Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/4A - Further Pure Mathematics 2

Monday 23 June 2025

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2025 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

The set $S = \{1, 3, 5, 9, 11, 13\}$ forms the group G under multiplication modulo 14. (a) Complete its Cayley table. (b) Write down a subgroup of order 2. The group H is given by

$*$	p	q	r	s	t	u
p	p	q	r	s	t	u
q	q	t	u	r	s	p
r	r	u	t	q	p	s
s	s	r	q	p	u	t
t	t	s	p	u	r	q
u	u	p	s	t	q	r

.

(c) Show that G and H are isomorphic.

Model solution

(a) Each entry is the product of its row and column headings, reduced modulo 14. For example, $5 \times 9 = 45 \equiv 3 \pmod{14}$.

$\times_{14}$	1	3	5	9	11	13
1	1	3	5	9	11	13
3	3	9	1	13	5	11
5	5	1	11	3	13	9
9	9	13	3	11	1	5
11	11	5	13	1	9	3
13	13	11	9	5	3	1

The completed table is as shown.

(3) (b) A subgroup of order 2 must contain the identity and one element whose square is the identity. Since $13^2 = 169 \equiv 1 \pmod{14}$, *the set is closed.*

$$\{1, 13\}.$$

$\{1, 13\}$ *is a subgroup of order 2.*

(1) (c) A useful way to show two groups of the same finite order are isomorphic is to show both are cyclic with the same order. In G, *successive powers of* 3 *are*

$$3^1 = 3, \quad 3^2 = 9, \quad 3^3 = 13, \quad 3^4 = 11, \quad 3^5 = 5, \quad 3^6 = 1 \pmod{14}.$$

Thus 3 *generates every element of* G. *From the table for* H,

$$q^1 = q, \quad q^2 = t, \quad q^3 = s, \quad q^4 = r, \quad q^5 = u, \quad q^6 = p.$$

So q *generates every element of* H. *Both groups are cyclic of order* 6, *so mapping equal powers of* 3 *to equal powers of* q *preserves the operation.*

$$1 \mapsto p, \quad 3 \mapsto q, \quad 9 \mapsto t, \quad 13 \mapsto s, \quad 11 \mapsto r, \quad 5 \mapsto u.$$

$G \cong H$.

(3)

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Question 2

Question 2

For $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, (a) determine the eigenvalues and (b) hence find an orthogonal matrix P and diagonal matrix D such that $D = P^T AP$.

Model solution

(a) Eigenvalues are found from the characteristic equation $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (1-\lambda)(4-\lambda) - 4 = 0$$

$$\lambda^2 - 5\lambda = 0 \quad \Rightarrow \quad \lambda(\lambda - 5) = 0.$$

$$\lambda = 0,\ 5.$$

(3) (b) Since A is symmetric, eigenvectors belonging to distinct eigenvalues are orthogonal. We find and then normalise them so that they can be the columns of an orthogonal matrix.

$$\lambda = 0: \quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x + 2y = 0, \quad \textit{so } \mathbf{v}_0 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

$$\lambda = 5: \quad (A - 5I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -4x + 2y = 0, \quad \textit{so } \mathbf{v}_5 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Both vectors have length $\sqrt{5}$, so their unit versions are $\frac{1}{\sqrt{5}}(2,-1)^T$ and $\frac{1}{\sqrt{5}}(1,2)^T$. Putting them in the matching eigenvalue order gives

$$P = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}.$$

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}.$$

(5)

Question 3

Question 3

The circle C satisfies $|z-5+4i| = |z-2-2i|$, where $z = x+iy$. (a) Show that $x^2-2x+y^2-7=0$. The half-line $\arg(z-a) = 3\pi/4$, where a is real, is tangent to C. (b) Find a.

Model solution

(a) Write each modulus as a distance in Cartesian coordinates, then square to remove the square roots.

$$|z-5+4i|^2 = (x-5)^2+(y+4)^2, \qquad |z-2-2i|^2 = (x-2)^2+(y-2)^2.$$

$$(x-5)^2+(y+4)^2 = (x-2)^2+(y-2)^2$$

$$x^2-10x+25+y^2+8y+16 = x^2-4x+4+y^2-4y+4.$$

$$\therefore \quad x^2-2x+y^2-7=0.$$

$$\boxed{x^2-2x+y^2-7=0.}$$

(3) (b) The argument $3\pi/4$ gives a gradient $\tan(3\pi/4) = -1$. Since the half-line starts at the real point a, its equation is $y = -(x-a) = -x+a$. A tangent meets the circle at one repeated point, so substitution must give a quadratic with discriminant zero.

$$x^2-2x+(-x+a)^2-7=0$$

$$2x^2-(2+2a)x+a^2-7=0.$$

$$\Delta = [-2(1+a)]^2-4(2)(a^2-7)=0$$

$$4(1+a)^2-8(a^2-7)=0 \Rightarrow a^2-2a-15=0 \Rightarrow (a-5)(a+3)=0.$$

The algebra gives $a=5$ or $a=-3$. The stated object is a half-line with direction $3\pi/4$: from $a=5$ it reaches the circle, whereas the ray from -3 in that direction does not.

$$\boxed{a=5.}$$

(6)

Question 4

Question 4

The sequence satisfies $u_{n+2} = 6u_{n+1} - 8u_n$ for $n \geqslant 1$, with $u_1 = 399.6$, $u_2 = 798.4$. (a) Prove by induction that $u_n = 200(2^n) - \frac{1}{10}(4^n)$. (b) Hence find the first negative term.

