Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/4A - Further Pure Mathematics 2

Monday 26 June 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Let $A = \begin{pmatrix} -1 & a \\ 3 & 8 \end{pmatrix}$. (a) Find its characteristic equation. (b) Use Cayley–Hamilton to find a, b such that $A^3 = A + bI$.

Model solution

(a) The characteristic equation comes from $\det(\lambda I - A) = 0$.

$$\det \begin{pmatrix} \lambda + 1 & -a \\ -3 & \lambda - 8 \end{pmatrix} = (\lambda + 1)(\lambda - 8) - 3a = 0.$$

$$\lambda^2 - 7\lambda - 8 - 3a = 0.$$

$$\lambda^2 - 7\lambda - 8 - 3a = 0.$$

(2)

(b) Cayley--Hamilton lets the characteristic equation be used with A *in place of* λ*. Multiplying by* A *will produce* A^3*, as required.*

$$A^2 - 7A - (8 + 3a)I = 0, \quad \text{so} \quad A^2 = 7A + (8 + 3a)I,$$

$$A^3 = 7A^2 + (8 + 3a)A = 7\{7A + (8 + 3a)I\} + (8 + 3a)A = (57 + 3a)A + (56 + 21a)I.$$

Matching this with $A^3 = A + bI$ *gives*

$$57 + 3a = 1 \implies a = -\frac{56}{3}, \quad \text{so} \quad b = 56 + 21\left(-\frac{56}{3}\right) = -336.$$

$$a = -\frac{56}{3} \text{ and } b = -336.$$

(4)

Question 2

Question 2

The complex number z satisfies $|z-6| = 2|z+3i|$. (a) Show that the locus passes through 0, -4 and $-8i$. (b) Sketch the locus. (c) Shade the region $|z-6| \geqslant 2|z+3i|$, $|z| \leqslant 4$.

Model solution

(a) To verify a point is on a modulus locus, substitute it and compare the two distances.

$$z = 0: \quad |0-6| = 6 = 2|3i|;$$

$$z = -4: \quad |-10| = 10 = 2|-4+3i| = 2\sqrt{16+9};$$

$$z = -8i: \quad |-6-8i| = 10 = 2|-5i|.$$

All three stated points lie on the locus.

(2)

(b) Writing $z = x + iy$ identifies the circle needed for an accurate sketch.

$$(x-6)^2 + y^2 = 4\{x^2 + (y+3)^2\} \implies x^2 + y^2 + 4x + 8y = 0,$$

$$(x+2)^2 + (y+4)^2 = 20.$$

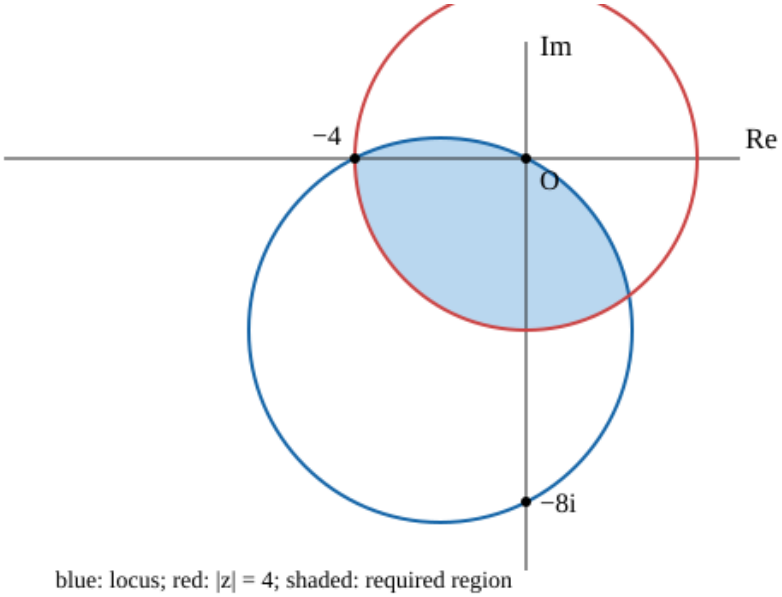
Sketch the circle with centre $(-2,-4)$ and radius $\sqrt{20}$, marking O, $(-4,0)$ and $(0,-8)$.

