Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/3C - Further Mechanics 1

Friday 7 June 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

In this question, **i** *and* **j** *are horizontal perpendicular unit vectors.*
Particles A and B, of masses 3 kg and 2 kg respectively, collide directly on a smooth horizontal plane. Immediately before the collision,

$$\mathbf{u}_A = 3\mathbf{i} - \mathbf{j}, \qquad \mathbf{u}_B = -6\mathbf{i} + 2\mathbf{j} \quad \text{m s}^{-1}.$$

Immediately after the collision, A has velocity $-2\mathbf{i} + \frac{2}{3}\mathbf{j}$ m s^{-1}.
(a) Find the total kinetic energy before the collision.
(b) Find the impulse exerted on A by B, in terms of **i** and **j**.
(c) Find the velocity of B immediately after the collision, in terms of **i** and **j**.

Model solution

(a) Kinetic energy depends on speed, so first square the magnitude of each velocity vector.

$$\begin{aligned} \textit{total KE} &= \frac{1}{2}(3)\left(3^2 + (-1)^2\right) + \frac{1}{2}(2)\left((-6)^2 + 2^2\right) \\ &= 15 + 40 = 55. \end{aligned}$$

$$55 \ J$$

(3)

(b) The impulse on A *is its change in momentum, so use mass times (final velocity − initial velocity).*

$$\begin{aligned} \mathbf{J}_A &= 3\left[\left(-2\mathbf{i} + \frac{2}{3}\mathbf{j}\right) - (3\mathbf{i} - \mathbf{j})\right] \\ &= 3\left(-5\mathbf{i} + \frac{5}{3}\mathbf{j}\right) = -15\mathbf{i} + 5\mathbf{j}. \end{aligned}$$

$$\mathbf{J}_A = (-15\mathbf{i} + 5\mathbf{j}) \ N\ s$$

(3)

(c) The impulse on B *is equal in magnitude and opposite in direction to that on* A*. Apply impulse--momentum to* B*.*

$$\begin{aligned} 2\mathbf{v}_B - 2(-6\mathbf{i} + 2\mathbf{j}) &= 15\mathbf{i} - 5\mathbf{j}, \\ 2\mathbf{v}_B + 12\mathbf{i} - 4\mathbf{j} &= 15\mathbf{i} - 5\mathbf{j}, \\ 2\mathbf{v}_B &= 3\mathbf{i} - \mathbf{j}. \end{aligned}$$

$$\mathbf{v}_B = \left(\tfrac{3}{2}\mathbf{i} - \tfrac{1}{2}\mathbf{j}\right) \ m\ s^{-1}$$

(3)

Question 2

Question 2

A rough plane is inclined at an angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$. A particle P of mass m is projected up a line of greatest slope with speed $2\sqrt{ag}$. The coefficient of friction is $\frac{1}{7}$, and the particle comes to rest after travelling distance d. Air resistance is negligible.
(a) Show that the magnitude of the frictional force is $\frac{4}{35}mg$.
(b) Using the work–energy principle, find d in terms of a.

Model solution

(a) The given tangent gives a 3--4--5 triangle, so $\cos\theta = \frac{4}{5}$. Resolve perpendicular to the plane: there is no acceleration in this direction.

$$R = mg\cos\theta = \frac{4}{5}mg.$$

Friction has magnitude μR and acts down the plane because the motion is up the plane.

$$F = \frac{1}{7}\left(\frac{4}{5}mg\right) = \frac{4}{35}mg.$$

$$F = \frac{4}{35}mg$$

(3)

(b) Use work--energy because the particle moves through a known distance under constant forces. Its initial kinetic energy is used against the gain in gravitational potential energy and the work done against friction.

$$\frac{1}{2}m(2\sqrt{ag})^2 = mgd\sin\theta + \frac{4}{35}mgd.$$

$$2mga = mgd\left(\frac{3}{5} + \frac{4}{35}\right) = \frac{5}{7}mgd.$$

$$d = \frac{7}{5}a$$

(4)

Question 3

Question 3

A 1000 kg car travels on a horizontal road at constant speed 72 km h^{-1} against a constant resistance of 900 N. Its engine works at a constant rate of P kW.
(a) Find P.
The car subsequently travels uphill, where $\sin\alpha = \frac{2}{49}$. In a refined model, non-gravitational resistance has magnitude $20v$ N, where v is in m s^{-1}.
(b) When the power is 30 kW and the uphill speed is 10 m s^{-1}, find the acceleration a.
(c) Later, the power is again 30 kW and the car travels uphill at constant speed U m s^{-1}. Find U.

