Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/4C - Further Mechanics 2

Tuesday 27 June 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Three particles of masses $3m$, $4m$ and km are at $(2a, 3a)$, $(a, 5a)$ and $(2\mu a, \mu a)$ respectively. Their centre of mass is $(2a, 4a)$. Find k and μ.

Model solution

Use the centre-of-mass formula separately in the horizontal and vertical directions. The given coordinates of the centre of mass supply two equations for the two unknown constants.

$$\frac{3m(2a) + 4m(a) + km(2\mu a)}{(3 + 4 + k)m} = 2a,$$

so, after cancelling ma,

$$10 + 2k\mu = 2(7 + k). \qquad (1)$$

Similarly, using vertical coordinates,

$$\frac{3m(3a) + 4m(5a) + km(\mu a)}{(7 + k)m} = 4a,$$

and hence

$$29 + k\mu = 4(7 + k). \qquad (2)$$

Equation (1) gives $k\mu = k + 2$*, while (2) gives* $k\mu = 4k - 1$*. Equating these expressions,*

$$k + 2 = 4k - 1 \quad \Longrightarrow \quad k = 1.$$

Substitution into $k\mu = k + 2$ *gives* $\mu = 3$*.*

$$k = 1 \text{ and } \mu = 3.$$

(6)

Question 2

Question 2

A particle of mass 2 kg moves on a smooth horizontal surface under a force $F = v + 2$ newtons, where v is its speed in m s^{-1}. Its speed increases from 5 to 10 m s^{-1} in T seconds and over distance D metres. Find T and D exactly.

Model solution

(a) Since the force is expressed in terms of speed and the time is required, apply Newton's second law with $a = \dfrac{dv}{dt}$.

$$2\frac{dv}{dt} = v + 2 \quad \Longrightarrow \quad \frac{dv}{dt} = \frac{v+2}{2}.$$

Separating variables is appropriate because the right-hand side depends only on v*:*

$$\int_5^{10} \frac{2}{v+2}\,dv = \int_0^T dt.$$

$$2[\ln(v+2)]_5^{10} = T \quad \Longrightarrow \quad T = 2\ln\left(\frac{12}{7}\right).$$

$$T = 2\ln\left(\frac{12}{7}\right) \text{ s.}$$

(4)

(b) For distance, change the acceleration to $a = v\dfrac{dv}{dx}$*. This avoids first finding* v *as a function of time.*

$$v\frac{dv}{dx} = \frac{v+2}{2} \quad \Longrightarrow \quad \frac{dx}{dv} = \frac{2v}{v+2} = 2 - \frac{4}{v+2}.$$

Therefore

$$D = \int_5^{10} \left(2 - \frac{4}{v+2}\right) dv = [2v - 4\ln(v+2)]_5^{10} = 10 - 4\ln\left(\frac{12}{7}\right).$$

$$D = 10 - 4\ln\left(\frac{12}{7}\right) \text{ m.}$$

(4)

Question 3

Question 3

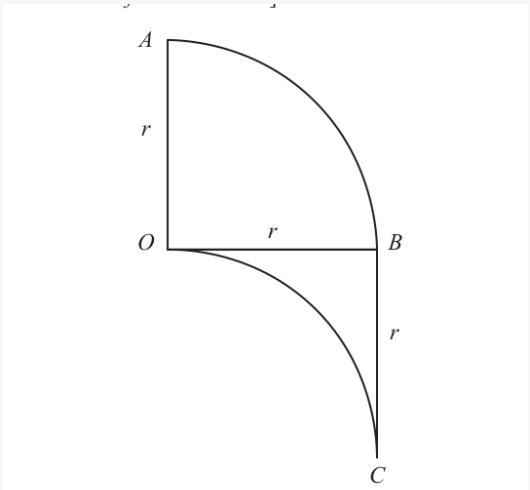


Figure 1

'orm wire are joined together to form the rigid **framewo**
vhere

are straight, with $OA = OB = BC = r$

arter of a circle with centre O and radius r

arter of a circle of radius r

A uniform wire framework consists of straight pieces OA, OB, BC, each of length r, and quarter-circle arcs AB and OC, each of radius r. Find the stated centre-of-mass results and the horizontal force F needed when the framework is pivoted at A.

