Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/3A - Further Pure Mathematics 1

Wednesday 19 June 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(**a**) Given $y = \ln(3 + x^2)$, complete the table at $x = 3$.

(**b**) Use Simpson's rule to estimate $\int_2^5 \ln(3 + x^2)\,dx$.

(**c**) Hence estimate $\int_2^5 \ln\sqrt{3 + x^2}\,dx$.

Model solution

(a) Substitute the missing x-value into the given formula, then round to four significant figures.

$$y = \ln(3 + 3^2) = \ln 12 = 2.484906\ldots$$

$$y = 2.485$$

(1)

(b) There are six equal intervals from 2 to 5, so Simpson's rule is appropriate and uses $h = 0.5$. The first and last ordinates have coefficient 1, the interior odd ordinates coefficient 4, and the interior even ordinates coefficient 2.

$$\begin{aligned}\int_2^5 \ln(3 + x^2)\,dx &\approx \frac{0.5}{3}\left[1.946 + 3.332 + 4(2.225 + 2.725 + 3.146) + 2(2.485 + 2.944)\right] \\ &= 8.086\ldots\end{aligned}$$

$$\int_2^5 \ln(3 + x^2)\,dx \approx 8.09$$

(3)

(c) Use $\ln\sqrt{A} = \frac{1}{2}\ln A$. Thus the new integrand is one half of that in part (b), so its integral is also one half.

$$\int_2^5 \ln\sqrt{3 + x^2}\,dx = \frac{1}{2}(8.09) = 4.045.$$

$$4.05$$

(1)

Question 2

Question 2

Use algebra to determine the values of x for which $|x^2 - 2x| \leqslant x$.

Model solution

Since an absolute value is non-negative, the right-hand side forces $x \geqslant 0$. To find the boundaries, remove the modulus by considering its two possible signs.

$$x^2 - 2x = x \implies x(x-3) = 0 \implies x = 0 \text{ or } x = 3.$$

$$x^2 - 2x = -x \implies x(x-1) = 0 \implies x = 0 \text{ or } x = 1.$$

For $0 < x < 1$, $|x^2 - 2x| = 2x - x^2 > x$, so this interval is rejected. For $1 \leqslant x \leqslant 3$, the inequality holds; for $x > 3$, $x^2 - 2x > x$. Include the equality case $x = 0$.

$$x = 0 \text{ or } 1 \leqslant x \leqslant 3$$

(4)

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Question 3

Question 3

Use L'Hospital's rule to show that $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=0$.

Model solution

Combining the two fractions produces a single quotient. Both its numerator and denominator tend to zero, which is the required indeterminate form for L'Hospital's rule.

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}.$$

Differentiate numerator and denominator. The denominator still gives zero at $x=0$, so apply L'Hospital's rule once more.

$$\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}=\lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x}.$$

$$\lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x}=\frac{\sin 0}{2\cos 0-0\sin 0}=\frac{0}{2}=0.$$

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=0$$

(6)

Question 4

Question 4

The curve satisfies $\cos x\, y'' + y^2 y' + \sin x = 0$ and $(\pi/4, 1)$ is stationary.
(**a**) Determine its nature. (**b**) Show that $y''' = \sqrt{2} - 2$ there.
(**c**) Find the Taylor series about $x = \pi/4$ through the cubic term.

Model solution

(a) At a stationary point $y' = 0$. Substitution into the differential equation directly gives y'', whose sign determines the nature.

$$\cos\frac{\pi}{4}y'' - 1^2(0) + \sin\frac{\pi}{4} = 0 \quad \Longrightarrow \quad \frac{\sqrt{2}}{2}y'' + \frac{\sqrt{2}}{2} = 0,$$

$$y'' = -1 < 0.$$

The stationary point is a local maximum.

