Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/3B - Further Statistics 1

November 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Contents

Question 1 2

Question 2 3

Question 3 5

Question 4 6

Question 5 8

Question 6 10

Question 7 12

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Question 1

Question 1

Kelly throws a tetrahedral die repeatedly. The expected frequency of each face is 43 if it is unbiased; the observed frequencies for faces $1, 2, 3, 4$ are $47, 34, 36, x$. (a) Show that $x = 55$. (b) Explain why the goodness-of-fit test has 3 degrees of freedom. (c) At the 5% level, test whether there is evidence that the die is unbiased. **(7)**

Model solution

(a) The expected frequencies total $4 \times 43 = 172$, so the observed frequencies must have the same total.

$$x = 172 - (47 + 34 + 36) = 55.$$

$$x = 55.$$

(1)

(b) There are four categories and no parameters have been estimated from the data. The sole constraint is that the four observed frequencies must add to the fixed total.

$$\nu = 4 - 1 = 3.$$

There are 3 degrees of freedom.

(1)

(c) This is a goodness-of-fit test because the observed face frequencies are being compared with those expected from an unbiased, discrete-uniform die.

H_0 : *the tetrahedral die is unbiased,* H_1 : *the tetrahedral die is biased.*

$$\chi^2 = \sum \frac{(O - E)^2}{E} = \frac{(47-43)^2 + (34-43)^2 + (36-43)^2 + (55-43)^2}{43} = 6.744\ldots$$

At the 5% level with 3 degrees of freedom, the upper-tail critical value is $\chi^2_{3,0.05} = 7.815$. A large value would contradict H_0.

$$6.744 < 7.815.$$

Do not reject H_0. There is insufficient evidence that the tetrahedral die is biased; the data are consistent with an unbiased die.

(5)

Question 2

Question 2

Weekday garage calls occur randomly at a mean rate of 1.25 per 10 minutes. (a) Show that the probability of at least 2 calls in 30 minutes is 0.888 (3 d.p.). (b) Find the probability that this happens in fewer than 4 of 6 randomly selected non-overlapping 30-minute periods. (c) From 150 such periods, use a Poisson approximation to find the probability that at least 3 periods contain exactly 8 calls, showing it is 0.664 (3 s.f.). (d) Explain why the approximation is reasonable. (e) State hypotheses for testing whether Saturday calls differ from weekday calls. (f) In four Saturday hours, 40 calls occur: test at 5%. **(14)**

Model solution

(a) A count in a fixed interval with random independent arrivals is Poisson. Thirty minutes is three 10-minute intervals, so its mean is $1.25 \times 3 = 3.75$*. Use the complement because "at least 2" is easiest via 0 or 1.*

$$C \sim \text{Poisson}(3.75), \qquad P(C \geqslant 2) = 1 - P(C = 0) - P(C = 1) = 1 - e^{-3.75}(1 + 3.75) = 0.88829\ldots$$

$$P(C \geqslant 2) = 0.888 \textit{ (3 d.p.).}$$

(2)

(b) Each non-overlapping period is a trial whose success is "at least 2 calls", with probability found in part (a). The required number of successes is fewer than 4.

$$D \sim B(6, 0.88829\ldots), \qquad P(D < 4) = \sum_{r=0}^{3} \binom{6}{r} (0.88829\ldots)^r (0.11171\ldots)^{6-r} = 0.02163\ldots$$

$$0.0216 \textit{ (approximately).}$$

(2)

(c) First find the small probability that one 30-minute period has exactly 8 calls. Then count such periods among the 150 periods.

$$P(C = 8) = e^{-3.75}\frac{3.75^8}{8!} = 0.02281\ldots$$

$$E \sim B(150, 0.02281\ldots), \qquad E(E) = 150(0.02281\ldots) = 3.4215\ldots$$

