Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/4C - Further Mechanics 2

Tuesday 25 June 2019

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2019 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

A bead moves with constant angular speed ω in a horizontal circle on the smooth inside of a fixed hemispherical shell of radius a. The centre of its circle is $\frac{1}{4}a$ below the rim. Find ω in terms of a and g.

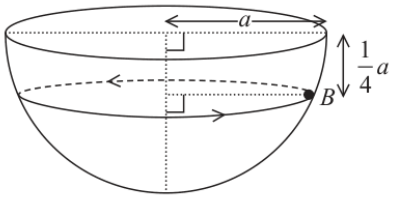


Figure 1

rical shell of radius a is fixed with its rim uppermost and ho
noving with constant angular speed, ω, in a horizontal circle

Model solution

Let θ be the angle between the inward normal reaction R and the upward vertical. The bead has no vertical acceleration, while horizontally it has centripetal acceleration towards the axis of its circular path.

$$R\cos\theta = mg, \qquad R\sin\theta = mr\omega^2.$$

The radius r of the horizontal circle is found from the right-angled geometry of the sphere. Its centre is $\frac{1}{4}a$ below the rim, so

$$\tan\theta = \frac{r}{a/4} = \frac{4r}{a}.$$

Dividing the horizontal equation by the vertical equation eliminates the unknown reaction.

$$\tan\theta = \frac{r\omega^2}{g} \quad \Longrightarrow \quad \frac{4r}{a} = \frac{r\omega^2}{g}.$$

$$\omega^2 = \frac{4g}{a}.$$

$$\omega = 2\sqrt{\frac{g}{a}}.$$

(6)

Question 2

Question 2

A particle of mass 0.4 kg moves in the positive x-direction under the single force $\frac{k}{v}$ N, where v is its speed. At $(x, v) = (3, 2)$ and $(6, 2.5)$:

(a) show that $v^3 = \frac{61}{24}x + \frac{9}{24}$;

(b) show that the time for the speed to increase from 2 to 2.5 m s^{-1} is $\frac{81}{61}$ s.

Model solution

(a) Because the force is given in terms of speed but the data are positions, use $a = v\,dv/dx$ *in Newton's second law.*

$$\frac{k}{v} = 0.4\left(v\frac{dv}{dx}\right) \quad \Longrightarrow \quad 0.4v^2\,dv = k\,dx.$$

$$\frac{0.4}{2}v^2 = kx + C \quad \Longrightarrow \quad \frac{0.4}{3}v^3 = kx + C.$$

Substitute each given position--speed pair to determine the two constants.

$$\frac{3.2}{3} = 3k + C, \qquad \frac{25/4}{3} = 6k + C.$$

$$k = \frac{61}{180}, \qquad C = \frac{1}{20}.$$

$$v^3 = \frac{3}{0.4}\left(\frac{61}{180}x + \frac{1}{20}\right) = \frac{61}{24}x + \frac{9}{24}.$$

$$v^3 = \frac{61}{24}x + \frac{9}{24}.$$

(6)

(b) Time is required, so now use $a = dv/dt$. *The force law gives a separable differential equation in* v *and* t.

$$\frac{k}{v} = 0.4\frac{dv}{dt} \quad \Longrightarrow \quad v\,dv = \frac{k}{0.4}\,dt = \frac{61}{72}\,dt.$$

$$\int_2^{2.5} v\,dv = \frac{61}{72}\int_0^T dt.$$

$$\left[\frac{1}{2}v^2\right]_2^{2.5} = \frac{61}{72}T \quad \Longrightarrow \quad \frac{9}{8} = \frac{61}{72}T.$$

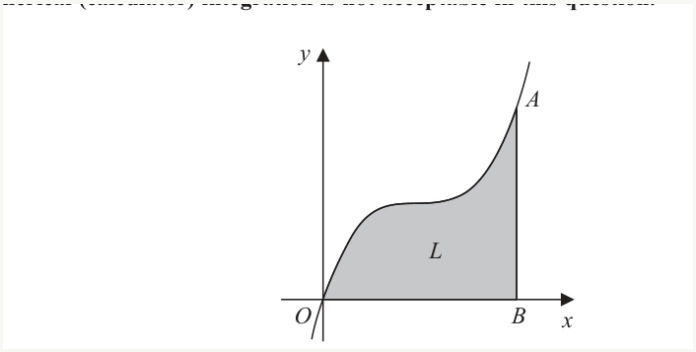
$$T = \frac{81}{61}\ \textbf{s}.$$

(4)

Question 3

Question 3

The region under $y = \frac{1}{4}(x-2)^3 + 2$, above the x-axis and between $x = 0$ and $x = 4$, is a uniform lamina of area 8 cm^2. Its centre of mass is $(\bar{x}, \bar{y})$.
(a) Show that $\bar{y} = \frac{8}{7}$.
(b) The lamina is suspended from $A = (4, 4)$. When it is in equilibrium, AB makes an angle θ with the downward vertical. Find θ.