(2)

(b) Differentiate the differential equation implicitly, using the product rule on both products. Then substitute the known values at the stationary point.

$$\cos x\, y''' - \sin x\, y'' + 2y(y')^2 + y^2 y'' + \cos x = 0.$$

$$\frac{\sqrt{2}}{2}y''' - \frac{\sqrt{2}}{2}(-1) + 2(1)(0)^2 + 1^2(-1) + \frac{\sqrt{2}}{2} = 0.$$

$$\frac{\sqrt{2}}{2}y''' + \sqrt{2} - 1 = 0 \quad \Longrightarrow \quad y''' = \sqrt{2} - 2.$$

$$y''' = \sqrt{2} - 2$$

(4)

(c) Taylor's formula uses the function value and successive derivatives at the centre $a = \pi/4$. The stationary-point condition makes the linear term zero.

$$\begin{aligned} y &= 1 + 0\left(x - \frac{\pi}{4}\right) + \frac{-1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}-2}{3!}\left(x - \frac{\pi}{4}\right)^3 + \cdots \\ &= 1 - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}-2}{6}\left(x - \frac{\pi}{4}\right)^3 + \cdots. \end{aligned}$$

$$y = 1 - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}-2}{6}\left(x - \frac{\pi}{4}\right)^3$$

(2)

Question 5

Question 5

For $y = e^{3x}\sin x$: **(a)** use Leibnitz's theorem to find $y^{(4)}$; **(b)** write it as $e^{3x}R\sin(x+\alpha)$.

Model solution

(a) Leibnitz's theorem is designed for the fourth derivative of a product. Let $u = e^{3x}$ *and* $v = \sin x$*, and list the derivatives needed.*

$$u' = 3e^{3x}, \quad u'' = 9e^{3x}, \quad u''' = 27e^{3x}, \quad u^{(4)} = 81e^{3x},$$

$$v' = \cos x, \quad v'' = -\sin x, \quad v''' = -\cos x, \quad v^{(4)} = \sin x.$$

$$\begin{aligned} y^{(4)} &= u^{(4)}v + 4u'''v' + 6u''v'' + 4u'v''' + uv^{(4)} \\ &= 81e^{3x}\sin x + 4(27e^{3x})\cos x + 6(9e^{3x})(-\sin x) \\ &\quad + 4(3e^{3x})(-\cos x) + e^{3x}\sin x \\ &= e^{3x}(28\sin x + 96\cos x). \end{aligned}$$

$$\frac{d^4y}{dx^4} = e^{3x}(28\sin x + 96\cos x)$$

(6)

(b) Expand $R\sin(x+\alpha)$ *and match the coefficients of* $\sin x$ *and* $\cos x$*.*

$$R\sin(x+\alpha) = R\cos\alpha\sin x + R\sin\alpha\cos x,$$

$$R\cos\alpha = 28, \qquad R\sin\alpha = 96.$$

$$R = \sqrt{28^2 + 96^2} = 100, \qquad \tan\alpha = \frac{96}{28} = \frac{24}{7}.$$

$$\alpha = \tan^{-1}\left(\frac{24}{7}\right) = 1.287\ldots$$

$$\frac{d^4y}{dx^4} = 100e^{3x}\sin\left(x + \tan^{-1}\frac{24}{7}\right)$$

(3)

Question 6

Question 6

The ellipse is $x^2/25 + y^2/9 = 1$. The hyperbola is $x^2/a^2 - y^2/b^2 = 1$; its eccentricity is reciprocal to that of the ellipse and their foci are the same. Find **(i)** a and **(ii)** b.

Model solution

For the ellipse, $a_E = 5$ *and* $b_E = 3$*, so* $e_E^2 = 1 - b_E^2/a_E^2$*. Its focal distance is* $a_E e_E$.

$$e_E = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}, \qquad c = 5\left(\frac{4}{5}\right) = 4.$$

(i) The hyperbola has eccentricity $e_H = 1/e_E = 5/4$*. Since its foci are* $(\pm 4, 0)$*, use* $c = ae_H$.

$$4 = a\left(\frac{5}{4}\right) \quad \Longrightarrow \quad a = \frac{16}{5}.$$

$$a = \frac{16}{5}$$

(3)

(ii) For a hyperbola $c^2 = a^2 + b^2$*, so the shared focal distance now determines* b.

$$b^2 = 4^2 - \left(\frac{16}{5}\right)^2 = \frac{144}{25} \quad \Longrightarrow \quad b = \frac{12}{5},$$

taking the positive root because b *is stated to be positive.*

$$b = \frac{12}{5}$$

(3)

Question 7

Question 7

(a) With $t = \tan(\theta/2)$, transform $\displaystyle\int \frac{d\theta}{2\sin\theta + \cos\theta + 2}$.