Since this binomial has large n *and small* p*, approximate it by* $E \sim \text{Poisson}(3.4215\ldots)$*.*

$$P(E \geqslant 3) = 1 - P(E \leqslant 2) = 1 - e^{-3.4215\ldots}\left(1 + 3.4215\ldots + \frac{(3.4215\ldots)^2}{2}\right) = 0.6639\ldots$$

$$P(E \geqslant 3) = 0.664 \textit{ (3 s.f.).}$$

(4)

(d) The Poisson approximation to the binomial is appropriate because $n = 150$ *is large and* $p = P(C = 8) = 0.02281\ldots$ *is small.* **(1)**

(e) In four weekday hours there are 24 *ten-minute intervals, so the null mean is* $24(1.25) = 30$. *``Different'' gives a two-sided alternative.*

$$H_0 : \lambda = 30, \qquad H_1 : \lambda \neq 30,$$

where λ *is the mean number of calls in four hours.* **(1)**

(f) Under H_0, *the Saturday four-hour count has the weekday model* $X \sim \text{Poisson}(30)$. *The observed value is in the upper tail, so calculate its upper-tail probability and compare it with* 0.025, *one tail of a 5% two-sided test.*

$$P(X \geqslant 40) = 1 - P(X \leqslant 39) = 0.04625\ldots > 0.025.$$

Do not reject H_0. *There is insufficient evidence at the 5% level that the Saturday number of calls is different from the weekday number.*

(4)

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Question 3

Question 3

A courier's daily number of successfully delivered parcels is $X \sim B(400, 0.64)$. A random sample $X_1, \ldots, X_{100}$ is taken. Estimate $P(\overline{X} > 257)$. **(4)**

Model solution

The sample is large, so the central limit theorem gives an approximately normal distribution for the sample mean. Start with the mean and variance of one day.

$$E(X) = 400(0.64) = 256, \qquad \text{Var}(X) = 400(0.64)(0.36) = 92.16.$$

$$\overline{X} \sim N\left(256, \frac{92.16}{100}\right) = N(256, 0.9216).$$

$$P(\overline{X} > 257) = P\left(Z > \frac{257 - 256}{\sqrt{0.9216}}\right) = P(Z > 1.0417\ldots) = 0.1492\ldots$$

$$P(\overline{X} > 257) \approx 0.149.$$

(4)

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Question 4

Question 4

For the number N of photographs entered, $P(N = n)$ for $n = 0, 1, 2, 3, 4, 5$ is respectively $a, 0.2, 0.05, 0.25, b, c$. Given $E(4N + 2) = 14.8$ and $P(N = 5 \mid N > 2) = \frac{1}{2}$: (a) show that $\text{Var}(N) = 2.76$; (b) find the expected entry fee in pence, where fees for $0, 1, 2, 3, 4, 5$ photographs are $0, 50, 70, 90, 100, 100$; (c) give a limitation of a Poisson model. **(10)**

Model solution

(a) The transformed expectation first determines $E(N)$, then use the probability total and the conditional probability to determine the unknown probabilities.

$$4E(N) + 2 = 14.8 \quad \Rightarrow \quad E(N) = 3.2.$$

$$a + b + c = 1 - (0.2 + 0.05 + 0.25) = 0.5.$$

$$0.2 + 0.1 + 0.75 + 4b + 5c = 3.2 \quad \Rightarrow \quad 4b + 5c = 2.15.$$

$$P(N = 5 \mid N > 2) = \frac{c}{0.25 + b + c} = \frac{1}{2} \quad \Rightarrow \quad c = 0.25 + b.$$

$$4b + 5(0.25 + b) = 2.15 \quad \Rightarrow \quad b = 0.10, \qquad c = 0.35.$$

$$a = 0.5 - b - c = 0.05.$$

Now calculate $E(N^2)$, because $\text{Var}(N) = E(N^2) - [E(N)]^2$.

$$E(N^2) = 1(0.2) + 4(0.05) + 9(0.25) + 16(0.1) + 25(0.35) = 13.$$

$$\text{Var}(N) = 13 - (3.2)^2 = 2.76.$$

$$\text{Var}(N) = 2.76.$$

(6)

(b) Form the distribution of the fee and take its expectation. A member entering four or five photographs pays the capped fee of 100p.

$$E(F) = 50(0.2) + 70(0.05) + 90(0.25) + 100(0.1) + 100(0.35) = 81.$$

The expected entry fee is 81p.