Model solution

(a) The centre of mass of a circular arc lies on its line of symmetry. For a quarter-circle, the given arc formula first gives its distance from the centre O; resolving this at $45°$ gives its perpendicular distance from OA.

$$\alpha = \frac{\pi}{4}, \qquad \frac{r\sin\alpha}{\alpha} = \frac{r(1/\sqrt{2})}{\pi/4} = \frac{2\sqrt{2}r}{\pi}.$$

Thus its distance from OA is

$$\frac{2\sqrt{2}r}{\pi}\cos\frac{\pi}{4} = \frac{2r}{\pi}.$$

$$\text{The centre of mass of arc AB is } \frac{2r}{\pi} \text{ from OA.}$$

(2)

(b) Take moments of the separate pieces about OA. For uniform wire, mass is proportional to length, so lengths may be used in place of masses. The two arcs each have length $\pi r/2$ and each has perpendicular distance $2r/\pi$ from OA.

$$r\left(\frac{r}{2}\right) + r(r) + \frac{\pi r}{2}\left(\frac{2r}{\pi}\right) + \frac{2r}{\pi}\left(\frac{\pi r}{2}\right) = (3r + \pi r)d.$$

The left side is $\frac{7}{2}r^2$, while $3r + \pi r$ is the total wire length. Hence

$$\frac{7}{2}r^2 = (3+\pi)rd \quad \Longrightarrow \quad d = \frac{7r}{2(3+\pi)}.$$

$$\boxed{d = \frac{7r}{2(3+\pi)}.}$$

(4)

(c) With AO vertical, take moments about the pivot A. The weight acts vertically, at horizontal distance d from AO, and the horizontal force at C has perpendicular distance $AC = 2r$.

$$W\left(\frac{7r}{2(3+\pi)}\right) = F(2r).$$

Cancelling r gives

$$F = \frac{7W}{4(3+\pi)}.$$

$$\boxed{F = \frac{7W}{4(3+\pi)}.}$$

(3)

Question 4

Question 4

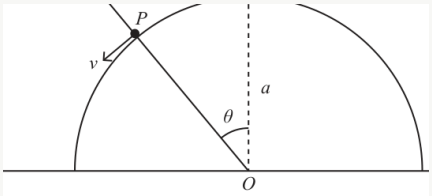


Figure 2

e of radius a is fixed on a horizontal surface with its pla
ace. The centre of the plane face of the hemisphere is O

M is disturbed from rest at the highest point of the hem

A particle of mass M slides from rest at the top of a smooth fixed hemisphere of radius a. At angle θ from the vertical its speed is v and normal reaction is R. Show the expression for R and find the speed when it leaves the surface.

Model solution

(a) The hemisphere is smooth, so the normal reaction does no work. Use conservation of energy to find v^2, then resolve radially inwards because the acceleration towards the centre is v^2/a.

$$\frac{1}{2}Mv^2 = Mga(1-\cos\theta), \qquad \textit{so} \qquad \frac{Mv^2}{a} = 2Mg(1-\cos\theta).$$

Taking inwards as positive, the inward component of weight is $Mg\cos\theta$, whereas R acts outwards:

$$\frac{Mv^2}{a} = Mg\cos\theta - R.$$

Therefore

$$R = Mg\cos\theta - 2Mg(1-\cos\theta) = Mg(3\cos\theta - 2).$$

$$\boxed{R = Mg(3\cos\theta - 2).}$$

(6)

(b) The particle leaves the surface precisely when contact is lost, so $R = 0$.

$$3\cos\theta - 2 = 0 \quad \Longrightarrow \quad \cos\theta = \frac{2}{3}.$$

Substitute this into the energy result:

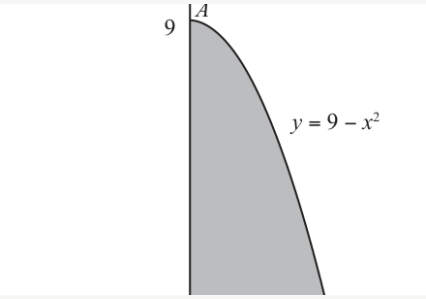
$$v^2 = 2ga\left(1 - \frac{2}{3}\right) = \frac{2ga}{3}.$$

$$\boxed{v = \sqrt{\frac{2ga}{3}}\ m\ s^{-1}.}$$

(3)

Question 5

Question 5



The uniform lamina is bounded by the axes and $y = 9 - x^2$, for $x \geqslant 0$, and has area 18 m^2. It hangs from vertical strings at O and A; the tension at A is λW.

Model solution

(a) The line OB is the x-axis, so the required distance is the vertical coordinate of the centre of mass. Find the moment about OB by using vertical strips: each strip has its centre at height $y/2$.

$$M_{OB} = \frac{1}{2}\int_0^3 (9 - x^2)^2\,dx = \frac{1}{2}\int_0^3 (81 - 18x^2 + x^4)\,dx = \frac{324}{5}.$$

Thus

$$\bar{y} = \frac{M_{OB}}{18} = \frac{324/5}{18} = \frac{18}{5} = 3.6.$$

The centre of mass is 3.6 *m* from OB.