(2)

(c) Squaring is valid because both sides are non-negative. The inequality selects the inside of the first circle; $|z| \leqslant 4$ selects the inside of a circle centred at the origin.

$$|z-6| \geqslant 2|z+3i| \iff (x+2)^2 + (y+4)^2 \leqslant 20, \qquad |z| \leqslant 4 \iff x^2 + y^2 \leqslant 16.$$

Shade the intersection of these two closed discs.



blue: locus; red: |z| = 4; shaded: required region

(2)

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Question 3

Question 3

Subscribers satisfy the stated weekly cancellation and gain model. (a) Explain the recurrence. (b) Prove the supplied formula for U_n. (c) Evaluate the model after six months.

Model solution

(a) At the end of week n, 20% cancel, so 80% of U_n remains. The new subscribers gained during week $n+1$ are $20(n+1)$.

$$U_1 = 25, \qquad U_{n+1} = 0.8U_n + 20(n+1) \quad (n = 1, 2, \ldots).$$

The recurrence follows from retaining $0.8U_n$ subscribers and adding $20(n+1)$.

(2)

(b) Use induction because the proposed expression is indexed by every positive integer. The base case confirms the start; the inductive step checks that the recurrence carries the formula forward. For $n = 1$,

$$325\left(\frac{4}{5}\right)^0 + 100(1) - 400 = 325 + 100 - 400 = 25,$$

so the formula is true initially. Assume, for $n = k$, that

$$U_k = 325\left(\frac{4}{5}\right)^{k-1} + 100k - 400.$$

Then the recurrence gives

$$\begin{aligned} U_{k+1} &= \frac{4}{5}\left[325\left(\frac{4}{5}\right)^{k-1} + 100k - 400\right] + 20(k+1) \\ &= 325\left(\frac{4}{5}\right)^{k} + 80k - 320 + 20k + 20 \\ &= 325\left(\frac{4}{5}\right)^{k} + 100(k+1) - 400. \end{aligned}$$

Thus it is true for $k+1$ whenever it is true for k; with the base case, it holds for all $n \geqslant 1$.

$$U_n = 325\left(\frac{4}{5}\right)^{n-1} + 100n - 400.$$

(5)

(c) Six months is about 26 weeks, so compare the model at $n = 26$ with the observed value.

$$U_{26} = 325\left(\frac{4}{5}\right)^{25} + 2600 - 400 = 2201.2\ldots$$

The model predicts about 2201 subscribers, about 401 too many; it overestimates and is not a very good model after six months.

(2)

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Question 4

Question 4

(a) Use the Euclidean algorithm for $168, 66$. (b) Find a, b with $168a + 66b = 6$. (c) Explain why $168x + 66y = 10$ has no integer solutions. (d) Solve $11v \equiv 8 \pmod{28}$.

Model solution

(a) The Euclidean algorithm repeatedly replaces the larger number by a remainder; the last non-zero remainder is the HCF.

$$168 = 2(66) + 36, \quad 66 = 1(36) + 30, \quad 36 = 1(30) + 6, \quad 30 = 5(6) + 0.$$

$$\text{hcf}(168, 66) = 6.$$

(2)

(b) Work backwards from the final non-zero remainder to express it using 168 and 66.

$$6 = 36 - 30 = 36 - (66 - 36) = 2(168 - 2 \cdot 66) - 66 = 2(168) - 5(66).$$

$$a = 2, \ b = -5.$$

(3)

(c) Every value of $168x + 66y$ is a multiple of their HCF.

It is always divisible by 6, whereas 10 is not; hence no integer solutions exist.

(1)

(d) Part (b), divided by 6, supplies a multiplicative inverse of 11 modulo 28.

$$1 = 2(28) - 5(11) \implies 11^{-1} \equiv -5 \pmod{28}.$$

$$v \equiv -5 \cdot 8 = -40 \equiv 16 \pmod{28}.$$

$$v \equiv 16 \pmod{28}.$$

(3)

Question 5

Question 5

(i) Count the security codes. (ii) Determine the combination-lock code from the four stated conditions.