(b) Hence evaluate it from $\theta = \pi/2$ to $\theta = 2\pi/3$.

Model solution

(a) The tangent half-angle substitution turns both sine and cosine into rational functions of t, making this integral algebraic.

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}, \qquad d\theta = \frac{2\,dt}{1+t^2}.$$

$$\begin{aligned}\int \frac{d\theta}{2\sin\theta + \cos\theta + 2} &= \int \frac{2\,dt}{4t + (1-t^2) + 2(1+t^2)} \\ &= \int \frac{2\,dt}{t^2 + 4t + 3}.\end{aligned}$$

$$a = 2,\ b = 4,\ c = 3$$

(3)

(b) Change the limits to t-values before integrating: $\theta = \pi/2$ gives $t = 1$, and $\theta = 2\pi/3$ gives $t = \sqrt{3}$. Factor the quadratic and use partial fractions.

$$\frac{2}{t^2+4t+3} = \frac{2}{(t+1)(t+3)} = \frac{1}{t+1} - \frac{1}{t+3}.$$

$$\begin{aligned}\int_{\pi/2}^{2\pi/3} \frac{d\theta}{2\sin\theta + \cos\theta + 2} &= \left[\ln\left(\frac{t+1}{t+3}\right)\right]_1^{\sqrt{3}} \\ &= \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}+3}\right) - \ln\left(\frac{2}{4}\right) \\ &= \ln\left(\frac{2(\sqrt{3}+1)}{\sqrt{3}+3}\right) \\ &= \ln\left(\frac{2\sqrt{3}}{3}\right).\end{aligned}$$

$$\ln\left(\frac{2\sqrt{3}}{3}\right)$$

(4)

Question 8

Question 8

For $y^2 = 4ax$, point $A = (at^2, 2at)$. **(a)** Find its tangent.
For $B = (2k^2, 4k)$ and $C = (2k^2, -4k)$, the enclosed triangle has area 432. **(b)** Find B, C.

Model solution

(a) Differentiate implicitly to get the gradient at the parametric point, then use the point--gradient form of a line.

$$2y\frac{dy}{dx} = 4a \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{2a}{y}.$$

$$\left.\frac{dy}{dx}\right|_A = \frac{2a}{2at} = \frac{1}{t}.$$

$$y - 2at = \frac{1}{t}(x - at^2) \quad \Longrightarrow \quad yt = x + at^2.$$

$$\boxed{yt = x + at^2}$$

(3)

(b) Since B and C lie on $y^2 = 4ax$, substitution gives $a = 2$. Use part (a) with parameters k and $-k$ to find their tangent intersection D, then use base times perpendicular height.

$$(4k)^2 = 4a(2k^2) \quad \Longrightarrow \quad a = 2.$$

$$yk = x + 2k^2, \qquad -yk = x + 2k^2.$$

$$2yk = 0 \Longrightarrow y = 0, \qquad x = -2k^2,$$

$$D = (-2k^2, 0).$$

The vertical base BC has length $8k$, and its perpendicular distance from D is $2k^2 - (-2k^2) = 4k^2$.

$$432 = \frac{1}{2}(8k)(4k^2) = 16k^3 \quad \Longrightarrow \quad k = 3.$$

$$\boxed{B = (18, 12), \qquad C = (18, -12)}$$

(5)

Question 9

Question 9

(i) $\mathbf{r} = (2,-3,1) + \lambda(3,4,-1)$ and $\mathbf{r} = (13,5,8) + \mu(1,-2,5)$. Find their intersection and containing plane.
(ii) Find the two lines through the origin making angles $60°, 45°$ with the x, y axes respectively.