(3)

(c) A Poisson random variable has no upper bound, whereas a member may enter at most five photographs.

A Poisson model assigns positive probability to $N > 5$, which is impossible here.

(1)

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Question 5

Question 5

Asha's red-counter probability is 0.07. (a) Find the probability of at least 3 whites before a red. (b) Find the probability that her second red is on draw 9. Davinda draws until n reds; if the mean and standard deviation of her draw count D are 4400 and 660, find her red probability p. Jerry tests whether his red proportion is smaller than Asha's: J is the draw number of his first red. (d) At 10%, find the critical region. (e) Interpret $J = 34$. (f) If Jerry's red probability is 0.011, show that the test power is 0.702 (3 s.f.). **(18)**

Model solution

(a) Each draw is replaced, so draws are independent. To get at least three whites before any red, the first three draws must all be white.

$$P(\textit{at least 3 whites before a red}) = (1-0.07)^3 = 0.804357\ldots$$

$$0.804 \textit{ (approximately).}$$

(2)

(b) For the second red to be on draw 9, among the first eight draws there must be exactly one red, and draw 9 must be red.

$$\binom{8}{1}(0.07)^1(0.93)^7(0.07) = 0.02358\ldots$$

$$0.0236 \textit{ (approximately).}$$

(2)

(c) D has a negative-binomial distribution: it counts draws needed for n successes, each with probability p. Its mean and variance are n/p and $n(1-p)/p^2$.

$$\frac{n}{p} = 4400, \qquad \frac{n(1-p)}{p^2} = 660^2.$$

Dividing the variance equation by the mean equation eliminates n.

$$\frac{1-p}{p} = \frac{660^2}{4400} = 99 \quad \Rightarrow \quad \frac{1}{p} = 100 \quad \Rightarrow \quad p = 0.01.$$

$$p = 0.010.$$

(4)

(d) A smaller red probability makes the first red take longer, so this is an upper-tail geometric test.

$$H_0 : p = 0.07, \qquad H_1 : p < 0.07, \qquad J \sim \text{Geo}(0.07) \textit{ under } H_0.$$

Choose the smallest integer cutoff c *for which the upper-tail probability is at most 0.10.*

$$P(J \geqslant c) = (0.93)^{c-1} \leqslant 0.10.$$

$$c - 1 \geqslant \frac{\log(0.10)}{\log(0.93)} = 31.72\ldots,$$

$$c \geqslant 32.72\ldots \quad \Rightarrow \quad c = 33.$$

Critical region: $J \geqslant 33$.

(5)

(e) The observation $J = 34$ *lies in the critical region, so it is sufficiently unusual under Asha's red probability.*

Reject H_0. **There is evidence that Jerry's bag contains a smaller proportion of red counters than Asha's bag.**

(2)

(f) The power is the probability of rejecting H_0 *when the true red probability is* 0.011. *Use the same critical region, but this new geometric distribution.*

$$\text{Power} = P_{p=0.011}(J \geqslant 33) = (1 - 0.011)^{32} = 0.7019\ldots$$

Power $= 0.702$ **(3 s.f.).**

(3)

Question 6

Question 6

$G_X(t) = k(1+2t)^5$. (a) Show $k = \frac{1}{243}$. (b) Find $P(X = 2)$. (c) Find the PGF of $W = 2X + 3$. Also $G_Y(t) = \dfrac{t(1+2t)^2}{9}$, independently of X. (d) Find the PGF of $U = X + Y$ in simplest form. (e) Use calculus to find $\text{Var}(U)$. **(14)**

