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Edexcel A Level Further Mathematics

9FM0/3D - Decision Mathematics 1

Friday 21 June 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

The numbers $17, 8, 16, 12, 24, 19, 23, 11, 20, 13, 4$ are packed by first fit into bins of size n:

1	$17, 8, 12$
2	$16, 24$
3	$19, 11, 4$
4	$23, 13$
5	20

(a) Explain why $n = 40$. (b) Quick-sort the original list into descending order, showing pivots. (c) Use first-fit decreasing with bins of size 40. **(7)**

Model solution

(a) First fit tries an item in every earlier bin before opening or using a later bin. Thus a rejected fit gives useful information about the capacity. The item 4 is put into Bin 3 rather than Bin 2. At that stage Bin 2 contains

$$16 + 24 = 40.$$

Thus $n \geqslant 40$. Also, 16 was not put into Bin 1, whose earlier contents were 17 and 8; therefore $n < 17 + 8 + 16 = 41$. Since n is an integer, $40 \leqslant n < 41$ forces the capacity.

$$n = 40.$$

(1)

(b) Use the middle-right item as each pivot, partitioning into entries greater than the pivot followed by entries less than the pivot. This produces descending order directly.

$$\begin{aligned} 17, 8, 16, 12, 24, \boxed{19}, 23, 11, 20, 13, 4 &\longrightarrow 24, 23, 20, \boxed{19}, 17, 8, 16, 12, 11, 13, 4 \\ &\longrightarrow 24, 23, 20, 19, 17, 16, 13, 12, 8, 11, 4 \quad (\textit{pivots } 23, 12) \\ &\longrightarrow 24, 23, 20, 19, 17, 16, 13, 12, 11, 8, 4 \quad (\textit{pivots } 16, 11) \\ &\longrightarrow 24, 23, 20, 19, 17, 16, 13, 12, 11, 8, 4 \quad (\textit{pivot } 4). \end{aligned}$$

This final pass partitions the remaining non-singleton sublists; the resulting sublists are the singletons

$$24 \mid 23 \mid 20 \mid 19 \mid 17 \mid 16 \mid 13 \mid 12 \mid 11 \mid 8 \mid 4.$$

Hence the list is fully sorted.

$$24, 23, 20, 19, 17, 16, 13, 12, 11, 8, 4.$$

(4)

(c) First-fit decreasing processes that descending list, placing each number in the first bin with sufficient remaining space.

Bin	*contents*	*total*
1	$24, 16$	40
2	$23, 17$	40
3	$20, 19$	39
4	$13, 12, 11, 4$	40
5	8	8

$\{24, 16\}, \{23, 17\}, \{20, 19\}, \{13, 12, 11, 4\}, \{8\}.$

(2)

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Question 2

Question 2

The distances (miles) are

	A	B	C	D	E	F	G	H	J
A	–	50	59	26	50	40	87	63	59
B	50	–	28	61	79	63	45	64	48
C	59	28	–	33	57	35	70	36	45
D	26	61	33	–	24	64	71	37	33
E	50	79	57	24	–	40	64	30	31
F	40	63	35	64	40	–	47	70	71
G	87	45	70	71	64	47	–	34	67
H	63	64	36	37	30	70	34	–	33
J	59	48	45	33	31	71	67	33	–

(a) Use Prim's algorithm from D. (b) State its weight. (c) Draw the tree. A historian needs a shortest Hamiltonian cycle: (d) give the doubled-tree upper bound; (e) use nearest neighbour from D; (f) choose the best upper bound, also comparing 352 miles from F; (g) delete A to obtain a lower bound; (h) choose the best lower bound, also comparing 274 miles from deleting J. **(13)**

Model solution

(a) Prim's algorithm grows one connected tree. At each stage choose the shortest arc from a tree vertex to a vertex not yet in the tree. Starting at D*, the selected arcs, in order, are*

$$\mathrm{DE}(24),\ \mathrm{AD}(26),\ \mathrm{EH}(30),\ \mathrm{EJ}(31),\ \mathrm{CD}(33),\ \mathrm{BC}(28),\ \mathrm{GH}(34),\ \mathrm{CF}(35).$$

They connect all nine vertices without forming a cycle.

MST arcs: DE, AD, EH, EJ, CD, BC, GH, CF.

(3)

(b) Add the weights of the eight selected arcs.

$$24 + 26 + 30 + 31 + 33 + 28 + 34 + 35 = 241.$$

241 *miles.*

(1)

(c) The required drawing is the tree from part (a): join exactly the pairs DE, AD, EH, EJ, CD, BC, GH, CF.

MST as specified in part (a).

(1)

(d) Traversing every MST arc twice gives a closed route, so twice the MST weight is an initial upper bound.

$$2(241) = 482.$$

Initial upper bound $= 482$ miles.