Model solution

(a) This is a second-order recurrence, so induction needs two consecutive base cases and assumes two consecutive cases. For $n = 1$ and $n = 2$, respectively,

$$200(2) - \frac{1}{10}(4) = 399.6 = u_1, \qquad 200(4) - \frac{1}{10}(16) = 798.4 = u_2.$$

Assume, for some $k \geqslant 1$, that

$$u_k = 200(2^k) - \frac{1}{10}(4^k), \qquad u_{k+1} = 200(2^{k+1}) - \frac{1}{10}(4^{k+1}).$$

Use the recurrence to produce the next term.

$$\begin{aligned} u_{k+2} &= 6u_{k+1} - 8u_k \\ &= 6\left(200\,2^{k+1} - \frac{1}{10}4^{k+1}\right) - 8\left(200\,2^k - \frac{1}{10}4^k\right) \\ &= 200(12-8)2^k - \frac{1}{10}(24-8)4^k \\ &= 200(2^{k+2}) - \frac{1}{10}(4^{k+2}). \end{aligned}$$

Thus, if the formula holds for two successive values $k, k+1$, it holds for $k+2$. Together with the two base cases, it is true for every $n \in \mathbb{N}$ by induction.

$$u_n = 200(2^n) - \frac{1}{10}4^n \quad (n \in \mathbb{N}).$$

(6) (b) Factor the closed form to locate where it changes sign.

$$u_n = 200(2^n) - \frac{1}{10}(2^{2n}) = \frac{2^n}{10}(2000 - 2^n).$$

As $2^n/10 > 0$, $u_n < 0$ exactly when $2^n > 2000$. Since $2^{10} = 1024 < 2000 < 2048 = 2^{11}$, the first such index is $n = 11$.

$$u_{11} = 200(2048) - \frac{1}{10}(4194304) = -9830.4.$$

The first negative term is $u_{11} = -9830.4$.

(3)

Question 5

Question 5

(i) A password uses one capital letter, three distinct digits and one of $@, \$, \$, \&$. Find the number when (a) the characters are in that order and (b) they can be in any order. (ii) Use Fermat's Little Theorem to find the least positive residue of 4^{50} modulo 13. (iii) Find the number of positive odd numbers less than 1000 containing the digit 2.

Model solution

(i)(a) Apply the multiplication principle in the specified order. There are 26 *letter choices, then* $10, 9, 8$ *choices for the distinct digits, and* 4 *symbol choices.*

$$26 \times 10 \times 9 \times 8 \times 4 = 74880.$$

74880 ***passwords.***

(2) (i)(b) Each selection from part (a) has five distinct character types, so its five positions can be permuted in $5!$ *ways. The three digit slots were indistinguishable for this rearrangement, so divide by* $3!$*.*

$$74880 \times \frac{5!}{3!} = 1497600.$$

1497600 ***passwords.***

(1) (ii) Fermat's Little Theorem applies because 13 *is prime and* $13 \nmid 4$*:* $4^{12} \equiv 1 \pmod{13}$*. Write the exponent using a multiple of* 12*.*

$$4^{50} = 4^{48}4^2 = (4^{12})^4 \cdot 16 \equiv 1^4 \cdot 3 \equiv 3 \pmod{13}.$$

The least positive residue is 3*.*

(3) (iii) Counting the complement is simpler. There are 500 *positive odd numbers below* 1000*. Count those which do not contain* 2*: for one digit there are* 5 *choices; for two digits there are* $8 \times 5 = 40$ *choices; for three digits there are* $8 \times 9 \times 5 = 360$ *choices. Here the final digit has five choices* $1, 3, 5, 7, 9$*.*

$$500 - (5 + 40 + 360) = 95.$$

95 ***positive odd numbers.***

(4)

Question 6

Question 6

$I_n = \int_0^2 (4-x^2)^n \, dx$. (a) Prove, for $n \geqslant 1$, that $I_n = \frac{8n}{2n+1} I_{n-1}$. (b) Hence find n when $I_n = \frac{65536}{315}$.

