{dt}.}$$

(2)

(b) Substitute the two derivative identities into the original equation, eliminating derivatives with respect to t.

$$(h_{xx} - h_x) - 2h_x + 2h = e^{3x},$$

$$\boxed{\frac{d^2h}{dx^2} - 3\frac{dh}{dx} + 2h = e^{3x}.}$$

(1)

(c) This is now a constant-coefficient differential equation. The complementary function comes from its auxiliary equation.

$$m^2 - 3m + 2 = 0 \quad \Rightarrow \quad (m-1)(m-2) = 0, \qquad h_{CF} = Ae^x + Be^{2x}.$$

The forcing term is e^{3x}, so try a particular integral $h_{PI} = ke^{3x}$.

$$9ke^{3x} - 9ke^{3x} + 2ke^{3x} = e^{3x} \quad \Rightarrow \quad k = \frac{1}{2}.$$

$$h = Ae^{x} + Be^{2x} + \frac{1}{2}e^{3x} = At + Bt^2 + \frac{1}{2}t^3.$$

$$h = At + Bt^2 + \frac{1}{2}t^3.$$

(6)

(d) Apply each condition to determine the two constants before using the model at $t = 5$.

$$h(1) = A + B + \frac{1}{2} = 2.5 \quad \Rightarrow \quad A + B = 2.$$

$$\frac{dh}{dt} = A + 2Bt + \frac{3}{2}t^2, \qquad -1 = A + 4B + 6 \quad \Rightarrow \quad A + 4B = -7.$$

$$3B = -9 \Rightarrow B = -3, \qquad A = 5.$$

$$h(5) = 5(5) - 3(5^2) + \frac{1}{2}(5^3) = 12.5.$$

The passenger is 12.5 m above the ground after 5 seconds.

(5)

Question 3

Question 3

Use algebra to determine the values of x for which $\dfrac{x^2-2x-24}{|x+6|} < 5-4x$.

Model solution

The absolute value in the denominator is the clue to split the work into $x > -6$ and $x < -6$. The point $x = -6$ is excluded because the expression is undefined.

$$\textit{For } x > -6, \quad |x+6| = x+6,$$

$$x^2-2x-24 < (5-4x)(x+6) \Rightarrow 5x^2+17x-54 < 0 \Rightarrow (5x+27)(x-2) < 0.$$

$$\therefore \quad -\frac{27}{5} < x < 2.$$

For $x < -6$, $|x+6| = -(x+6)$. Multiplying by $x+6$, which is negative, reverses the inequality.

$$-(x^2-2x-24) > (5-4x)(x+6) \Rightarrow 3x^2+21x-6 > 0.$$

$$3x^2+21x-6 = 0 \Rightarrow x = \frac{-7 \pm \sqrt{57}}{2}.$$

This upward-opening quadratic is positive outside its roots. Intersecting with $x < -6$ leaves only the left-hand interval.

$$x < \frac{-7-\sqrt{57}}{2} \quad \textit{or} \quad -\frac{27}{5} < x < 2.$$

(7)

Question 4

Question 4

The ellipse E is $\frac{x^2}{16}+\frac{y^2}{9}=1$. Points $P(4\cos\theta, 3\sin\theta)$ and $Q(4\cos\theta, -3\sin\theta)$ lie on E. (a) Find the eccentricity. (b) Show the normal at P is $4x\sin\theta - 3y\cos\theta = 7\sin\theta\cos\theta$. (c) Find its intersection R with OQ. (d) Show the locus of R has the same eccentricity as E.

Model solution

(a) The larger semi-axis is $a=4$ and the other is $b=3$, so use $e^2 = 1 - b^2/a^2$.

$$e = \sqrt{1-\frac{3^2}{4^2}} = \frac{\sqrt{7}}{4}.$$

$$e = \frac{\sqrt{7}}{4}.$$

(2)

(b) Implicit differentiation supplies the tangent gradient, whose negative reciprocal is the normal gradient.

$$\frac{x^2}{16}+\frac{y^2}{9}=1 \Rightarrow \frac{x}{8}+\frac{2y}{9}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx} = -\frac{9x}{16y}.$$

$$\left.\frac{dy}{dx}\right|_P = -\frac{3\cos\theta}{4\sin\theta}, \qquad m_{\text{normal}} = \frac{4\sin\theta}{3\cos\theta}.$$

$$y - 3\sin\theta = \frac{4\sin\theta}{3\cos\theta}(x - 4\cos\theta) \Rightarrow 4x\sin\theta - 3y\cos\theta = 7\sin\theta\cos\theta.$$

$$l_1 : 4x\sin\theta - 3y\cos\theta = 7\sin\theta\cos\theta.$$

(4)

(c) Since OQ passes through the origin and Q, its gradient is $-3\sin\theta/(4\cos\theta)$. Substitute its equation into the normal equation.

$$y = -\frac{3\sin\theta}{4\cos\theta}x.$$

$$4x\sin\theta - 3\left(-\frac{3\sin\theta}{4\cos\theta}x\right)\cos\theta = 7\sin\theta\cos\theta \Rightarrow \frac{25}{4}x\sin\theta = 7\sin\theta\cos\theta.$$

$$x = \frac{28}{25}\cos\theta, \qquad y = -\frac{21}{25}\sin\theta.$$

$$R\left(\frac{28}{25}\cos\theta, -\frac{21}{25}\sin\theta\right).$$

(4)

(d) The parametric coordinates have the standard ellipse form $x = a\cos\theta,\ y = b\sin\phi$, *with* $\phi = -\theta$. *Thus its semi-axes are* $28/25$ *and* $21/25$.

$$e = \sqrt{1 - \frac{(21/25)^2}{(28/25)^2}} = \sqrt{1 - \frac{441}{784}} = \frac{\sqrt{7}}{4}.$$

$$\text{The locus is an ellipse, } \frac{x^2}{(28/25)^2} + \frac{y^2}{(21/25)^2} = 1, \text{ with eccentricity } \frac{\sqrt{7}}{4}.$$

(2)

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