Model solution

(a) For a vertical strip, the moment about the x-axis is $\frac{1}{2}y^2\,dx$. Dividing the total moment by the given area gives $\bar{y}$.

$$8\bar{y} = \frac{1}{2}\int_0^4 \left(\frac{1}{4}(x-2)^3 + 2\right)^2 dx.$$

$$8\bar{y} = \int_0^4 \left(\frac{1}{32}(x-2)^6 + \frac{1}{2}(x-2)^3 + 2\right) dx.$$

$$8\bar{y} = \left[\frac{(x-2)^7}{224} + \frac{(x-2)^4}{8} + 2x\right]_0^4 = \frac{8}{7} + 0 + 8 = \frac{64}{7}.$$

$$\bar{y} = \frac{8}{7} \text{ cm}.$$

(4)

(b) First find $\bar{x}$, using moments about the y-axis. The vertical line through the centre of mass must pass through the suspension point A, which then gives the angle.

$$8\bar{x} = \int_0^4 x\left(\frac{1}{4}(x-2)^3 + 2\right) dx = \left[\frac{(x-2)^5}{20} + \frac{(x-2)^4}{8} + (x-2)^2 + 4(x-2)\right]_0^4 = \frac{96}{5}.$$

$$\bar{x} = \frac{12}{5}.$$

Relative to $A = (4, 4)$, the centre of mass is $4 - \frac{12}{5} = \frac{8}{5}$ cm horizontally towards the interior and $4 - \frac{8}{7} = \frac{20}{7}$ cm below A.

$$\tan\theta = \frac{\frac{8}{5}}{\frac{20}{7}} = \frac{14}{25}.$$

$$\theta = 29.2^\circ (\text{to 3s.f.}).$$

(7)

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Question 4

Question 4

A 4 m flagpole is modelled as a non-uniform rod with mass per unit length $18 - 3x$ kg m^{-1}, where x is measured from its fixed end A.
(a) Show that its mass is 48 kg.
The pole is held at 45° to a vertical wall by a cable perpendicular to the pole at its midpoint. A 4 kg ball is attached at B. Find the tension T, then give one reason why it is unlikely to be the true value.



Figure 3

d *A* of the flagpole is fixed to a point on a vertical wall. A cable ha
lpoint of the flagpole and the other end attached to a point on the w
4. The cable is perpendicular to the flagpole. The flagpole and the
l plane that is perpendicular to the wall. A small ball of mass 4 kg i
e at *B*. The cable holds the flagpole and ball in equilibrium, with tl
l, as shown in Figure 3.

ısion in the cable is *T* newtons.

ble is modelled as a light inextensible string and the ball is modelle

Model solution

(a) Integrate the mass density over the length of the pole.

$$M = \int_0^4 (18 - 3x)\,dx = \left[18x - \frac{3}{2}x^2\right]_0^4 = 72 - 24 = 48.$$

$$\text{Mass} = 48 \text{ kg}.$$

(3)

(b) The density is non-uniform, so find the pole's centre of mass before taking moments. Its moment about A is

$$\int_0^4 x(18 - 3x)\,dx = \left[9x^2 - x^3\right]_0^4 = 80.$$

$$48d = 80 \quad \Longrightarrow \quad d = \frac{5}{3} \text{ m}.$$

Take moments about A, so the unknown wall reaction has no moment. The cable is perpendicular to the pole, giving moment $2T$. Each weight is vertical, so its perpendicular distance from A is its distance along the pole multiplied by $\cos 45°$.

$$2T = 48g\left(\frac{5}{3}\right)\cos 45° + 4g(4)\cos 45°.$$

$$2T = 96g\cos 45° = 48\sqrt{2}g, \qquad T = 24\sqrt{2}g.$$

$$T = 24\sqrt{2}g \quad N = 333 \; N (3\text{s.f.}).$$

(8)

(c) The model treats the ball as a particle, whereas its mass is distributed over a finite size.