(1)

(e)(i) Nearest neighbour always visits the nearest unvisited castle, then returns to its start. From D, the successive choices are:

$$\mathrm{D} \to \mathrm{E} \to \mathrm{H} \to \mathrm{J} \to \mathrm{C} \to \mathrm{B} \to \mathrm{G} \to \mathrm{F} \to \mathrm{A} \to \mathrm{D}.$$

$$24 + 30 + 33 + 45 + 28 + 45 + 47 + 40 + 26 = 318.$$

Upper bound $= 318$ miles.

(2)

(e)(ii) The route found by the algorithm is the route producing that total.

$$\mathrm{D} - \mathrm{E} - \mathrm{H} - \mathrm{J} - \mathrm{C} - \mathrm{B} - \mathrm{G} - \mathrm{F} - \mathrm{A} - \mathrm{D}.$$

(1)

(f) The tightest upper bound is the smallest valid route length.

$$318 < 352 < 482.$$

Best upper bound $= 318$ miles, because it is the smallest of the three upper bounds.

(1)

(g) For the deleted-vertex lower-bound method, remove A, find the MST weight left, then add the two shortest arcs incident with A, namely AD and AF.

$$(241 - 26) + 26 + 40 = 281.$$

Lower bound $= 281$ miles.

(2)

(h) A lower bound is most useful when it is as large as possible.

$$281 > 274.$$

Best lower bound $= 281$ miles, because it is the larger lower bound.

(1)

Question 3

Question 3

Figure 1 is the weighted cycle-track network with total weight 413 km. Its arcs are AB25, AC18, AD47, BC5, BD17, CD24, DE20, DF34, DG26, EF12, EJ41, EK37, FG4, FH15, FJ27, FK25, GH21, HJ9, H
(a) Use Dijkstra from A to J. (b) Starting at G, find whether an open route finishing at J or K is shorter, state repeated tracks and total length. (c) With BC and HK closed, compare Tarig's open route from G to H with Abi's. **(13)**

Model solution

(a) Dijkstra's algorithm is appropriate because all arc lengths are non-negative. Permanently label the smallest tentative distance, then update all of its neighbours.

permanent vertex	*distance from* A	*useful updates*
A	0	B : 25, C : 18, D : 47
C	18	B : 23, D : 42
B	23	D : 40
D	40	E : 60, F : 74, G : 66
E	60	F : 72, J : 101, K : 97
G	66	F : 70, H : 87
F	70	H : 85, J : 97, K : 95
H	85	J : 94, K : 91
K	91	J : 94
J	94	

Tracing the labels backwards gives $J \leftarrow H \leftarrow F \leftarrow G \leftarrow D \leftarrow B \leftarrow C \leftarrow A$.

Shortest path $A - C - B - D - G - F - H - J$, *of length* 94 *km.*

(5)

(b) In an open route, the start and finish are allowed to be odd; every other odd vertex must be paired by repeated shortest paths. The odd vertices are A, B, C, G, J, K, *and* G *is already the start.*
If the finish is J, *pair* A, B, C, K:

pairing	*extra distance (km)*
$AB + CK$	$23 + 73 = 96$
$AC + BK$	$18 + 68 = 86$
$AK + BC$	$91 + 5 = 96$

If the finish is K, *pair* A, B, C, J:

$AB + CJ$	$23 + 76 = 99$
$AC + BJ$	$18 + 71 = 89$
$AJ + BC$	$94 + 5 = 99$

The least extra distance is 86, *so finish at* J. *The shortest paths for* AC *and* BK *repeat the tracks*

$$AC,\ BD,\ DG,\ GF,\ FH,\ HK.$$

$$\textit{route length} = 413 + 86 = 499\ \textit{km}.$$

Finish at J*; repeat* AC, BD, DG, GF, FH, HK*; minimum length* 499 *km.*

(6)

(c) Removing BC *and* HK *reduces the network weight by* $5 + 6$*. With* BC *unavailable, the shortest* A*-to-*J *connection becomes* $94 + 2 = 96$ *km; this is the extra repeated distance required for Tarig's open-route pairing.*

$$\textit{Tarig's length} = (413 - 5 - 6) + (94 + 2) = 498 \textit{ km}.$$

Tarig's route is shorter: 498 *km compared with Abi's* 499 *km.*

(2)

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Question 4

Question 4

(a) Explain why no graph has six vertices of degrees $1, 2, 3, 4, 5, 6$. A tree T has degrees $1, 2, 4 - x, 2x - 5, 4x - 11, 3x - 5$, where x is an integer. (b) Explain why T is not Eulerian. (c) Find x, then decide whether T is semi-Eulerian. (d) Give a non-isomorphic graph with degree sequence $1, 2, 3, 3, 3, 4$. **(8)**

Model solution

(a) By the handshaking lemma, the number of odd-degree vertices in a graph must be even. The proposed degrees contain three odd values, $1, 3, 5$, which is impossible.