Model solution

(a) At constant speed the resultant force is zero. Convert the speed to SI units before using power = driving force × speed.

$$v = 72 \times \frac{1000}{3600} = 20 \ \textit{m s}^{-1}.$$

$$F = 900 \ \textit{N}, \qquad 1000P = Fv = 900(20) = 18000.$$

$$\boxed{P = 18 \ \textit{kW}}$$

(3)

(b) Take up the slope as positive. The engine force comes from $P = Fv$; resistance and the component of weight both act down the slope.

$$F_{\text{engine}} = \frac{30000}{10} = 3000 \ \textit{N}, \qquad R = 20(10) = 200 \ \textit{N}.$$

$$3000 - 200 - 1000g\left(\frac{2}{49}\right) = 1000a.$$

$$3000 - 200 - 400 = 1000a.$$

$$\boxed{a = 2.4 \ \textit{m s}^{-2}}$$

(4)

(c) Constant speed again means zero resultant force. This time both the driving force and resistance depend on the unknown speed U.

$$\frac{30000}{U} - 20U - 1000g\left(\frac{2}{49}\right) = 0.$$

$$\frac{30000}{U} - 20U - 400 = 0.$$

$$30000 - 20U^2 - 400U = 0 \quad \Longrightarrow \quad U^2 + 20U - 1500 = 0.$$

$$(u + 50)(u - 30) = 0.$$

Speed must be positive, so the negative root is rejected.

$$u = 30 \ m\ s^{-1}$$

(5)

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Question 4

Question 4

Particle A, of mass $2m$, moves with speed $3u$ on a smooth horizontal plane and collides directly with stationary particle B, of mass m. The coefficient of restitution is e, where $e > 0$.
(a) Show that the speed of B immediately after the collision is $2u(1+e)$.
After this collision, B hits a smooth fixed vertical wall perpendicular to its motion. The coefficient of restitution between B and the wall is $\frac{1}{2}$.
(b) Show that there will be a second collision between A and B.
(c) Find the magnitude of the impulse received by B from the wall.
(d) Find the loss in kinetic energy of B in its collision with the wall.

Model solution

(a) Let v_A and v_B be the velocities after the direct collision, taking the initial direction of A as positive. Conservation of linear momentum and the restitution law give two equations for the two unknown velocities.

$$2m(3u) = 2mv_A + mv_B, \qquad \textit{so} \qquad 6u = 2v_A + v_B. \tag{1}$$

$$v_B - v_A = e(3u - 0) = 3eu. \tag{2}$$

$$v_B = v_A + 3eu.$$

$$6u = 2v_A + v_A + 3eu \quad \Longrightarrow \quad v_A = u(2-e).$$

$$v_B = u(2-e) + 3eu = 2u(1+e).$$

$$\boxed{v_B = 2u(1+e)}$$

(6)

(b) The velocity of A found above determines whether it continues towards the wall after B. Since a coefficient of restitution satisfies $0 < e \leqslant 1$,

$$v_A = u(2-e) > 0.$$

Thus A continues towards the wall. Particle B reaches the wall first and rebounds towards A, while A is still moving towards the wall. They therefore move towards one another and there is a second collision.

$$\boxed{\textit{There will be a second collision between A and B.}}$$

(3)

(c) Immediately before hitting the wall, B has speed $2u(1+e)$. A fixed wall reverses the direction of the normal component, and restitution $\frac{1}{2}$ halves its speed.

$$v_{B,\text{after}} = -\frac{1}{2}\big[2u(1+e)\big] = -u(1+e).$$

$$|\textit{impulse}| = m\,|v_{B,\text{after}} - v_{B,\text{before}}| = m\,|-u(1+e) - 2u(1+e)|\,.$$

$$3mu(1+e)\ \textbf{\textit{N s}}$$

(3)

(d) Find the decrease in kinetic energy using the speeds immediately before and after impact with the wall.

$$\textit{KE loss} = \frac{1}{2}m\big[2u(1+e)\big]^2 - \frac{1}{2}m\big[u(1+e)\big]^2$$
$$= \frac{1}{2}mu^2(1+e)^2(4-1).$$

$$\tfrac{3}{2}mu^2(1+e)^2$$

(3)

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Question 5

Question 5

A light elastic string has natural length $2a$ and modulus of elasticity $2mg$. It hangs vertically from a fixed point A, with a particle P of mass m attached. In equilibrium $AE = 3a$. From E, P is projected vertically downwards with speed $\frac{3}{2}\sqrt{ag}$. Air resistance is negligible.
Find the elastic energy stored in the string when P first comes to instantaneous rest, in the form $kmga$.

