Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/4A - Further Pure Mathematics 2

October 2020

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2020 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

A club has 12 adult and 14 junior members. Select a team of 8: (i) without restriction; (ii) with 4 adults and 4 juniors; (iii) with more than half adults.

Model solution

(i) Teams are unordered selections, so choose 8 from all 26 members.

$$\binom{26}{8} = 1\,562\,275.$$

$1\,562\,275 \text{teams.}$

(1) (ii) Choose the specified numbers independently from the adult and junior groups.

$$\binom{12}{4}\binom{14}{4} = 495(1001) = 495\,495.$$

$495\,495 \text{teams.}$

(2) (iii) More than half of 8 means 5, 6, 7 or 8 adults. Add these mutually exclusive cases.

$$\binom{12}{5}\binom{14}{3} + \binom{12}{6}\binom{14}{2} + \binom{12}{7}\binom{14}{1} + \binom{12}{8}\binom{14}{0} = 383\,955.$$

$383\,955 \text{teams.}$

(3)

Question 2

Question 2

Solve $u_1 = 1,\ u_2 = 4,\ 9u_{n+2} - 12u_{n+1} + 4u_n = 3n.$

Model solution

Solve the homogeneous recurrence first, then add a particular solution for the linear forcing term.

$$9r^2 - 12r + 4 = 0 \quad \Longrightarrow \quad (3r-2)^2 = 0,$$

so

$$u_n^{(h)} = (A + Bn)\left(\frac{2}{3}\right)^n.$$

Because the right side is linear in n*, try* $u_n^{(p)} = an + b.$

$$9[a(n+2)+b] - 12[a(n+1)+b] + 4(an+b) = 3n,$$

$$an + 6a + b = 3n,$$

giving $a = 3,\ b = -18.$ *Hence*

$$u_n = (A + Bn)\left(\frac{2}{3}\right)^n + 3n - 18.$$

The initial values now determine A *and* $B.$

$$\frac{2}{3}(A+B) - 15 = 1 \Rightarrow A + B = 24, \qquad \frac{4}{9}(A+2B) - 12 = 4 \Rightarrow A + 2B = 36.$$

Thus $A = B = 12$*, so*

$$u_n = 12(n+1)\left(\tfrac{2}{3}\right)^n + 3n - 18.$$

(9)

Question 3

Question 3

$$\mathbf{M} = \begin{pmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

(a) Show that a characteristic equation is $\lambda^3 - (2k+13)\lambda + 5(k+6) = 0$. Given $\det \mathbf{M} = 5$, (b)(i) find k, (ii) use Cayley–Hamilton to find $\mathbf{M}^{-1}$.

Model solution

(a) Form $\det(\lambda\mathbf{I} - \mathbf{M})$, as this is the definition of the characteristic equation.

$$\begin{aligned} 0 &= \begin{vmatrix} \lambda - 1 & -k & 2 \\ -2 & \lambda + 4 & -1 \\ -1 & -2 & \lambda - 3 \end{vmatrix} \\ &= (\lambda - 1)\big((\lambda + 4)(\lambda - 3) - 2\big) + k\big(-2(\lambda - 3) - 1\big) + 2(\lambda + 8) \\ &= \lambda^3 - (2k + 13)\lambda + 5(k + 6). \end{aligned}$$

$$\lambda^3 - (2k+13)\lambda + 5(k+6) = 0$$

(3) (b)(i) The constant term of $\det(\lambda\mathbf{I} - \mathbf{M})$ is $-\det \mathbf{M}$.

$$5(k+6) = -5$$

$$k = -7$$

(2) (b)(ii) Substitute $k = -7$ and use Cayley--Hamilton. Since $\det \mathbf{M} \neq 0$, multiply through by $\mathbf{M}^{-1}$.

$$\mathbf{M}^3 + \mathbf{M} - 5\mathbf{I} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{M}^{-1} = \frac{1}{5}(\mathbf{M}^2 + \mathbf{I}).$$

$$\mathbf{M}^2 + \mathbf{I} = \begin{pmatrix} -14 & 17 & -15 \\ -5 & 5 & -5 \\ 8 & -9 & 10 \end{pmatrix}.$$

$$\mathbf{M}^{-1} = \begin{pmatrix} -\frac{14}{5} & \frac{17}{5} & -3 \\ -1 & 1 & -1 \\ \frac{8}{5} & -\frac{9}{5} & 2 \end{pmatrix}.$$

(5)

Question 4

Question 4

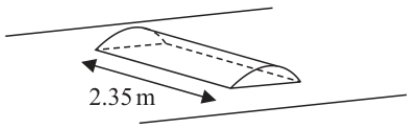


Figure 1

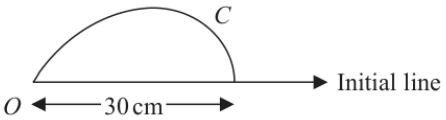


Figure 2

Figure 1 shows a sketch of a design for a road speed bump of width 2.35 metres. The speed bump has a uniform cross-section with vertical ends and its length is 30 cm. A side profile of the speed bump is shown in Figure 2.