No such graph exists, since it would have an odd number of odd-degree vertices.

(1)

(b) An Eulerian graph must have every vertex of even degree. Every tree has leaves, hence at least one vertex of degree 1.

T cannot be Eulerian because it has a degree-1 vertex.

(1)

(c)(i) A tree with six vertices has $6 - 1 = 5$ arcs, so the sum of its degrees is $2 \times 5 = 10$.

$$1 + 2 + (4 - x) + (2x - 5) + (4x - 11) + (3x - 5) = 10,$$

$$8x - 14 = 10 \quad \Rightarrow \quad 8x = 24 \quad \Rightarrow \quad x = 3.$$

$$x = 3.$$

(3)

(c)(ii) Substitute this value to count the odd degrees, which is the test for a semi-Eulerian graph.

$$1,\ 2,\ 4 - 3,\ 2(3) - 5,\ 4(3) - 11,\ 3(3) - 5 = 1, 2, 1, 1, 1, 4.$$

There are four odd-degree vertices. A connected graph is semi-Eulerian only when exactly two vertices have odd degree.

T is not semi-Eulerian.

(2)

(d) One valid construction is specified by its edge set; this is equivalent to drawing it. Let the vertices be A, B, C, D, E, F.

$$\{AB, AC, AD, AE, BC, CD, DF, FB\}.$$

Its degrees are $4, 3, 3, 3, 2, 1$ for A, B, C, D, F, E, respectively. It is not isomorphic to the given graph: here the degree-1 vertex E is adjacent to the degree-4 vertex A, whereas in Figure 2 the degree-1 vertex is adjacent to a degree-3 vertex.

For example, use edges AB, AC, AD, AE, BC, CD, DF, FB.

(1)

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Question 5

Question 5

Positive x, y satisfy $x - y \geqslant 100$, $x \leqslant 5y$, and $2x + 3y \geqslant 350$, while $x + y$ is minimised. (a) Set up the initial big-M tableau using one slack, two surplus and two artificial variables. (b) A final row has basic variable x, with $x = 130$: state x and deduce y. **(9)**

Model solution

(a) The two $\geqslant$ constraints require a surplus variable and an artificial variable to start the simplex method. The $\leqslant$ constraint requires one slack variable. Since the aim is minimisation, maximise $P = -(x + y)$, penalising artificial variables by $-M$.

$$\begin{aligned} x - y - s_1 + a_1 &= 100, \\ x - 5y + s_2 &= 0, \\ 2x + 3y - s_3 + a_2 &= 350, \\ P &= -x - y - M(a_1 + a_2). \end{aligned}$$

Substituting a_1 and a_2 from the first and third equations makes the objective row free of artificial-variable columns:

$$P + (1 - 3M)x + (1 - 2M)y + Ms_1 + Ms_3 = -450M.$$

b.v.	x	y	s_1	s_2	s_3	a_1	a_2	*Value*
a_1	1	-1	-1	0	0	1	0	100
s_2	1	-5	0	1	0	0	0	0
a_2	2	3	0	0	-1	0	1	350
P	$1-3M$	$1-2M$	M	0	M	0	0	$-450M$

Initial tableau as shown.

(6)

(b)(i) In the final tableau a basic variable equals the value in its row when all non-basic variables are zero.

$$x = 130.$$

(1)

(b)(ii) Substitute $x = 130$ into the original clues, then combine the resulting bounds on y.

$$130 - y \geqslant 100 \Rightarrow y \leqslant 30, \qquad 130 \leqslant 5y \Rightarrow y \geqslant 26,$$

$$260 + 3y \geqslant 350 \Rightarrow y \geqslant 30, \qquad y < 130.$$

The simultaneous bounds force $y = 30$.

$$y = 30.$$

(2)

Question 6

Question 6

The precedences are A, B, C start; D after A; E, F after A, B, C; G after C; H, I, J after D, E; K after F, G, J; and L after F, G. The three-worker schedule is

worker	activity interval (hours)			
1	C : 0–7	E : 7–12	J : 12–15	K : 15–20
2	A : 0–4	D : 4–8	G : 8–12	H : 12–14, I : 14–18
3	B : 0–5	F : 7–9	L : 12–15	

(a) Draw the activity-on-arc network with the minimum number of dummies. (b) From the schedule state the critical path, completion time and floats on B, G. (c) If I lasts 8 hours, decide whether three workers can still finish on time. **(11)**

Model solution

(a) In an activity-on-arc network, each activity is an arrow. Dummies are needed only where separate predecessor sets must be merged without accidentally imposing an extra precedence. The following event--arc list is an unambiguous network with the minimum four dummies.

arc	*activity*	*arc*	*activity*
$0 \to 1$	A	$0 \to 2$	B
$0 \to 3$	C	$1 \to 2$	*dummy*
$3 \to 2$	*dummy*	$1 \to 4$	D
$2 \to 4$	E	$2 \to 5$	F
$3 \to 5$	G	$4 \to 6$	H
$4 \to 9$	I	$4 \to 7$	J
$5 \to 7$	*dummy*	$7 \to 9$	K
$5 \to 9$	L	$6 \to 9$	*dummy*

An activity-on-arc network represented by the event--arc list above.

