Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/4A - Further Pure Mathematics 2

Monday 24 June 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Use Fermat's Little Theorem to determine the least positive residue of $21^{80} \pmod{23}$.

Model solution

Since 23 is prime and $23 \nmid 21$, Fermat's Little Theorem is the intended way to reduce the large exponent: $21^{22} \equiv 1 \pmod{23}$. Write the exponent as a multiple of 22, plus a small remainder.

$$80 = 3 \times 22 + 14, \qquad 21^{80} \equiv 21^{14} \pmod{23}.$$

Using $21 \equiv -2 \pmod{23}$ makes the remaining power particularly simple.

$$21^{14} \equiv (-2)^{14} = 2^{14} = (2^7)^2.$$

$$2^7 = 128 \equiv 13 \pmod{23}, \qquad 2^{14} \equiv 13^2 = 169 \equiv 8 \pmod{23}.$$

The least positive residue is 8.

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Question 2

Question 2

Determine a closed form for $u_1 = 4,\ u_2 = 6$, where $u_{n+2} = 6u_{n+1} - 9u_n$ for $n = 1, 2, 3, \ldots$.

Model solution

This is a linear recurrence with constant coefficients, so try a solution $u_n = r^n$. The recurrence then becomes an auxiliary equation.

$$r^{n+2} = 6r^{n+1} - 9r^n \quad \Longrightarrow \quad r^2 - 6r + 9 = 0 \quad \Longrightarrow \quad (r-3)^2 = 0.$$

The repeated root $r = 3$ means the two independent forms are 3^n and $n3^n$. Hence

$$u_n = (A + Bn)3^n.$$

Use the two given initial terms to find A and B.

$$4 = 3(A + B), \qquad 6 = 9(A + 2B).$$

$$A + B = \frac{4}{3}, \qquad A + 2B = \frac{2}{3} \quad \Longrightarrow \quad B = -\frac{2}{3}, \quad A = 2.$$

$$u_n = \left(2 - \frac{2n}{3}\right)3^n = (6 - 2n)3^{n-1}.$$

$$\boxed{u_n = 2(3 - n)3^{n-1}.}$$

(5)

Question 3

Question 3

(a) Use the Euclidean Algorithm to determine $\text{hcf}(234, 96) = h$.
(b) Hence find integers a, b such that $234a + 96b = h$.
(c) Solve $96x \equiv 36 \pmod{234}$.

Model solution

(a) The Euclidean Algorithm repeatedly replaces a pair by a smaller remainder; the final non-zero remainder is the highest common factor.

$$234 = 2(96) + 42, \quad 96 = 2(42) + 12, \quad 42 = 3(12) + 6, \quad 12 = 2(6).$$

$$h = 6.$$

(3)

(b) Work backwards from the final non-zero remainder. This expresses the hcf as an integer combination of the original two numbers.

$$6 = 42 - 3(12) = 42 - 3(96 - 2 \times 42) = 7(42) - 3(96)$$

$$= 7(234 - 2 \times 96) - 3(96) = 7(234) - 17(96).$$

$$a = 7, \ b = -17, \textit{ since } 234(7) + 96(-17) = 6.$$

(3)

(c) A congruence $ax \equiv b \pmod{m}$ *can be divided by* $\gcd(a, m)$ *when that gcd also divides* b. *Here* $6 \mid 36$, *so divide the congruence and the modulus by* 6.

$$16x \equiv 6 \pmod{39}.$$

From part (b), dividing $234(7) - 96(17) = 6$ *by* 6 *gives* $39(7) - 16(17) = 1$. *Thus* $-17 \equiv 22 \pmod{39}$ *is the inverse of* 16.

$$x \equiv 22 \times 6 = 132 \equiv 15 \pmod{39}.$$

$$x \equiv 15 \pmod{39}. \textit{ Equivalently, modulo } 234, \ x \equiv 15, 54, 93, 132, 171, 210.$$

(5)

Question 4

Question 4

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & p & -2 \\ 0 & -2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \text{ is an eigenvector of } A.$$

(a) Find its eigenvalue. (b) Hence show $p = 3$. (c) Find the remaining eigenvalues and corresponding eigenvectors. (d) Find P, D such that $A = PDP^T$.

Model solution

(a) For an eigenvector v, $Av = \lambda v$. The first row does not contain p, so it immediately gives λ.

$$4(2) + 2(-1) + 0(2) = 2\lambda \implies 6 = 2\lambda \implies \lambda = 3.$$

The eigenvalue is 3.

(2)

(b) Now compare the second component of $Av = 3v$.

$$2(2) + p(-1) - 2(2) = 3(-1) \implies -p = -3 \implies p = 3.$$

$$p = 3.$$

(1)

(c)(i) With $p = 3$, use the characteristic equation to obtain the other eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 & 0 \\ 2 & 3-\lambda & -2 \\ 0 & -2 & 2-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 6\lambda) = 0.$$

$$\lambda = 0,\ 3,\ 6.$$

The remaining eigenvalues are 0 and 6.