Model solution

(i)(a) Repetition of digits is allowed, while the three letters must be distinct and are ordered.

$$5^4 \times {}^4P_3 = 625 \times 24 = 15000.$$

15000 *codes.*

(2)

(i)(b) With no adjacent letters, the only pattern is _L_L_L_*: choose the three letter positions from the five gaps around four digits.*

$$\binom{5}{3} \times 15000 = 150000.$$

Increase $= 150000 - 15000 = 135000$ *codes.*

(2)

(ii)(a) The possible odd digits are $1, 3, 5, 7, 9$*. A number divisible by 9 has digit sum divisible by 9.*

$$1 + 3 + 5 + 7 + 9 = 25, \quad 25 - 7 = 18,$$

and 18 *is divisible by* 9*. No other single omission gives a multiple of* 9*.*

The digits are $1, 3, 5, 9$ *(so* $e = 7$*).*

(2)

(ii)(b) The order condition leaves only 1359 *and* 9531*. Test the congruence using their first two digits.*

$$1359 \equiv 7 \pmod{13} \quad (13 \cdot 104 = 1352), \qquad 9531 \equiv 31 \not\equiv 7 \pmod{95}.$$

The code is 1359*.*

(2)

Question 6

Question 6

Find a closed form for $u_0 = 1$, $u_1 = 4$, $u_{n+2} = 2u_{n+1} - \frac{4}{3}u_n + n$.

Model solution

The non-homogeneous term is linear in n, so use a linear particular solution after solving the homogeneous recurrence.

$$3\lambda^2 - 6\lambda + 4 = 0 \implies \lambda = 1 \pm \frac{i}{\sqrt{3}} = \frac{2}{\sqrt{3}}e^{\pm i\pi/6}.$$

Thus

$$u_n^{(c)} = \left(\frac{2}{\sqrt{3}}\right)^n \left(P\cos\frac{n\pi}{6} + Q\sin\frac{n\pi}{6}\right).$$

Put $u_n^{(p)} = kn + l$. Substitution gives

$$k(n+2) + l = 2[k(n+1) + l] - \frac{4}{3}(kn + l) + n,$$

so $k = 3$ and $l = 0$. Hence

$$u_n = \left(\frac{2}{\sqrt{3}}\right)^n \left(P\cos\frac{n\pi}{6} + Q\sin\frac{n\pi}{6}\right) + 3n.$$

Using $u_0 = 1$ gives $P = 1$; using $u_1 = 4$ gives $Q = 0$.

$$u_n = \left(\frac{2}{\sqrt{3}}\right)^n \cos\left(\frac{n\pi}{6}\right) + 3n.$$

(7)

Question 7

Question 7

$G = \mathbb{R}\setminus\{-\frac{3}{2}\}$ has $x \bullet y = 3(x+y+1)+2xy$. Find the identity and inverse, and explain the exclusion.

Model solution

(a) An identity e must make $x \bullet e = x$ for every x.

$$3(x+e+1)+2xe = x \implies (2e+2)x+3e+3 = 0.$$

Therefore $e = -1$.

The identity is -1.

(2)

(b) Set $x \bullet y = -1$, since y is the inverse of x.

$$3(x+y+1)+2xy = -1 \implies y(2x+3) = -3x-4 \implies y = -\frac{3x+4}{2x+3}.$$

$$x^{-1} = -\frac{3x+4}{2x+3}.$$

(3)

(c) The inverse formula, and the identity equation, require division by $2x+3$.

For $x = -\frac{3}{2}$, $2x+3 = 0$, and $-\frac{3}{2} \bullet y = -\frac{3}{2}$ for every y, so this element has no inverse.

(1)

Question 8

Question 8

$I_n = \int_0^2 (x-2)^n e^{4x}\,dx$. (a) Prove the reduction formula. (b) Evaluate I_2.