For example, the ball's centre of mass need not be exactly at B, so its moment (and h

(1)

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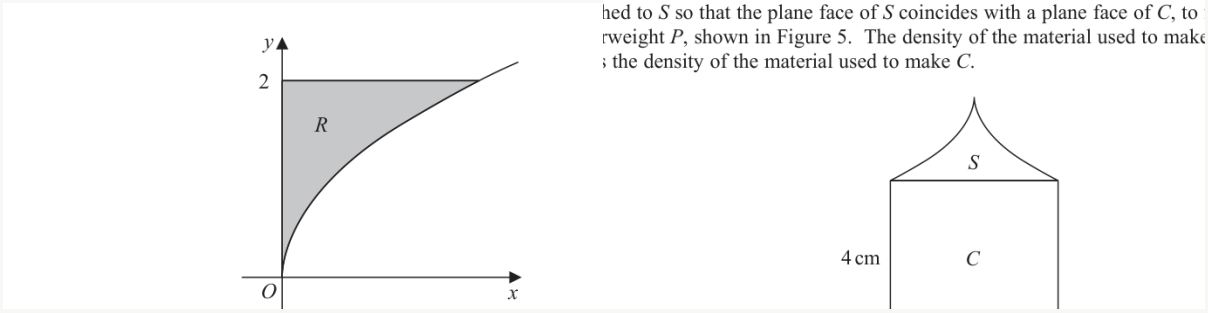
Question 5

Question 5

The region bounded by $y^2 = 2x$, $y = 2$, and the y-axis is rotated about the y-axis to form a uniform solid S, of volume $\frac{8}{5}\pi$ cm^3.
(a) Show that its centre of mass is $\frac{1}{3}$ cm from its plane face.
A cylinder of radius 2 cm, height 4 cm, and one-third of the density of S is attached to this plane face. The resulting paperweight is on the point of toppling on a rough plane inclined at θ. Find θ.

how that the centre of mass of S is $\frac{1}{3}$ cm from its plane face.

iform solid cylinder, C, has base radius 2 cm and height 4 cm. The cylin
hed to S so that the plane face of S coincides with a plane face of C, to
rweight P, shown in Figure 5. The density of the material used to make
s the density of the material used to make C.



Model solution

(a) Use horizontal discs: at height y, $x = \frac{1}{2}y^2$, *so the disc area is* $\pi x^2 = \frac{1}{4}\pi y^4$. *Taking moments about the base plane* $y = 0$,

$$\frac{8}{5}\pi\bar{y} = \int_0^2 y\left(\frac{1}{4}\pi y^4\right)dy = \frac{\pi}{4}\left[\frac{y^6}{6}\right]_0^2 = \frac{8}{3}\pi.$$

$$\bar{y} = \frac{5}{3} \text{ cm from the base.}$$

The plane face is at $y = 2$, *so the required distance from that face is* $2 - \frac{5}{3}$.

$$\text{Centre of mass of S is } \frac{1}{3} \text{ cm from its plane face.}$$

(4)

(b) Let the cylinder density be one unit. Then S has density three units. Measure distances from the circular base of the cylinder resting on the plane.

$$M_S = 3\left(\frac{8}{5}\pi\right) = \frac{24}{5}\pi, \text{quad} d_S = 4 + \frac{1}{3} = \frac{13}{3},$$

$$M_C = 16\pi, \text{quad} d_C = 2.$$

The combined centre-of-mass distance d *from the base is found by moments.*

$$\left(\frac{24}{5}\pi + 16\pi\right)d = \frac{24}{5}\pi\left(\frac{13}{3}\right) + 16\pi(2).$$

$$d = \frac{33}{13} \text{ cm.}$$

At the point of toppling, the vertical through the centre of mass passes through the downhill rim of the base. The horizontal offset is the radius 2 *cm, hence* $\tan\theta = 2/\mathrm{d}$.

$$\tan\theta = \frac{2}{33/13} = \frac{26}{33}.$$

$$\theta = 38.2^\circ (3\text{s.f.}).$$

(7)

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Question 6

Question 6

Points A and B, 4.5 m apart on a smooth horizontal surface, are joined by a light elastic string of natural length 1.5 m and modulus 15 N. A particle of mass 0.2 kg is attached at the equilibrium point AP = 1.5 m. It is pulled towards A until AP = 1 m, then released.
(a) Prove that the motion is SHM.
(b) Find (i) its maximum speed and (ii) its maximum acceleration.
(c) Find the total time in one complete oscillation for which its speed exceeds 5 m s^{-1}.