Model solution

(i)(a) At the intersection, the coordinate expressions from the two vector equations are equal. Solve the resulting simultaneous equations for the two parameters.

$$2 + 3\lambda = 13 + \mu, \qquad -3 + 4\lambda = 5 - 2\mu, \qquad 1 - \lambda = 8 + 5\mu.$$

$$\lambda = 3, \qquad \mu = -2.$$

$$P = (2,-3,1) + 3(3,4,-1) = (11,9,-2).$$

$$\boxed{P = (11,9,-2)}$$

(2)

(i)(b) A plane containing both lines has a normal perpendicular to both direction vectors, so take their cross product. Then use the point--normal equation through P.

$$(3,4,-1) \times (1,-2,5) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -1 \\ 1 & -2 & 5 \end{vmatrix}$$
$$= (18,-16,-10) = 2(9,-8,-5).$$

$$(9,-8,-5) \cdot (x - 11, y - 9, z + 2) = 0.$$

$$9x - 8y - 5z = 37.$$

$$\boxed{9x - 8y - 5z = 37}$$

(4)

(ii) The direction cosines are the cosines of the angles with the coordinate axes. The identity $l^2 + m^2 + n^2 = 1$ *determines the two possible* z *direction cosines.*

$$l = \cos 60° = \frac{1}{2}, \qquad m = \cos 45° = \frac{1}{\sqrt{2}},$$

$$n^2 = 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}, \quad \text{qquad} n = \pm\frac{1}{2}.$$

Multiplying either direction vector by 2 *avoids fractions.*

$$\frac{x}{1} = \frac{y}{\sqrt{2}} = \frac{z}{1} \quad \textit{and} \quad \frac{x}{1} = \frac{y}{\sqrt{2}} = \frac{z}{-1}$$

(4)

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Question 10

Question 10

$t^2x'' - 2t(t+1)x' + 2(t+1)x = 8t^3e^t$. With $x = tu$: (**a**) obtain the transformed equation; (**b**) find the particular solution given $x = 0$ at $t = \ln 3, \ln 5$.

Model solution

(a) The substitution $x = tu$ is given. Differentiate it twice using the product rule, then substitute so that the terms in u and u' cancel.

$$x = tu, \qquad x' = u + tu', \qquad x'' = u' + u' + tu'' = 2u' + tu''.$$

$$\begin{aligned} t^2x'' - 2t(t+1)x' + 2(t+1)x &= t^2(2u' + tu'') - 2t(t+1)(u + tu') + 2(t+1)tu \\ &= 2t^2u' + t^3u'' - 2t(t+1)u - 2t^2(t+1)u' + 2t(t+1)u \\ &= t^3(u'' - 2u'). \end{aligned}$$

$$t^3(u'' - 2u') = 8t^3e^t \quad \Longrightarrow \quad u'' - 2u' = 8e^t.$$

$$\boxed{u'' - 2u' = 8e^t}$$

(4)

(b) Solve the constant-coefficient transformed equation first. Its complementary function comes from the auxiliary equation; for the exponential forcing term, try a multiple of e^t.

$$m^2 - 2m = 0 \quad \Longrightarrow \quad m = 0, 2,$$

$$u_{CF} = A + Be^{2t}, \qquad u_{PI} = \lambda e^t.$$

$$u''_{PI} - 2u'_{PI} = \lambda e^t - 2\lambda e^t = -\lambda e^t = 8e^t,$$

$$\lambda = -8, \quad \text{qquadu} = A + Be^{2t} - 8e^t.$$

Finally, return to $x = tu$ and apply the two positions where the particle is at the origin.

$$x = t(A + Be^{2t} - 8e^t).$$

$$\begin{aligned} t = \ln 3: &\quad A + 9B = 24, \\ t = \ln 5: &\quad A + 25B = 40. \end{aligned}$$

$$A = 15, \qquad B = 1.$$

$$x = t\left(15 + e^{2t} - 8e^t\right)$$

$$\boxed{x = t\left(15 + e^{2t} - 8e^t\right)}$$

(8)