(4)

(b) Both string tensions and the weight are vertical. Taking moments about O removes the unknown tension at O. When OA is horizontal, its length is 9 m and the weight has moment arm 3.6 m.

$$(\lambda W)(9) = W(3.6).$$

Hence

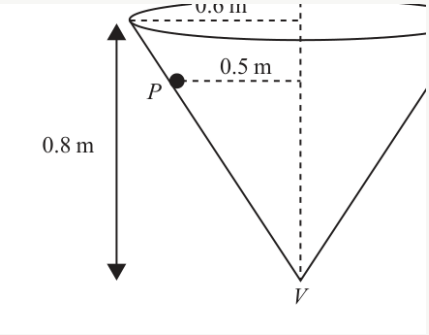
$$\lambda = 0.4.$$

$$\lambda = 0.4.$$

(3)

Question 6

Question 6



A particle of mass m moves in a horizontal circle of radius 0.5 m on the rough inner surface of a cone of base radius 0.6 m and height 0.8 m. The coefficient of friction is 0.25. Find the greatest angular speed ω.

Model solution

The cone's axial cross-section is a 3-4-5 *triangle, so for the angle* θ *shown between the surface and the vertical,* $\sin\theta = 3/5$ *and* $\cos\theta = 4/5$. *At the greatest speed the particle tends to move up the cone, so limiting friction* $F = \mu R = R/4$ *acts down the slope.*

$$F\cos\theta + mg = R\sin\theta \quad \Longrightarrow \quad \frac{4}{5}F + mg = \frac{3}{5}R,$$

or

$$4F + 5mg = 3R. \tag{1}$$

The horizontal components provide the centripetal force, directed towards the axis. Since the circle radius is 0.5 *m, its required force is* $m(0.5)\omega^2$.

$$F\sin\theta + R\cos\theta = m(0.5)\omega^2 \quad \Longrightarrow \quad 3F + 4R = 2.5m\omega^2. \tag{2}$$

At limiting friction, $F = R/4$. *Substituting in (1),*

$$R + 5mg = 3R \quad \Longrightarrow \quad R = \frac{5mg}{2}.$$

Then (2) gives

$$\frac{3}{4}R + 4R = 2.5m\omega^2 \quad \Longrightarrow \quad 19R = 10m\omega^2.$$

Hence

$$\omega^2 = \frac{19g}{4}, \qquad \omega = \frac{1}{2}\sqrt{19g} = 6.82\ldots$$

$$\omega_{\max} = 6.82 \text{ rad s}^{-1} (3\text{s.f.}).$$

(9)

Question 7

Question 7

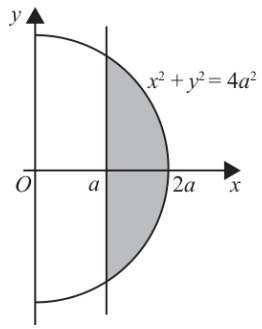


Figure 5

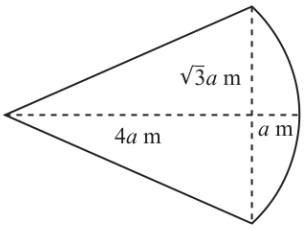


Figure 6

The spherical dome is formed by rotating the region $a \leqslant x \leqslant 2a$, $x^2 + y^2 \leqslant 4a^2$ about the x-axis. Its density is λ/x^2 kg m^{-3}. It is joined to a uniform cone as shown. Find the requested centre-of-mass results and k.

Model solution

(a) A slice perpendicular to the x-axis is a disc of area $\pi(4a^2 - x^2)$. Multiplying its volume by the stated density gives its mass, so integration gives the mass and first moment.

$$M = \int_a^{2a} \frac{\lambda}{x^2}\pi(4a^2 - x^2)\,dx = \pi\lambda \int_a^{2a} \left(\frac{4a^2}{x^2} - 1\right) dx = \pi\lambda a.$$

Its first moment about the plane $x = 0$ is

$$\begin{aligned} \mathcal{M} &= \pi\lambda \int_a^{2a} x\left(\frac{4a^2}{x^2} - 1\right) dx \\ &= \pi\lambda \left[4a^2 \ln x - \frac{x^2}{2}\right]_a^{2a} = \pi\lambda a^2 \left(4\ln 2 - \frac{3}{2}\right). \end{aligned}$$

Thus the centre has x-coordinate $a(4\ln 2 - 3/2)$. The plane face is at $x = a$, so its distance from that face is

$$a\left(4\ln 2 - \frac{3}{2}\right) - a = \left(4\ln 2 - \frac{5}{2}\right) a.$$

$$\text{Distance} = \left(4\ln 2 - \frac{5}{2}\right) a \ m.$$

(6)

(b) Measure signed distances from the common plane face, positive towards the dome. The dome's centre is at $\left(4\ln 2 - \frac{5}{2}\right) a$, while the uniform cone's centre is one quarter of its height from its base, namely a, on the other side. Take moments about the plane face.