Model solution

(a) Every probability generating function satisfies $G(1) = 1$, since the probabilities sum to 1.

$$1 = G_X(1) = k(1+2)^5 = 243k,$$

$$\boxed{k = \frac{1}{243}.}$$

(2)

(b) In a PGF, $P(X = 2)$ is the coefficient of t^2.

$$G_X(t) = \frac{1}{243}(1+2t)^5, \quad \text{qquad} P(X=2) = \frac{1}{243}\binom{5}{2}(2)^2 = \frac{40}{243}.$$

$$\boxed{P(X = 2) = \frac{40}{243}.}$$

(2)

(c) For $W = 2X + 3$, substitute t^2 for t to represent $2X$, then multiply by t^3 to represent the added 3.

$$G_W(t) = t^3 G_X(t^2) = \frac{t^3(1+2t^2)^5}{243}.$$

$$\boxed{G_W(t) = \frac{t^3(1+2t^2)^5}{243}.}$$

(2)

(d) Independent random variables add by multiplying their PGFs.

$$G_U(t) = \frac{(1+2t)^5}{243} \cdot \frac{t(1+2t)^2}{9} = \frac{t(1+2t)^7}{2187}.$$

$$\boxed{G_U(t) = \frac{t(1+2t)^7}{2187}.}$$

(2)

(e) For a PGF, $G'(1) = E(U)$ ***and*** $G''(1) = E[U(U-1)]$. ***Hence*** $\text{Var}(U) = G''(1) + G'(1) - [G'(1)]^2$. ***Differentiate the PGF from part (d).***

$$G'_U(t) = \frac{(1+2t)^7 + 14t(1+2t)^6}{2187}, \qquad G'_U(1) = \frac{3^7 + 14 \cdot 3^6}{2187} = \frac{17}{3}.$$

$$G''_U(t) = \frac{28(1+2t)^6 + 168t(1+2t)^5}{2187}, \qquad G''_U(1) = 28.$$

$$\text{Var}(U) = 28 + \frac{17}{3} - \left(\frac{17}{3}\right)^2 = \frac{14}{9}.$$

$$\text{Var}(U) = \frac{14}{9}.$$

(6)

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Question 7

Question 7

Lollipop-stick length is normal with unknown mean μ and standard deviation 0.2. Farhan tests at 1%: $H_0 : \mu = 15$, $H_1 : \mu < 15$. (a) State the test size. Given the true mean is 14.9, (b)(i) find the minimum n for which the Type II error probability is below 0.05; (ii) state the effect on this probability of using the same n at 5%. **(8)**

Model solution

(a) The size is the significance level: the maximum probability of a Type I error under H_0.

$$0.01.$$

(1)

(b)(i) Because the alternative is lower-tailed, reject for a sufficiently small sample mean: $\overline{X} < k$. *Under* H_0, $\overline{X} \sim N(15, 0.2^2/n)$. *The 1% lower-tail normal value is* -2.3263.

$$\frac{k - 15}{0.2/\sqrt{n}} = -2.3263 \quad \Rightarrow \quad k = 15 - \frac{0.46526}{\sqrt{n}}.$$

A Type II error when $\mu = 14.9$ *is failing to reject, namely* $\overline{X} \geqslant k$. *To make this probability below 0.05, the corresponding standardised cutoff must exceed* 1.6449.

$$\frac{k - 14.9}{0.2/\sqrt{n}} > 1.6449.$$

$$\frac{0.1 - 0.46526/\sqrt{n}}{0.2/\sqrt{n}} > 1.6449 \quad \Rightarrow \quad 0.5\sqrt{n} - 2.3263 > 1.6449.$$

$$\sqrt{n} > 7.9424\ldots \quad \Rightarrow \quad n > 63.08\ldots$$

The minimum sample size is $n = 64$.

(6)

(b)(ii) Raising the significance level enlarges the rejection region. With the same sample size, it is less likely to fail to reject when $\mu = 14.9$.

The probability of a Type II error decreases.

(1)