Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/4A - Further Pure Mathematics 2

Tuesday 28 June 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

For the permutation group S_4, answer the following. Let

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

(a) Write down the identity. (b) Find a^{-1}. (c) Demonstrate associativity using a, b, c. (d) Explain why a subgroup of order 4 is possible.

Model solution

(a) The identity permutation leaves every member unchanged.

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

(1)

(b) To invert a permutation, reverse each mapping. Since a sends $1,2,3,4$ to $3,4,2,1$, its inverse sends $3,4,2,1$ back to $1,2,3,4$.

$$a^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}.$$

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(1)

(c) We calculate both bracketings. Composition is read from right to left, so the rightmost permutation acts first.

$$a \circ b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

$$(a \circ b) \circ c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

$$b \circ c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \qquad a \circ (b \circ c) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

$(a \circ b) \circ c = a \circ (b \circ c)$, *so these elements demonstrate associativity.*

(2)

(d) Lagrange's theorem says that a subgroup order must divide the order of its finite group. There are $4!$ permutations in S_4.

$$|S_4| = 4! = 24, \qquad 4 \mid 24.$$

A subgroup of order 4 is possible because 4 is a factor of 24.

(2)

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Question 2

Question 2

$$M = \begin{pmatrix} 1 & 0 & a \\ -3 & b & 1 \\ 0 & 1 & a \end{pmatrix}, \qquad a, b \in \mathbb{Z}, \quad a < b.$$

Its characteristic equation is $\lambda^3 - 7\lambda^2 + 13\lambda + c = 0$. (a) Find a, b, c. (b) Hence find M^{-1} using Cayley–Hamilton.

Model solution

(a) The coefficients of the given characteristic polynomial are the clue to expand $\det(M - \lambda I)$ *and compare coefficients.*

$$\begin{aligned} 0 &= \det \begin{pmatrix} 1-\lambda & 0 & a \\ -3 & b-\lambda & 1 \\ 0 & 1 & a-\lambda \end{pmatrix} \\ &= (1-\lambda)\big((b-\lambda)(a-\lambda) - 1\big) - 3a \\ &= -\lambda^3 + (a+b+1)\lambda^2 - (a+b+ab-1)\lambda + ab - 3a - 1. \end{aligned}$$

Multiplying by -1 *gives the monic polynomial, so comparison gives*

$$a + b + 1 = 7, \qquad a + b + ab - 1 = 13.$$

Thus $a + b = 6$ *and* $ab = 8$. *The two values are* 2 *and* 4*;* $a < b$ *fixes* $a = 2$, $b = 4$. *The constant term is* $3a + 1 - ab = -1$.

$$a = 2, \ b = 4, \ c = -1.$$

(5)

(b) Cayley--Hamilton lets the characteristic equation hold with M *in place of* λ. *Because the constant is non-zero, rearranging then multiplying by* M^{-1} *isolates the inverse.*

$$M^3 - 7M^2 + 13M - I = 0 \quad \Longrightarrow \quad M^{-1} = M^2 - 7M + 13I.$$

$$M^2 = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 & 6 \\ -15 & 17 & 0 \\ -3 & 6 & 5 \end{pmatrix}.$$

$$M^{-1} = \begin{pmatrix} 7 & 2 & -8 \\ 6 & 2 & -7 \\ -3 & -1 & 4 \end{pmatrix}.$$

$$M^{-1} = \begin{pmatrix} 7 & 2 & -8 \\ 6 & 2 & -7 \\ -3 & -1 & 4 \end{pmatrix}.$$

(3)

Question 3

Question 3

A frog starts on lily pad A and, at every hop, moves with equal probability to either of the other two pads. If p_n is the probability it is at A after n hops, then $p_{n+1} = \frac{1}{2}(1 - p_n)$ for $n \geqslant 1$, with $p_1 = 0$. (a) Explain $p_1 = 0$. (b) Prove by induction that $p_n = \frac{2}{3}(-\frac{1}{2})^n + \frac{1}{3}$. (c) Explain the long-term probability.