Model solution

(a) Integration by parts is appropriate because differentiating $(4-x^2)^n$ reduces its power. Take $u = (4-x^2)^n$ and $dv = dx$.

$$\begin{aligned} I_n &= \int_0^2 (4-x^2)^n \, dx \\ &= \left[x(4-x^2)^n\right]_0^2 + 2n \int_0^2 x^2 (4-x^2)^{n-1} \, dx. \end{aligned}$$

The boundary term is zero. To express the remaining integral in terms of the Is, use $x^2 = 4 - (4-x^2)$.

$$I_n = 2n \int_0^2 \left[4 - (4-x^2)\right] (4-x^2)^{n-1} \, dx = 2n(4I_{n-1} - I_n).$$

$$(2n+1)I_n = 8nI_{n-1} \quad \Rightarrow \quad I_n = \frac{8n}{2n+1} I_{n-1}.$$

$$I_n = \frac{8n}{2n+1} I_{n-1}.$$

(4) (b) Start from an easily evaluated base integral, then apply the reduction formula only for the even indices needed.

$$I_0 = \int_0^2 1 \, dx = 2.$$

$$I_2 = \frac{16}{5} I_1 = \frac{16}{5} \left(\frac{8}{3} I_0\right) = \frac{256}{15},$$

$$I_4 = \frac{32}{9} I_3 = \frac{32}{9} \left(\frac{24}{7} I_2\right) = \frac{32}{9} \cdot \frac{24}{7} \cdot \frac{256}{15} = \frac{65536}{315}.$$

$$n = 4.$$

(4)

Question 7

Question 7

(i) Explain why $21x \equiv 11 \pmod{36}$ has no solutions. (ii) Solve $36y \equiv 192 \pmod{40}$ using modular arithmetic.

Model solution

(i) A congruence $ax \equiv b \pmod{m}$ is soluble only if $\gcd(a, m)$ divides b.

$$\gcd(21, 36) = 3, \qquad 3 \nmid 11.$$

There are no solutions.

(2) (ii) The coefficient, right-hand side and modulus have common factor 4. Divide all three by 4; this is why the modulus becomes 10.

$$36y \equiv 192 \pmod{40} \Rightarrow 9y \equiv 48 \equiv 8 \pmod{10}.$$

The inverse of 9 modulo 10 is 9, since $9^2 = 81 \equiv 1 \pmod{10}$.

$$y \equiv 9 \cdot 8 \equiv 72 \equiv 2 \pmod{10}.$$

$y \equiv 2 \pmod{10}$, *i.e.* $y \equiv 2, 12, 22, 32 \pmod{40}$.

(4)

Question 8

Question 8

A satellite dish is modelled by rotating $y^2 = Ax$, $0 \leqslant x \leqslant 10$, through π radians about the x-axis. Its largest diameter is 60 cm and its depth is 10 cm. (a) Find A. (b) Find the curved inner surface area to 2 significant figures. (c) State a limitation of the model.

Model solution

(a) At the rim, the depth is $x = 10$ cm and the radius is half the diameter, $y = 30$ cm. Substitute this known point into the model.

$$30^2 = A(10) \quad \Rightarrow \quad A = 90.$$

$$A = 90.$$

(2) (b) For a curve revolved about the x-axis, the surface area is $2\pi \int y\sqrt{1 + (dy/dx)^2}\, dx$. Here $y = \sqrt{90x}$, so implicit differentiation of $y^2 = 90x$ gives

$$2y\frac{dy}{dx} = 90 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{45}{y}.$$

$$\begin{aligned} SA &= 2\pi \int_0^{10} y\sqrt{1 + \left(\frac{45}{y}\right)^2}\, dx \\ &= 2\pi \int_0^{10} \sqrt{y^2 + 2025}\, dx = 2\pi \int_0^{10} \sqrt{90x + 2025}\, dx. \end{aligned}$$

The integrand has the form $(90x + 2025)^{1/2}$, so use the reverse chain rule.

$$SA = 2\pi \left[\frac{(90x + 2025)^{3/2}}{135}\right]_0^{10} = \frac{2\pi}{135}\left(2925^{3/2} - 2025^{3/2}\right) \approx 3098.0.$$

$$\textit{curved inner area} = 3.1 \times 10^3 \text{ cm}^2 \textit{ (2 s.f.).}$$

(7) (c) The equation assumes the inner surface is an exact smooth paraboloid of revolution, whereas a real dish can have imperfections and may not follow this curve exactly.

For example, the real dish may not be perfectly smooth/parabolic.

(1)

Question 9

Question 9

The transformation T is $w = \dfrac{z}{z+2i}$, $z \neq -2i$, where $z = x + iy$ and $w = u + iv$. The circle $|z| = 4$ maps onto circle C. Find the centre and radius of C.

Model solution

To obtain the image locus in the w-plane, first make z the subject, then impose the original condition $|z| = 4$.

$$w(z+2i) = z \Rightarrow wz + 2iw = z \Rightarrow z(w-1) = -2iw$$

$$z = \frac{2iw}{1-w}.$$

Taking moduli removes the complex argument while retaining the locus condition.

$$|z| = \frac{2|w|}{|1-w|} = 4 \quad \Rightarrow \quad |w| = 2|1-w|.$$

Now write $w = u + iv$ and square both sides.

$$u^2 + v^2 = 4\big((1-u)^2 + v^2\big)$$

$$3u^2 - 8u + 3v^2 + 4 = 0 \quad \Rightarrow \quad u^2 - \frac{8}{3}u + v^2 + \frac{4}{3} = 0.$$

Complete the square in u to read the centre and radius directly.

$$\left(u - \frac{4}{3}\right)^2 + v^2 = \frac{4}{9}.$$

Centre $\left(\frac{4}{3}, 0\right)$; radius $\frac{2}{3}$.

(8)