Model solution

Let x be the further distance that P falls from E before first coming to rest. Energy conservation is appropriate because the only forces are gravity and the elastic tension, both conservative. At E the extension is a, and at the lowest point it is $a + x$.

$$\textit{EPE} = \frac{\lambda(\textit{extension})^2}{2l}, \qquad \lambda = 2mg, \quad l = 2a.$$

The particle loses gravitational potential energy mgx. Equate its initial kinetic energy, initial elastic energy, and this lost gravitational potential energy to the final elastic energy:

$$\frac{1}{2}m\left(\frac{3}{2}\sqrt{ag}\right)^2 + \frac{2mg\,a^2}{2(2a)} + mgx = \frac{2mg(a+x)^2}{2(2a)}.$$

$$\frac{9}{8}mga + \frac{1}{2}mga + mgx = \frac{mg(a+x)^2}{2a}.$$

$$\frac{13}{8}a + x = \frac{(a+x)^2}{2a}.$$

$$2a\left(\frac{13}{8}a + x\right) = (a+x)^2 \quad \Longrightarrow \quad x^2 = \frac{9}{4}a^2.$$

$$x = \frac{3}{2}a$$

The negative root would put the first rest point above E, so it is not physically relevant. The extension at the first rest point is therefore $a + x = \frac{5}{2}a$.

$$\textit{EPE} = \frac{2mg\left(\frac{5}{2}a\right)^2}{2(2a)} = \frac{25}{8}mga.$$

$$\frac{25}{8}mga$$

(7)

Question 6

Question 6

In this question, $\mathbf{i}$ *and* $\mathbf{j}$ *are horizontal perpendicular unit vectors.*
A particle P moves with velocity $4\mathbf{i}-\mathbf{j}$ m s^{-1} on a smooth horizontal plane. It collides with a smooth vertical wall and rebounds with velocity $\mathbf{i}+3\mathbf{j}$ m s^{-1}. The coefficient of restitution is e.
(a) Find e.
Afterwards P collides with a second smooth vertical wall parallel to $\mathbf{i}$. The coefficient of restitution with this wall is $\frac{1}{3}$. The deflection angle is $\alpha°$.
(b) Find α, to the nearest whole number.

Model solution

(a) A smooth wall exerts an impulse normal to itself. The change in velocity is parallel to this impulse, so it identifies a normal direction.

$$(\mathbf{i}+3\mathbf{j})-(4\mathbf{i}-\mathbf{j})=-3\mathbf{i}+4\mathbf{j}.$$

Use the unit normal $\mathbf{n}=\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$. *Resolve both velocities in this normal direction, because restitution only concerns the normal components.*

$$(4\mathbf{i}-\mathbf{j})\cdot\mathbf{n}=\frac{-12-4}{5}=-\frac{16}{5},$$

$$(\mathbf{i}+3\mathbf{j})\cdot\mathbf{n}=\frac{-3+12}{5}=\frac{9}{5}.$$

Thus the approach speed normal to the wall is $\frac{16}{5}$ *and the separation speed is* $\frac{9}{5}$.

$$e=\frac{\textit{separation speed}}{\textit{approach speed}}=\frac{9/5}{16/5}.$$

$$e=\frac{9}{16}$$

(6)

(b) The second wall is parallel to $\mathbf{i}$, *so the* $\mathbf{i}$ *component is unchanged. Its normal is parallel to* $\mathbf{j}$, *so that component reverses direction and its magnitude is multiplied by* $\frac{1}{3}$.

$$\mathbf{v}_{\text{before}}=\mathbf{i}+3\mathbf{j}, \qquad \mathbf{v}_{\text{after}}=\mathbf{i}-\mathbf{j}.$$

The angle of deflection is the angle between these two velocity vectors. Use the scalar product.

$$\cos\alpha=\frac{(\mathbf{i}+3\mathbf{j})\cdot(\mathbf{i}-\mathbf{j})}{|\mathbf{i}+3\mathbf{j}|\,|\mathbf{i}-\mathbf{j}|}=\frac{-2}{\sqrt{10}\sqrt{2}}.$$

$$\alpha=\cos^{-1}\left(\frac{-2}{\sqrt{20}}\right)=116.565\ldots°.$$

$$\alpha=117°$$

(4)