Model solution

(a) On its first hop the frog must leave A; it cannot remain on the same lily pad.

$$p_1 = 0.$$

(1)

(b) Induction is appropriate because the statement concerns every integer $n \geqslant 1$ and a recurrence carries one case to the next. First check the base case:

$$\frac{2}{3}\left(-\frac{1}{2}\right)^1 + \frac{1}{3} = -\frac{1}{3} + \frac{1}{3} = 0 = p_1.$$

Now assume the formula is true when $n = k$, so $p_k = \frac{2}{3}(-\frac{1}{2})^k + \frac{1}{3}$. Substitute this into the recurrence to generate the $k+1$ case:

$$\begin{aligned} p_{k+1} &= \frac{1}{2}(1 - p_k) \\ &= \frac{1}{2}\left(1 - \left[\frac{2}{3}\left(-\frac{1}{2}\right)^k + \frac{1}{3}\right]\right) \\ &= \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^k = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1}. \end{aligned}$$

This is precisely the required formula with $n = k+1$. Since it is true for $n = 1$ and truth for k implies truth for $k+1$, it is true for every $n \geqslant 1$.

$$p_n = \frac{2}{3}\left(-\frac{1}{2}\right)^n + \frac{1}{3} \quad (n \geqslant 1).$$

(6)

(c) The alternating term becomes negligible because its base has magnitude less than 1.

$$\lim_{n\to\infty}\left(-\frac{1}{2}\right)^n = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} p_n = \frac{1}{3}.$$

In the long term the probability of being on A is $\frac{1}{3}$.

(1)

Question 4

Question 4

(a) Use the Euclidean algorithm to show that 124 and 17 are coprime. (b) Hence solve $124x+17y = 10$. (c) Solve $124x \equiv 6 \pmod{17}$.

Model solution

(a) Repeated division expresses the greatest common divisor as the final non-zero remainder.

$$124 = 7(17) + 5, \qquad 17 = 3(5) + 2, \qquad 5 = 2(2) + 1.$$

$$\gcd(124, 17) = 1, \text{ so } 124 \text{ and } 17 \text{ are coprime.}$$

(2)

(b) "Hence" signals that we should back-substitute the Euclidean algorithm to make 1*, then scale the identity to make* 10*.*

$$\begin{aligned} 1 &= 5 - 2(2) \\ &= 5 - 2(17 - 3 \times 5) = 7 \times 5 - 2 \times 17 \\ &= 7(124 - 7 \times 17) - 2 \times 17 \\ &= 7 \times 124 - 51 \times 17. \end{aligned}$$

Multiplying by 10 *gives* $10 = 70(124) - 510(17)$.

$$\text{One solution is } x = 70, \ y = -510.$$

(3)

(c) Reduce the coefficient modulo 17*, then use its inverse. From part (b),* $7(124) \equiv 1 \pmod{17}$*, so* 7 *is an inverse of* 124 *modulo* 17*.*

$$124x \equiv 6 \pmod{17} \implies x \equiv 7 \times 6 \equiv 42 \equiv 8 \pmod{17}.$$

$$x \equiv 8 \pmod{17}.$$

(2)

Question 5

Question 5

The locus $|z + ai| = 3|z - a|$, where a is an integer, is a circle whose centre is in the third quadrant and whose radius is $\frac{3}{2}\sqrt{2}$. Determine (a) a and (b) the centre.