(3)

(c)(ii) For each new eigenvalue solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

$$\lambda = 0: \quad 4x + 2y = 0, \quad -2y + 2z = 0 \implies y = -2x,\ z = -2x,$$

$$v_0 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}.$$

$$\lambda = 6: \quad -2x + 2y = 0, \quad -2y - 4z = 0 \implies y = x,\ z = -\frac{x}{2},$$

$$\boldsymbol{v}_6 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

Eigenvectors may be $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ *for* 0*, and* $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ *for* 6*.*

(3)

(d) The matrix is symmetric, so its mutually perpendicular eigenvectors can be normalised to form an orthogonal matrix P*. Put the matching eigenvalues in* D *in the same column order.*

$$\frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\mathrm{P} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ -1 & -2 & 2 \\ 2 & -2 & -1 \end{pmatrix}, \quad \mathrm{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

$\mathrm{A} = \mathrm{PDP}^{\mathrm{T}}$ *for the* P *and* D *above.*

(3)

Question 5

Question 5

(i) The locus C satisfies $|z-3i| = 2|z|$. (a) Find its Cartesian equation. (b) Shade $\{z \in \mathbb{C} : |z-3i| > 2|z|\}$.
(ii) $T : z \mapsto w = z^3$. (a) Describe T. (b) Sketch the image of $R = \{z \in C : 0 < \arg z < \pi/4\}$.

Model solution

(i)(a) Put $z = x + iy$ and square both moduli. Squaring is valid because both sides are non-negative, and it converts the locus into Cartesian algebra.

$$|z-3i|^2 = x^2 + (y-3)^2, \quad |z|^2 = x^2 + y^2.$$

$$x^2 + (y-3)^2 = 4(x^2+y^2) \Longrightarrow 3x^2 + 3y^2 + 6y - 9 = 0 \Longrightarrow x^2 + (y+1)^2 = 4.$$

C is the circle $x^2 + (y+1)^2 = 4$, centre $(0,-1)$, radius 2.

(3)

(i)(b) Use the same expansion as in part (a), but retain the inequality. The origin satisfies $|-3i| > 0$, so the required side is the side containing the origin.

Draw $x^2 + (y+1)^2 = 4$ and shade its interior (the boundary is excluded for the strict inequality).

(2)

(ii)(a) Write $z = re^{i\theta}$. Then $w = z^3 = r^3 e^{i3\theta}$, which gives the effect on modulus and argument directly.

The modulus is cubed and the argument is tripled: $|w| = |z|^3$, $\arg w = 3 \arg z$.

(2)

(ii)(b) The rays bounding R have arguments 0 and $\pi/4$. Tripling the arguments maps them to 0 and $3\pi/4$; the inequalities remain strict.

Shade the open sector $\{w : 0 < \arg w < 3\pi/4\}$, with vertex at the origin.

(2)

Question 6

Question 6

$$I_n = \int \frac{\cos(nx)}{\sin x}\,dx, \qquad n \geqslant 1.$$

(a) Show that $I_{n+2} = I_n + \dfrac{2\cos((n+1)x)}{n+1}$.

(b) Hence find $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\cos(5x)}{\sin x}\,dx$ in the form $a + b\ln c$.

Model solution

(a) Compare I_{n+2} with I_n. The compound-angle formula makes their difference simplify, which is the clue for a reduction formula.

$$I_{n+2} - I_n = \int \frac{\cos((n+2)x) - \cos(nx)}{\sin x}\,dx.$$

$$\cos((n+2)x) - \cos(nx) = -2\sin((n+1)x)\sin x,$$

$$I_{n+2} - I_n = -2\int \sin((n+1)x)\,dx = \frac{2\cos((n+1)x)}{n+1}.$$

$$\boxed{I_{n+2} = I_n + \frac{2\cos((n+1)x)}{n+1}.}$$

(6)

(b) Apply the reduction formula twice to reduce I_5 to I_1, which is elementary.

$$I_5 = I_3 + \frac{\cos 4x}{2} = I_1 + \cos 2x + \frac{\cos 4x}{2}.$$

$$I_1 = \int \frac{\cos x}{\sin x}\,dx = \ln(\sin x).$$

Now evaluate the resulting antiderivative at the stated limits.

$$\left[\ln(\sin x) + \cos 2x + \frac{1}{2}\cos 4x\right]_{\pi/4}^{\pi/3}$$

$$= \left(\ln\frac{\sqrt{3}}{2} - \frac{1}{2} - \frac{1}{4}\right) - \left(\ln\frac{\sqrt{2}}{2} + 0 - \frac{1}{2}\right) = -\frac{1}{4} + \frac{1}{2}\ln\frac{3}{2}.$$

$$\boxed{-\frac{1}{4} + \frac{1}{2}\ln\frac{3}{2}.}$$

(5)

Question 7

Question 7

Let $G = \{I, A, B, C, D, E\}$, under matrix multiplication modulo 2, where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) Show B has order 3. (b) Find the orders of the others. (c) Give reasons why G is not isomorphic to (i) a cyclic group of order 6, (ii) the symmetry group of a regular hexagon. (d) Give an isomorphism from G to the stated group $H \cong S_3$.