The curve C shown in Figure 2 is modelled by the polar equation

$$r = 30(1 - \theta^2) \qquad 0 \leqslant \theta \leqslant 1$$

The profile is $r = 30(1-\theta^2)$, $0 \leqslant \theta \leqslant 1$, with r in cm. The uniform bump length is 2.35 m. Find the visible painted area.

Model solution

The painted area is two identical vertical end faces plus the curved surface; the road-facing base is excluded. Use the polar-area formula for one end face.

$$A_{\text{end}} = \frac{1}{2}\int_0^1 [30(1-\theta^2)]^2 \, d\theta$$
$$= 450\left[\theta - \frac{2\theta^3}{3} + \frac{\theta^5}{5}\right]_0^1 = 240 \ cm^2.$$

The curved area is arc length times bump length. The polar arc-length formula applies because the profile is in polar coordinates.

$$\frac{dr}{d\theta} = -60\theta,$$

$$L = \int_0^1 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$
$$= 30\int_0^1 \sqrt{(1-\theta^2)^2 + 4\theta^2} \, d\theta = 30\int_0^1 (1+\theta^2) \, d\theta = 40 \ cm.$$

Since $2.35 \ m = 235 \ cm$,

$$A = 2(240) + 235(40) = 9880 \ cm^2.$$

$9880 \ cm^2$.

(10)

Question 5

Question 5

$$w = \frac{1-3z}{z+2i}, \quad z \neq -2i, \qquad |z+i| = 3.$$

(a) Show $3|w+3| = |1+(3-w)i|$. (b) Find a Cartesian equation, centre and radius of C.

Model solution

(a) Make z the subject so the original modulus condition can be rewritten in terms of w.

$$w(z+2i) = 1-3z \quad \Longrightarrow \quad z = \frac{1-2iw}{w+3},$$

$$z+i = \frac{1-2iw+i(w+3)}{w+3} = \frac{1+(3-w)i}{w+3}.$$

Thus $|z+i| = 3$ becomes

$$3|w+3| = |1+(3-w)i|.$$

(4) (b)(i) Put $w = u+iv$, then square the modulus equation to obtain a real Cartesian equation.

$$9\left((u+3)^2+v^2\right) = (1+v)^2+(3-u)^2,$$

$$8u^2+60u+8v^2-2v+71 = 0.$$

(3) (b)(ii) Complete the square to read the centre and radius directly.

$$\left(u+\frac{15}{4}\right)^2+\left(v-\frac{1}{8}\right)^2 = \frac{333}{64}.$$

Centre $\left(-\frac{15}{4}, \frac{1}{8}\right)$, *radius* $\frac{3\sqrt{37}}{8}$.

(3)

Question 6

Question 6

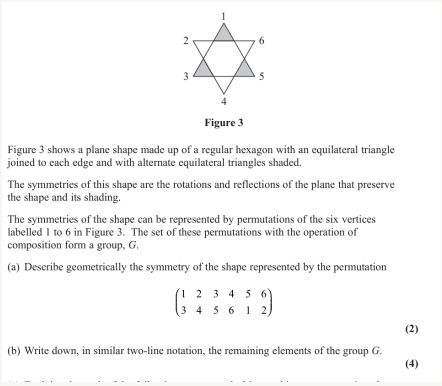


Figure 3

Figure 3 shows a plane shape made up of a regular hexagon with an equilateral triangle joined to each edge and with alternate equilateral triangles shaded.

The symmetries of this shape are the rotations and reflections of the plane that preserve the shape and its shading.

The symmetries of the shape can be represented by permutations of the six vertices labelled 1 to 6 in Figure 3. The set of these permutations with the operation of composition form a group, G.

(a) Describe geometrically the symmetry of the shape represented by the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$$

(2)

(b) Write down, in similar two-line notation, the remaining elements of the group G.

(4)

The symmetry group G acts on the labelled vertices. (a) Describe $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$. (b) Write the remaining elements. (c) Explain why no subgroup has order 4 and why G is not cyclic. (d) Give shading with symmetry group C_6.

Model solution

(a) Every vertex advances two positions, which is a rotation about the centre.

Rotation about the centre through $120°$ *anticlockwise.*

(2) (b) The group has the other non-identity rotation, three reflections and the identity.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

The five permutations above.