Model solution

(a) Write $z = x + iy$ and square the modulus equation. Squaring is valid because both moduli are non-negative, and it produces the Cartesian equation of the circle.

$$x^2 + (y + a)^2 = 9\big((x - a)^2 + y^2\big).$$

Expanding and collecting terms gives

$$8x^2 - 18ax + 8y^2 - 2ay + 8a^2 = 0,$$

or, after division by 8,

$$x^2 - \frac{9}{4}ax + y^2 - \frac{1}{4}ay + a^2 = 0.$$

Complete both squares:

$$\left(x - \frac{9}{8}a\right)^2 + \left(y - \frac{1}{8}a\right)^2 = \left(\frac{9}{8}a\right)^2 + \left(\frac{1}{8}a\right)^2 - a^2 = \frac{9}{32}a^2.$$

Hence $r = \frac{3\sqrt{2}}{8}|a|$. Given $r = \frac{3}{2}\sqrt{2}$, we have $|a| = 4$. The centre is $\left(\frac{9}{8}a, \frac{1}{8}a\right)$; to be in the third quadrant both coordinates must be negative, so $a = -4$.

$$a = -4.$$

(4)

(b) Substitute the determined value into the centre already obtained from completed squares.

$$\left(\frac{9}{8}a, \frac{1}{8}a\right) = \left(-\frac{9}{2}, -\frac{1}{2}\right).$$

The centre is $\left(-\frac{9}{2}, -\frac{1}{2}\right)$.

(2)

Question 6

Question 6

(a) Find the general solution of $u_n = 2u_{n-1} - u_{n-2} + 2^n$ for $n \geqslant 2$. (b) Hence solve it given $u_0 = 2u_1$ and $u_4 = 3u_2$.

Model solution

(a) Start with the homogeneous recurrence. Its repeated root tells us the complementary solution must contain both a constant and a term in n.

$$\lambda^2 - 2\lambda + 1 = 0 \quad \Longrightarrow \quad (\lambda - 1)^2 = 0, \qquad u_n^{(c)} = A + Bn.$$

The forcing term is a multiple of 2^n, and 2 is not a root of the auxiliary equation, so try $u_n^{(p)} = \mu 2^n$.

$$\mu 2^n = 2\mu 2^{n-1} - \mu 2^{n-2} + 2^n.$$

Dividing by 2^{n-2} gives $4\mu = 4\mu - \mu + 4$, hence $\mu = 4$. Therefore

$$u_n = A + Bn + 4(2^n).$$

$$\boxed{u_n = A + Bn + 4\,2^n.}$$

(4)

(b) Use the two given relations to determine the two arbitrary constants.

$$u_0 = A + 4, \qquad u_1 = A + B + 8.$$

Thus $u_0 = 2u_1$ gives

$$A + 4 = 2(A + B + 8) \quad \Longrightarrow \quad A + 2B = -12.$$

Also

$$u_2 = A + 2B + 16, \qquad u_4 = A + 4B + 64,$$

and $u_4 = 3u_2$ gives $2A + 2B = 16$. Solving,

$$A = 28, \qquad B = -20.$$

$$\boxed{u_n = 28 - 20n + 4(2^n).}$$

(2)

Question 7

Question 7

(i) $F(x) = px^4 + qx^3 + 2x^2 + rx + s$, where p, q, r, s are distinct. Find the number of possible quartics when they are chosen from (a) $\{-4, -2, 1, 3, 5\}$ and (b) $\{-4, -2, 0, 1, 3, 5\}$. (ii) A three-digit integer $N = abc$ is divisible by 11, has even digit sum, and satisfies $N \equiv 8 \pmod 9$. (a) Show $a - b + c = 0$. (b) Find all N.

Model solution

(i)(a) There are five available constants and four distinct coefficient positions, so use permutations rather than combinations: the positions matter.

$$^5P_4 = 5 \times 4 \times 3 \times 2 = 120.$$

$$120.$$

(1)

(i)(b) The leading coefficient p cannot be 0, otherwise F would not be quartic. There are five choices for p, then five, four and three choices for the remaining distinct coefficients.

$$5 \times 5 \times 4 \times 3 = 300.$$

$$300.$$

(1)