(5)

(b)(i) The critical path is the chain with zero float, hence it determines the project duration.

$\mathrm{C} - \mathrm{E} - \mathrm{J} - \mathrm{K}$.

(1)

(b)(ii) Reading the end of that critical path from the schedule gives the minimum completion time.

20 *hours.*

(1)

(b)(iii) Total float is how far an activity can be delayed without delaying the completion time.

Total float of $\mathrm{B} = 2$ *hours.*

(1)

(b)(iv) Apply the same schedule reading to activity G.

Total float of $G = 4$ *hours.*

(1)

(c) Keep one worker on the critical activities so that the 20-hour completion time is protected. A feasible reassignment is: worker 2 performs A, D, G, H *as scheduled and then* L *from time* 14 *to* 20*; worker 3 performs* B, F *as scheduled and then* I *from time* 12 *to* 20*. Activity* I *therefore receives its required 8 hours, while* L *still finishes by time 20 (it only needs to start by time 17).*

Yes. The project can still be completed by three workers in 20 *hours.*

(2)

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Question 7

Question 7

After iteration 1, the simplex tableau has rows

b.v.	x	y	z	s_1	s_2	s_3	Value
s_1	0	$-\frac{1}{2}$	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	30
x	1	$\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	10
s_3	0	1	1	0	0	1	26
P	0	$-\frac{1}{4}$	$-\frac{11}{4}$	0	$\frac{3}{4}$	0	30

(a) identify the first pivot column; (b) recover the original LP; (c) perform iteration 2; (d) assess optimality and state basic variables; (e) explain why a proposed third tableau with a 1 in the profit row's y-column is wrong. **(14)**

Model solution

(a) A pivot column becomes a unit column for the entering basic variable. The x-column is now the unit column associated with basic variable x.

The first pivot was in the x-column, because x is now basic.

(2)

(b) To recover the original constraints, write the row equations and eliminate the slack variables using the x-row. From the x-row,

$$x + \tfrac{1}{4}y - \tfrac{1}{4}z + \tfrac{1}{4}s_2 = 10 \quad \Rightarrow \quad 4x + y - z + s_2 = 40.$$

Thus $4x + y - z \leqslant 40$. Eliminating s_2 from the s_1-row gives

$$2x + z + s_1 = 50 \quad \Rightarrow \quad 2x + z \leqslant 50.$$

The s_3-row gives $y + z + s_3 = 26$, hence $y + z \leqslant 26$. Finally, substituting the x-row in the profit equation yields $P - 3x - y - 2z = 0$.

$$\begin{aligned} \textit{Maximise} \quad & P = 3x + y + 2z, \\ \textit{subject to} \quad & 2x + z \leqslant 50, \\ & 4x + y - z \leqslant 40, \\ & y + z \leqslant 26, \\ & x, y, z \geqslant 0. \end{aligned}$$

The linear programme is as displayed.

(5)

(c) The most negative profit-row entry is $-11/4$, so z enters. The ratio test uses positive entries in the z-column: $30/(3/2) = 20$ and $26/1 = 26$, so s_1 leaves. Make the pivot $3/2$ into 1, then clear the rest of the z-column.

$$R_1 \leftarrow \tfrac{2}{3}R_1, \qquad R_2 \leftarrow R_2 + \tfrac{1}{4}R_1, \qquad R_3 \leftarrow R_3 - R_1, \qquad R_P \leftarrow R_P + \tfrac{11}{4}R_1.$$

b.v.	x	y	z	s_1	s_2	s_3	Value
z	0	$-\frac{1}{3}$	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	20
x	1	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	0	15
s_3	0	$\frac{4}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$	1	6
P	0	$-\frac{7}{6}$	0	$\frac{11}{6}$	$-\frac{1}{6}$	0	85

Tableau T is as shown.

(4)

(d)(i) For this maximisation convention, an optimal tableau has no negative coefficient in the profit row. **The profit row still contains $-7/6$ and $-1/6$, so another improvement is possible.**

The solution is not yet optimal.

(1)

(d)(ii) The basic variables are read from the row labels and their Value entries.

$$z = 20,\ x = 15,\ s_3 = 6.$$

(1)

(e) When y enters as a basic variable, its column must be a unit column: one 1 in the y-row and zeros everywhere else, including the profit row.

The proposed tableau is wrong because its profit-row y-entry is 1, not 0.

(1)