Model solution

(a) Integration by parts reduces the power of $(x-2)$ by one. Take $u = (x-2)^n$ and $dv = e^{4x}\,dx$.

$$I_n = \left[\frac{1}{4}(x-2)^n e^{4x}\right]_0^2 - \frac{n}{4}\int_0^2 (x-2)^{n-1} e^{4x}\,dx.$$

For $n \geqslant 1$, the upper boundary term is zero and the lower term is $-\frac{1}{4}(-2)^n = -(-2)^{n-2}$. Therefore

$$I_n = -(-2)^{n-2} - \frac{n}{4} I_{n-1}.$$

$$a = -2, \text{ so } I_n = -(-2)^{n-2} - \frac{n}{4} I_{n-1}.$$

(4)

(b) Apply the reduction formula twice, then evaluate the remaining exponential integral.

$$I_2 = -1 - \frac{1}{2} I_1 = -1 - \frac{1}{2}\left[-\frac{1}{2} - \frac{1}{4} I_0\right] = -\frac{3}{4} + \frac{1}{8} I_0,$$

$$I_0 = \int_0^2 e^{4x}\,dx = \frac{1}{4}(e^8 - 1).$$

$$I_2 = -\frac{3}{4} + \frac{1}{32}(e^8 - 1) = \frac{e^8 - 25}{32}.$$

$$\int_0^2 (x-2)^2 e^{4x}\,dx = \frac{e^8 - 25}{32}.$$

(3)

Question 9

Question 9

An arc runs from $z_1 = 3 + 2i$ to $z_2 = a + 4i$ and contains $z_3 = 1 + 4i$. Its centre has real part -1. Find a and an equation for the arc.

Model solution

(a) Points z_2 and z_3 have the same imaginary part, so their perpendicular bisector is vertical. The centre has real coordinate -1, hence it is their midpoint in the real direction.

$$\frac{a+1}{2} = -1 \implies a = -3.$$

$$a = -3.$$

(2)

(b) An arc through fixed endpoints has a constant angle subtended by those endpoints. Evaluate it at the known point $z_3 = 1 + 4i$.

$$\arg\left(\frac{z_3 - z_1}{z_3 - z_2}\right) = \arg\left(\frac{-2+2i}{4}\right) = \frac{3\pi}{4}.$$

$$\arg\left(\frac{z-(3+2i)}{z-(-3+4i)}\right) = \frac{3\pi}{4}.$$

(3)

Question 10

Question 10

$y = \sqrt{1 + x^2/9}$, $-4 \leqslant x \leqslant 4$, is rotated about the x-axis. (a) Obtain the stated surface-area form. (b) Find the total area.

Model solution

(a) The curved surface is found from $2\pi \int y\sqrt{1 + (dy/dx)^2}\, dx$. *The solid also has two circular ends, each of radius* $y(4)$.

$$\frac{dy}{dx} = \frac{x}{9\sqrt{1 + x^2/9}}, \qquad 1 + \left(\frac{dy}{dx}\right)^2 = \frac{81 + 10x^2}{9(9 + x^2)}.$$

Thus

$$2\pi \int_{-4}^{4} y\sqrt{1 + (y')^2}\, dx = \frac{2\pi}{9} \int_{-4}^{4} \sqrt{81 + 10x^2}\, dx.$$

Since $y(4)^2 = 1 + 16/9 = 25/9$, *the two ends have total area* $2\pi(25/9) = 50\pi/9$. *Hence*

$$S = \frac{2\pi}{9} \int_{-4}^{4} \sqrt{81 + 10x^2}\, dx + \frac{50\pi}{9}.$$

$$p = \frac{2}{9},\ q = \frac{50}{9}.$$

(6)

(b) The suggested hyperbolic substitution makes $81 + 10x^2$ *a perfect square. Put* $x = \frac{9}{\sqrt{10}} \sinh u$, *so* $dx = \frac{9}{\sqrt{10}} \cosh u\, du$ *and* $\sqrt{81 + 10x^2} = 9 \cosh u$. *With* $u_0 = \operatorname{arsinh}(4\sqrt{10}/9)$,

$$\int_{-4}^{4} \sqrt{81 + 10x^2}\, dx = \frac{81}{\sqrt{10}} \int_{-u_0}^{u_0} \cosh^2 u\, du = \frac{81}{\sqrt{10}} \left[\frac{u}{2} + \frac{1}{4} \sinh 2u\right]_{-u_0}^{u_0}.$$

Therefore

$$S = \frac{2\pi}{9} \cdot \frac{81}{\sqrt{10}} \left(u_0 + \frac{1}{2} \sinh 2u_0\right) + \frac{50\pi}{9} = 81.211299\ldots \text{ cm}^2.$$

$$S = 81 \text{ cm}^2 \text{ to the nearest cm}^2.$$

(6)