(ii)(a) The divisibility test for 11 gives $a - b + c \equiv 0 \pmod{11}$. Since $1 \leqslant a \leqslant 9$, $0 \leqslant b \leqslant 9$ and $0 \leqslant c \leqslant 9$,

$$-8 \leqslant a - b + c \leqslant 18.$$

As $a - b + c$ is a multiple of 11, it must be 0 or 11. The digit sum is even, and

$$a - b + c = (a + b + c) - 2b,$$

so $a - b + c$ is even. It cannot equal the odd number 11.

$$a - b + c = 0.$$

(2)

(ii)(b) From part (a), $b = a + c$. The condition modulo 9 can be tested with the digit sum, since $N \equiv a + b + c \pmod 9$.

$$a + b + c \equiv 8 \pmod 9, \qquad 2b \equiv 8 \pmod 9.$$

As b is a digit, this gives $b = 4$, and hence $a + c = 4$. Since $a \geqslant 1$, the possibilities are

$$(a, c) = (1, 3), (2, 2), (3, 1), (4, 0).$$

$$N = 143,\ 242,\ 341,\ 440.$$

(4)

Question 8

Question 8

The locus of $z = x + iy$ satisfying

$$\arg\left(\frac{z-(8+5i)}{z-(2+5i)}\right) = \frac{\pi}{3}$$

is an arc of a circle C. (a) Sketch it. (b) Explain why the centre has x-coordinate 5. (c) Find its radius. (d) Find its y-coordinate.

Model solution

(a) The argument is the angle between the lines from z to $8+5i$ and $2+5i$. Thus the locus is the major arc above the endpoints $(2,5)$ and $(8,5)$, excluding the endpoints, from which the chord is seen at angle $\pi/3$.

Major arc through $(2,5)$ and $(8,5)$, above the chord.

(2)

(b) A circle centre is equidistant from the two endpoints of its chord, so it lies on the perpendicular bisector of the horizontal chord. That bisector is $x = \frac{2+8}{2} = 5$.

The centre has x-coordinate 5.

(1)

(c) The chord has length 6. The angle at the circumference is $\pi/3$, so the corresponding central angle is $2\pi/3$. Splitting the isosceles triangle in half gives a right triangle with opposite side 3.

$$\sin\left(\frac{\pi}{3}\right) = \frac{3}{r} \quad \Longrightarrow \quad r = \frac{6}{\sqrt{3}} = 2\sqrt{3}.$$

$$r = 2\sqrt{3}.$$

(2)

(d) The perpendicular distance from the centre to the chord is the adjacent side of the same right triangle. The centre is above the chord for the required major arc.

$$h = r\cos\left(\frac{\pi}{3}\right) = 2\sqrt{3}\left(\frac{1}{2}\right) = \sqrt{3}, \qquad y_{\text{centre}} = 5 + \sqrt{3}.$$

The y-coordinate is $5+\sqrt{3}$.

(2)

Question 9

Question 9

Let $I_n = \int_0^{\pi/2} \sin^n(2x)\,dx$. (a) Prove that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geqslant 2$. (b) Hence find exactly $\int_0^{\pi/2} 64\sin^5 x \cos^5 x\,dx$.

Model solution

(a) The exponent n suggests integration by parts to reduce it by two. Write $\sin^n(2x) = \sin^{n-1}(2x)\sin(2x)$, choosing the final sine as the differentiated factor.

$$\begin{aligned} I_n &= \int_0^{\pi/2} \sin^{n-1}(2x)\sin(2x)\,dx \\ &= \left[-\frac{1}{2}\sin^{n-1}(2x)\cos(2x)\right]_0^{\pi/2} + \frac{n-1}{2}\int_0^{\pi/2} \sin^{n-2}(2x)\cos^2(2x)\,dx. \end{aligned}$$

The boundary term is zero: at each limit either $\sin(2x) = 0$, or (when $n = 1$) the displayed reduction is not needed. Use $\cos^2(2x) = 1 - \sin^2(2x)$:

$$I_n = \frac{n-1}{2}(I_{n-2} - I_n).$$

Rearranging gives

$$2I_n = (n-1)I_{n-2} - (n-1)I_n \quad \Longrightarrow \quad nI_n = (n-1)I_{n-2}.$$

$$\boxed{I_n = \frac{n-1}{n} I_{n-2}.}$$

(4)