$$2W\left(4\ln 2 - \frac{5}{2}\right) a - kW(a) = (k+2)Wd.$$

Therefore

$$d = \frac{8\ln 2 - 5 - k}{k+2}a,$$

and a distance is non-negative, so

$$d = \frac{|k+5-8\ln 2|}{k+2}a.$$

$$\boxed{d = \frac{|k+5-8\ln 2|}{k+2}a.}$$

(4)

(c) When suspended, the line from the suspension point to the centre of mass is vertical. The point is $\sqrt{3}a$ from the central axis in the plane face, so the right triangle gives $\tan\alpha = d/(\sqrt{3}a)$.

$$\frac{d}{\sqrt{3}a} = \frac{1}{2\sqrt{3}} \quad \Longrightarrow \quad d = \frac{a}{2}.$$

Using the expression from part (b),

$$\frac{k+5-8\ln 2}{k+2} = \frac{1}{2} \quad \Longrightarrow \quad 2k+10-16\ln 2 = k+2.$$

Thus

$$k = 16\ln 2 - 8.$$

$$\boxed{k = 16\ln 2 - 8.}$$

(3)

Question 8

Question 8

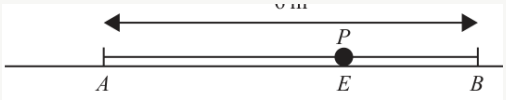


Figure 7

nd B lie on a smooth horizontal surface with $AB = 6\,\text{m}$.

s $0.3\,\text{kg}$.

astic string, of natural length 2 m and modulus of elasti
e other end is attached to A.

Fixed points A and B are 6 m apart. A particle of mass 0.3 kg is attached to strings AP and BP, each of natural length 2 m, with moduli 20 N and 40 N respectively. It is released from the midpoint of AB. Find the equilibrium point, establish SHM, and answer the timing questions.

Model solution

(a) Let $\mathrm{EB} = e$. *At equilibrium the tensions are equal. Since* $\mathrm{AE} = 6 - e$, *the extensions of the strings are* $4 - e$ *and* $e - 2$ *respectively.*

$$\frac{20}{2}(4 - e) = \frac{40}{2}(e - 2).$$

$$10(4 - e) = 20(e - 2) \quad \Longrightarrow \quad e = \frac{8}{3}.$$

$$\mathrm{EB} = \frac{8}{3}\ \textit{m}.$$

(3)

(b) Let x *be the displacement from* E *towards* A. *The extensions become* $4 - e - x$ *and* $e + x - 2$. *Resolve along* AB, *taking the direction towards* A *as positive.*

$$0.3\frac{d^2x}{dt^2} = 10(4 - e - x) - 20(e + x - 2).$$

The constant terms cancel because E *is the equilibrium position. With* $e = 8/3$,

$$0.3\frac{d^2x}{dt^2} = 10\left(\frac{4}{3} - x\right) - 20\left(\frac{2}{3} + x\right) = -30x,$$

so

$$\frac{d^2x}{dt^2} = -100x.$$

This has the standard SHM form $\frac{d^2x}{dt^2} = -\omega^2 x$, *so the motion is simple harmonic with* $\omega = 10$ *rad* s^{-1}.

$$\mathrm{P\ executes\ SHM\ about\ E, with}\ \omega = 10\ \textit{rad}\ s^{-1}.$$

(4)

(c) The midpoint is 3 m from B, whereas E is $8/3$ m from B, so the initial displacement from equilibrium is $1/3$ m. Released from rest means it starts at an extreme position. The first arrival at equilibrium is one quarter of a period.

$$T = \frac{2\pi}{10} = \frac{\pi}{5}, \qquad S = \frac{T}{4} = \frac{\pi}{20}.$$

$$S = \frac{\pi}{20} \text{ s}.$$

(3)

(d) The amplitude is $A = 1/3$ and $\omega = 10$. The speed formula $v^2 = \omega^2(A^2 - x^2)$ identifies the two positions at which the speed is 2 m s^{-1}.

$$4 = 100\left(\frac{1}{9} - x^2\right) \quad \Longrightarrow \quad x = \pm\frac{4}{15}.$$

Starting at an extreme, $x = \frac{1}{3}\cos(10t)$, so the first crossing occurs at

$$t_1 = \frac{1}{10}\cos^{-1}\left(\frac{4}{5}\right).$$

In each half-period, the speed exceeds 2 for the interval between the two crossings, of length $\frac{\pi}{10} - 2t_1$. There are two such intervals per full oscillation, so the required time is

$$2\left(\frac{\pi}{10} - \frac{1}{5}\cos^{-1}\left(\frac{4}{5}\right)\right) = 0.370918\ldots \text{ s}.$$

$$0.371 \text{ s} (3\text{s.f.}).$$

(4)