Question 7

Question 7

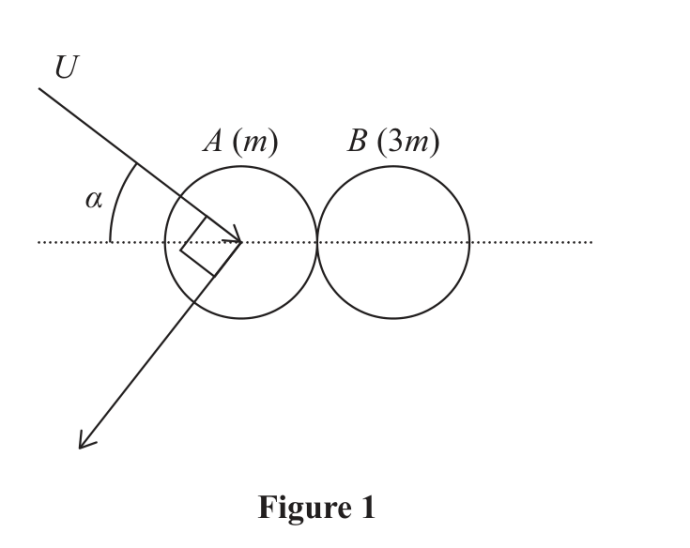


Figure 1

A smooth uniform sphere A of mass m and speed U collides obliquely on a smooth horizontal plane with a stationary smooth uniform sphere B of mass $3m$. The spheres have the same radius. Before collision, the direction of A makes an acute angle α with the line of centres. After collision, the direction of A is perpendicular to its original direction. The coefficient of restitution is e.

(a) Show that the speed of B after the collision is $\frac{1}{4}(1+e)U\cos\alpha$.

(b) Show that $e > \frac{1}{3}$.

(c) Show that $0 < \tan\alpha \leqslant \frac{1}{\sqrt{2}}$.

Model solution

(a) For smooth spheres, the impulse acts along the line of centres. Therefore the components perpendicular to this line are unchanged, and momentum and restitution are applied only along it. Let the post-collision components of A and B along the line of centres be v_A and v_B.

$$mU\cos\alpha = mv_A + 3mv_B. \quad (1)$$

$$v_B - v_A = e(U\cos\alpha - 0) = eU\cos\alpha. \quad (2)$$

Substitute $v_A = v_B - eU\cos\alpha$ from (2) into (1).

$$U\cos\alpha = v_B - eU\cos\alpha + 3v_B = 4v_B - eU\cos\alpha.$$

$$4v_B = (1+e)U\cos\alpha.$$

$$\boxed{v_B = \tfrac{1}{4}(1+e)U\cos\alpha}$$

(6)

(b) The perpendicular component of A is unchanged by the collision, so it remains $U\sin\alpha$. The diagram and the stated perpendicular deflection mean the post-collision component

of A along the line of centres must be directed opposite to the original positive direction. Thus the signed component v_A must be negative.

$$v_A = U\cos\alpha - 3v_B = U\cos\alpha - \frac{3}{4}(1+e)U\cos\alpha = \frac{1}{4}(1-3e)U\cos\alpha.$$

For this component to be negative,

$$\frac{1}{4}(1-3e)U\cos\alpha < 0.$$

Since $U > 0$ and $0° < \alpha < 90°$, $\cos\alpha > 0$. Hence the sign is determined by $1-3e$, giving

$$1-3e < 0.$$

$$e > \frac{1}{3}$$

(4)

(c) The component perpendicular to the line of centres is unchanged, so it is $U\sin\alpha$ after collision. The post-collision velocity of A therefore has components $(-v_A, U\sin\alpha)$, where $v_A = \frac{1}{4}(3e-1)U\cos\alpha$ is now a positive magnitude. Perpendicular directions have zero scalar product:

$$(U\cos\alpha, U\sin\alpha)\cdot(-v_A, U\sin\alpha) = 0.$$

$$-U\cos\alpha\, v_A + U^2\sin^2\alpha = 0.$$

Substitute the magnitude of the reversed line-of-centres component.

$$\frac{1}{4}(3e-1)U\cos\alpha = U\sin\alpha\tan\alpha.$$

$$\tan^2\alpha = \frac{1}{4}(3e-1).$$

A coefficient of restitution satisfies $e \leqslant 1$, so

$$\tan^2\alpha \leqslant \frac{1}{4}(3\cdot 1 - 1) = \frac{1}{2}.$$

Also $0° < \alpha < 90°$, so $\tan\alpha$ is positive and taking square roots preserves the inequality.

$$0 < \tan\alpha \leqslant \frac{1}{\sqrt{2}}$$

(5)