Model solution

(a) Let x be the displacement from equilibrium, positive towards B. At equilibrium the string portions have natural lengths 0.5 m and 1 m, because their common strain is $(4.5-1.5)/1.5 = 2$. Hooke's law therefore gives

$$T_A = 15\frac{1+x}{0.5} = 30(1+x), \qquad T_B = 15\frac{2-x}{1} = 15(2-x).$$

Taking positive force towards B, the resultant is the right-hand tension minus the left-hand tension.

$$0.2\frac{d^2x}{dt^2} = T_B - T_A = 15(2-x) - 30(1+x) = -45x.$$

$$\frac{d^2x}{dt^2} = -225x = -15^2x.$$

This has the form $\frac{d^2x}{dt^2} = -\omega^2 x$, so the motion is SHM with $\omega = 15$ *rad* s^{-1}.

(5)

(b) Release is from $x = -0.5$, so the amplitude is $a = 0.5$ m. For SHM, $v_{max} = a\omega$ and $a_{max} = a\omega^2$.

$$v_{max} = 0.5(15) = 7.5 \text{ m } s^{-1}, \qquad a_{max} = 0.5(15)^2 = 112.5 \text{ m } s^{-2}.$$

(i) 7.5 *m* s^{-1}; (ii) 112.5 *m* s^{-2}.

(3)

(c) A suitable displacement equation is $x = -0.5\cos(15t)$, giving speed magnitude $7.5|\sin(15t)|$. First find the time from a turning point until the speed first reaches 5 m s^{-1}.

$$7.5\sin(15t) = 5 \quad \Longrightarrow \quad t = \frac{\sin^{-1}(2/3)}{15} = 0.0486\ldots \text{ s}.$$

In one period $2\pi/15$, the speed is not greater than 5 for four such end intervals.

$$\text{required time} = \frac{2\pi}{15} - 4(0.0486\ldots) = 0.22428\ldots \text{ s}.$$

0.224 s(3s.f.).

(5)

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Question 7

Question 7

A particle of mass m is attached to a light rod of length L, freely rotating vertically about O. It is projected vertically downwards at speed u from a horizontal rod and first comes to rest at A. Given $u^2 = \frac{2}{3}gL$:
(a) explain why its acceleration at A is perpendicular to OA;
(b) find (i) its acceleration magnitude at A, and (ii) the angle θ its acceleration makes with the horizontal;
(c) find the rod tension when the particle is at its lowest point.

Model solution

(a) At A the particle is instantaneously at rest, so its centripetal acceleration v^2/L, along OA, is zero. Only tangential acceleration remains.

The acceleration is perpendicular to OA.

(1)

(b) Let ϕ be the angle that OA makes with the downward vertical. Since the particle is at rest at A, conservation of energy from the initial horizontal position to A gives

$$\frac{1}{2}mu^2 = mgL\cos\phi.$$

$$\frac{1}{2}m\left(\frac{2}{3}gL\right) = mgL\cos\phi \quad \Longrightarrow \quad \cos\phi = \frac{1}{3}.$$

At A, there is no radial acceleration, so the tangential component of gravity is the whole acceleration. The acceleration is perpendicular to OA, so it makes the same acute angle ϕ with the horizontal. Hence $\theta = \phi$.

$$|\mathbf{a}| = g\sin\phi = g\sqrt{1-\frac{1}{9}} = \frac{2\sqrt{2}}{3}g.$$

$$\theta = \phi = \cos^{-1}\left(\frac{1}{3}\right) = 70.5°.$$

$$\text{(i)}\ \frac{2\sqrt{2}}{3}g\ \textit{m s}^{-2} (= 9.24\ \textit{m s}^{-2}); \quad \text{(ii)}\ 70.5°.$$

(6)

(c) At the lowest point, resolve towards the centre O. The tension is upwards and the weight downwards, so

$$T - mg = \frac{mv^2}{L}.$$

Use energy again: from the initial horizontal position to the lowest point the particle falls a height L.

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mgL = \frac{1}{2}m\left(\frac{2}{3}gL\right) + mgL = \frac{4}{3}mgL.$$

$$v^2 = \frac{8}{3}gL.$$

$$T = mg + \frac{m}{L}\left(\frac{8}{3}gL\right) = \frac{11}{3}mg.$$

$$T = \frac{11}{3}mg \ N.$$

(5)

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