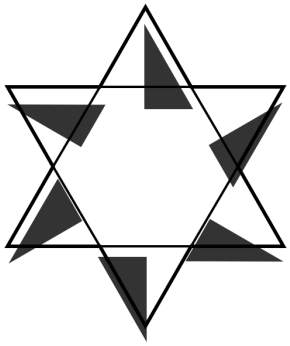
(4) (c)(i) By Lagrange's theorem, a subgroup order must divide $|G| = 6$.

$4 \nmid 6$, *so no subgroup of order 4 exists.*

(1) (c)(ii) A cyclic group of order 6 needs an element of order 6. Rotations here have order at most 3 and reflections order 2.

No element has order 6, so G is not cyclic.

(1) (d) Shade the same clockwise half of every outer point. Rotations through $60°$ *preserve*



this chiral pattern, while every reflection reverses it.

Only the six rotations remain, so the group is isomorphic to C_6.

(2)

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Question 7

Question 7

$$I_n = \int (4 - x^2)^{-n}\, dx, \qquad n > 0.$$

(a) Show that $I_{n+1} = \dfrac{x}{8n(4-x^2)^n} + \dfrac{2n-1}{8n} I_n$. (b) Find I_2.

Model solution

(a) Use integration by parts, taking $dv = dx$; the resulting x^2 can be rewritten in terms of $4 - x^2$.

$$\begin{aligned} I_n &= x(4-x^2)^{-n} - 2n \int x^2 (4-x^2)^{-n-1}\, dx \\ &= x(4-x^2)^{-n} - 2n \int \left[4 - (4-x^2)\right](4-x^2)^{-n-1}\, dx \\ &= x(4-x^2)^{-n} - 8n I_{n+1} + 2n I_n. \end{aligned}$$

Rearrangement gives

$$I_{n+1} = \tfrac{x}{8n(4-x^2)^n} + \tfrac{2n-1}{8n} I_n.$$

(5) (b) Evaluate I_1 first, then use the reduction formula with $n = 1$.

$$I_1 = \frac{1}{2} \operatorname{artanh}\left(\frac{x}{2}\right) + C,$$

$$I_2 = \frac{x}{8(4-x^2)} + \frac{1}{8} I_1 = \frac{x}{8(4-x^2)} + \frac{1}{16} \operatorname{artanh}\left(\frac{x}{2}\right) + C.$$

$$I_2 = \frac{x}{8(4-x^2)} + \frac{1}{16} \operatorname{artanh}\left(\frac{x}{2}\right) + C.$$

(3)

Question 8

Question 8

A four-digit number $n = abcd$ satisfies: $n \equiv 3 \pmod 7$; it is divisible by 9; $a + b = c + d$; b is smaller than every other digit; and c is even. Complete parts (a)–(d).

Model solution

(a) Use place value and reduce each coefficient modulo 7.

$$n = 1000a + 100b + 10c + d \equiv 6a + 2b + 3c + d \pmod 7,$$

because $1000 \equiv 6,\ 100 \equiv 2,\ 10 \equiv 3 \pmod 7$. *Hence*

$$6a + 2b + 3c + d \equiv 3 \pmod 7.$$

(2) (b) Divisibility by 9 means the digit sum is $9k$. *The equal-sums condition replaces* $c + d$ *by* $a + b$.

$$a + b + c + d = 9k, \qquad 2(a + b) = 9k.$$

Thus k *is even. Since* b *is smaller than all three other digits, the digit sum is from 3 to 35; the only possible* $9k$ *with even* k *is 18. Therefore*

$$a+b=9.$$

(4) (c) Use $b = 9 - a$ *and* $d = 9 - c$ *in the congruence from part (a).*

$$3 \equiv 6a + 2(9 - a) + 3c + (9 - c) = 4a + 2c + 27 \pmod 7.$$

So $2c \equiv -4a - 24 \pmod 7$. *Multiplying by* 4, *the inverse of* 2 *modulo 7, gives*

$$c \equiv 5(a - 1) \pmod 7.$$

(2) (d) Since $b < a$ *and* $a + b = 9$, *only* $a = 5, 6, 7, 8, 9$ *need checking. The congruence for even* c, *then* $d = 9 - c$, *eliminates all but one case.*

a	b	c	*reason*
9	0	5	c *is odd*
8	1	0 *or* 7	$0 < b$, 7 *odd*
7	2	2	$c = b$
6	3	4	$d = 5$ *works*
5	4	6	$d = 3 < b$

The surviving number is 6345, *and* $6345 \equiv 3 \pmod 7$, *verifying the first condition.*

$$n = 6345, \text{ uniquely.}$$

(4)