Model solution

(a) Calculate successive powers, reducing every entry modulo 2. The order is the least positive power which gives the identity.

$$B^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = E, \qquad B^3 = EB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Since $B \neq I$, B has order 3.

(2)

(b) The identity has order 1. Also $E = B^2 = B^{-1}$, so it has order 3. Squaring the remaining matrices tests whether they are involutions.

$$A^2 = C^2 = D^2 = I.$$

$\text{ord}(I) = 1$, $\text{ord}(A) = \text{ord}(C) = \text{ord}(D) = 2$, *and* $\text{ord}(E) = 3$.

(3)

(c)(i) Isomorphisms preserve element orders. A cyclic group of order 6 has a generator of order 6, whereas G has none.

G is not isomorphic to a cyclic group of order 6.

(1)

(c)(ii) Isomorphisms also preserve the number of elements.

A regular hexagon has 12 symmetries, whereas G has 6 elements; therefore they are not isomorphic.

(1)

(d) Match the identity, the two elements of order 3, and the three involutions, then check that multiplication is respected. One valid correspondence is

$$I \mapsto e, \qquad A \mapsto c, \qquad B \mapsto a, \qquad C \mapsto d, \qquad D \mapsto f, \qquad E \mapsto b.$$

The displayed bijection is an isomorphism $G \to H$.

(3)

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Question 8

Question 8

The bell section is modelled by rotating $y = \frac{9}{2}e^{x/9}$, $0 \leqslant x \leqslant 9$, about the x-axis.

(a) Show its external surface area is $K\int_0^9 e^{x/9}\sqrt{4+e^{2x/9}}\,dx$.

(b) With $u = e^{x/9}$, show the given split-integral form and find a, b.

(c) Hence find the area to 3 significant figures.

Model solution

(a) For a curve rotated about the x-axis, use $S = 2\pi\int y\sqrt{1+(dy/dx)^2}\,dx$. The curve is given explicitly as y in terms of x, so this formula is directly applicable.

$$\frac{dy}{dx} = \frac{1}{2}e^{x/9}.$$

$$S = 2\pi\int_0^9 \frac{9}{2}e^{x/9}\sqrt{1+\frac{1}{4}e^{2x/9}}\,dx = \frac{9\pi}{2}\int_0^9 e^{x/9}\sqrt{4+e^{2x/9}}\,dx.$$

$$K = \frac{9\pi}{2}.$$

(3)

(b) The exponential and its derivative occur in the integrand, so $u = e^{x/9}$ removes the exponentials.

$$u = e^{x/9}, \qquad du = \frac{1}{9}e^{x/9}\,dx = \frac{u}{9}\,dx, \qquad dx = \frac{9}{u}\,du.$$

$$x = 0 \Rightarrow u = 1, \qquad x = 9 \Rightarrow u = e.$$

$$\int_0^9 e^{x/9}\sqrt{4+e^{2x/9}}\,dx = 9\int_1^e \sqrt{4+u^2}\,du.$$

Split the integrand by writing $u^2+4 = (u^2+4)$. This creates a derivative-over-square-root term and a standard $1/\sqrt{u^2+4}$ term.

$$\sqrt{u^2+4} = \frac{u^2+4}{\sqrt{u^2+4}} = \frac{u^3+2u}{\sqrt{u^4+4u^2}} + \frac{2}{\sqrt{u^2+4}}.$$

$$\int_0^9 \cdots\,dx = 9\int_1^e \frac{2u+u^3}{\sqrt{4u^2+u^4}}\,du + 18\int_1^e \frac{1}{\sqrt{4+u^2}}\,du.$$

$$a = 1,\ b = e.$$

(5)

(c) Integrate the two terms obtained in part (b). For the first, let $v = 4u^2 + u^4$*, whose derivative is* $4(2u + u^3)$*. For the second use* $\int du/\sqrt{u^2+4} = \operatorname{arsinh}(u/2)$*.*

$$\int \frac{2u+u^3}{\sqrt{4u^2+u^4}}\,du = \frac{1}{2}\sqrt{4u^2+u^4}, \qquad \int \frac{du}{\sqrt{4+u^2}} = \operatorname{arsinh}\left(\frac{u}{2}\right).$$

$$S = \frac{9\pi}{2}\left[\frac{9}{2}\sqrt{4u^2+u^4} + 18\operatorname{arsinh}\left(\frac{u}{2}\right)\right]_1^e.$$

$$S = \frac{9\pi}{2}\left(\frac{9}{2}(e\sqrt{4+e^2} - \sqrt{5}) + 18\left[\operatorname{arsinh}\left(\frac{e}{2}\right) - \operatorname{arsinh}\left(\frac{1}{2}\right)\right]\right) = 602.370125\ldots \text{ cm}^2.$$

Evaluating the exact expression gives $602.370125\ldots$*, which rounds as required.*

The external surface area is $602\ \text{cm}^2$ *to* 3 *significant figures.*

(5)

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