(b) The factor 64 and the equal powers invite the identity $\sin(2x) = 2\sin x \cos x$.

$$64\sin^5 x\cos^5 x = 64\left(\frac{\sin(2x)}{2}\right)^5 = 2\sin^5(2x).$$

Therefore the required integral is $2I_5$. Apply the reduction twice, then evaluate I_1 directly:

$$I_5 = \frac{4}{5}I_3 = \frac{4}{5}\cdot\frac{2}{3}I_1, \qquad I_1 = \int_0^{\pi/2}\sin(2x)\,dx = \left[-\frac{1}{2}\cos(2x)\right]_0^{\pi/2} = 1.$$

$$2I_5 = 2\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{16}{15}.$$

$$\boxed{\frac{16}{15}.}$$

(3)

Question 10

Question 10

A plant pot has flat circular base radius 10 cm and height 15 cm. Its curved inner surface is formed by rotating, about the y-axis, the curve

$$x = 10 + 15t - 5t^3, \qquad y = 15t^2, \qquad 0 \leqslant t \leqslant 1.$$

(a) Show that its curved inner area is $150\pi \int_0^1 (2 + 3t + 2t^2 + 2t^3 - t^5)\,dt$. (b) Find total inner area. (c) A 12 m^2 tin paints both inside and outside; estimate complete pots. (d) Give a limitation.

Model solution

(a) For a parametric curve rotated about the y-axis, the radius is x and the surface-area element is $2\pi x\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$. This is appropriate because the curve gives the pot's inside profile.

$$\frac{dx}{dt} = 15 - 15t^2, \qquad \frac{dy}{dt} = 30t.$$

$$\begin{aligned} A &= 2\pi \int_0^1 (10 + 15t - 5t^3)\sqrt{(15 - 15t^2)^2 + (30t)^2}\,dt \\ &= 2\pi \int_0^1 (10 + 15t - 5t^3)\sqrt{225 + 450t^2 + 225t^4}\,dt \\ &= 2\pi \int_0^1 (10 + 15t - 5t^3)(15 + 15t^2)\,dt. \end{aligned}$$

Here $15 + 15t^2 > 0$, so it is the correct square root. Expanding and factoring 150 gives

$$A = 150\pi \int_0^1 (2 + 3t + 2t^2 + 2t^3 - t^5)\,dt.$$

$$\boxed{A = 150\pi \int_0^1 (2 + 3t + 2t^2 + 2t^3 - t^5)\,dt.}$$

(5)

(b) Integrate the polynomial for the curved surface, then add the flat base area $\pi(10)^2$.

$$\begin{aligned} A_{\text{curved}} &= 150\pi \left[2t + \frac{3}{2}t^2 + \frac{2}{3}t^3 + \frac{1}{2}t^4 - \frac{1}{6}t^6\right]_0^1 \\ &= 150\pi \left(2 + \frac{3}{2} + \frac{2}{3} + \frac{1}{2} - \frac{1}{6}\right) = 675\pi \ \textit{cm}^2. \end{aligned}$$

$$A_{\text{inner}} = 675\pi + 100\pi = 775\pi \ \textit{cm}^2.$$

$$\boxed{775\pi \ \textit{cm}^2 \ (\approx 2430 \ \textit{cm}^2).}$$

(4)

(c) Both sides are painted, so use twice the inner-area model. Convert $12\ m^2$ *to* $120\,000\ cm^2$ *before dividing.*

$$\frac{120\,000}{2(775\pi)} \approx 24.65.$$

Only complete pots can be painted.

24 *complete plant pots.*

(2)

(d) The calculation treats the outside as having exactly the same area as the inside, whereas a real pot has material thickness (and may also have a rim).

For example, the outside surface area is larger than the inside surface area because the pot has thickness.

(1)

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