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Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/4D - Decision Mathematics 2

Tuesday 25 June 2019

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2019 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Contents

Question 1 2

Question 2 4

Question 3 5

Question 4 7

Question 5 9

Question 6 10

Question 7 12

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Question 1

Question 1

Use the stepping-stone method to improve the given north-west-corner transportation solution, test the second solution for optimality, and find its cost.

Model solution

(a) The entering cell is specified as DQ. A closed stepping-stone route alternates between occupied cells and the one empty entering cell, so that row and column totals are unchanged.

$$\mathrm{DQ}(+),\ \mathrm{CQ}(-),\ \mathrm{CS}(+),\ \mathrm{DS}(-),\ \mathrm{DQ}.$$

The minus cells contain $5, 19$*, so the largest permissible change is*

$$\theta = \min(5, 19) = 5.$$

Thus the revised allocations are

	P	Q	R	S
A	23	0	0	0
B	2	40	0	0
C	0	0	12	6
D	0	5	0	14

$$\mathrm{The first improved solution has CQ} = 0, \mathrm{CS} = 6, \mathrm{DQ} = 5, \mathrm{and DS} = 14.$$

(2)

(b) Shadow costs are found from $c_{ij} = r_i + s_j$ *in each occupied cell. Set* $r_A = 0$*; this fixes the otherwise arbitrary scale.*

	P	Q	R	S	r_i
A	×	0	1	−7	0
B	×	×	5	−1	−5
C	4	7	×	×	−8
D	1	×	1	×	−1
s_j	15	14	16	18	

Here each displayed interior number is the improvement index $c_{ij} - (r_i + s_j)$ *for an unoccupied cell. The most negative index is* -7 *at AS, so AS enters. Its route is*

$$\mathrm{AS}(+),\ \mathrm{AP}(-),\ \mathrm{BP}(+),\ \mathrm{BQ}(-),\ \mathrm{DQ}(+),\ \mathrm{DS}(-),\ \mathrm{AS}.$$

The minus allocations are $23, 40, 14$*, hence* $\theta = 14$ *and DS exits. Applying alternating* $+\theta$ *and* $-\theta$ *gives*

	P	Q	R	S
A	9	0	0	14
B	16	26	0	0
C	0	0	12	6
D	0	19	0	0

Enter AS and exit DS; the second improved allocation is as displayed.

(4)

(c) To test optimality, calculate fresh shadow costs and improvement indices. A minimisation solution is optimal only if every unused-cell index is non-negative.

	P	Q	R	S	r_i
A	×	0	8	×	0
B	×	×	12	6	−5
C	−3	0	×	×	−1
D	1	×	8	7	−1
s_j	15	14	9	11	

Since the improvement index at CP is -3, sending units round its route would reduce the cost. Therefore the current solution is not optimal.

Notoptimal, becauseCPhasnegativeimprovementindex $-$ 3.

(3)

(d) Multiply each non-zero allocation by its unit transport cost and add.

$$9(15) + 14(11) + 16(10) + 26(9) + 12(8) + 6(10) + 19(13) = 1086.$$

Thecostofthesolutioninpart(b)is$1086.

(1)

Question 2

Question 2

Use the Hungarian algorithm, reducing rows first, to maximise the profit from assigning the four workers to the four tasks.

Model solution

(a) The Hungarian algorithm minimises a cost matrix. Because these are profits, subtract every entry from 203, the largest profit; minimising the resulting opportunity-cost matrix is equivalent to maximising profit.

	1	2	3	4
T	100	106	129	123
H	2	48	58	48
J	92	123	126	111
M	0	15	66	19

Reduce each row by its row minimum, then each column by its column minimum.

	1	2	3	4
T	0	6	29	23
H	0	46	56	46
J	0	31	34	19
M	0	15	66	19

$\longrightarrow$

	1	2	3	4
T	0	0	0	4
H	0	40	27	27
J	0	25	5	0
M	0	9	37	0

The zeros cannot yet be chosen independently for all four rows. Cover the zeros with the minimum number of lines; using the usual adjustment, subtract the least uncovered value 5 from uncovered entries and add it at intersections of two lines. This gives

	1	2	3	4
T	5	0	0	9
H	0	35	22	27
J	0	20	0	0
M	0	4	32	0

Now select independent zeros: $\mathrm{T} \to 2$, $\mathrm{H} \to 1$, $\mathrm{J} \to 3$, $\mathrm{M} \to 4$. *There is one selected zero in every row and every column, so this is optimal.*

$$\mathrm{T} \to 2,\ \mathrm{H} \to 1,\ \mathrm{J} \to 3,\ \mathrm{M} \to 4.$$

(6)

(b) Return to the original profit table to calculate the actual objective value.

$$97 + 201 + 77 + 184 = 559.$$

$$\text{Maximum total profit} = \$559.$$

(1)

Question 3

Question 3

For the directed network in Figure 1, explain why Dijkstra's algorithm is unsuitable and use dynamic programming to determine x and the minimum route.

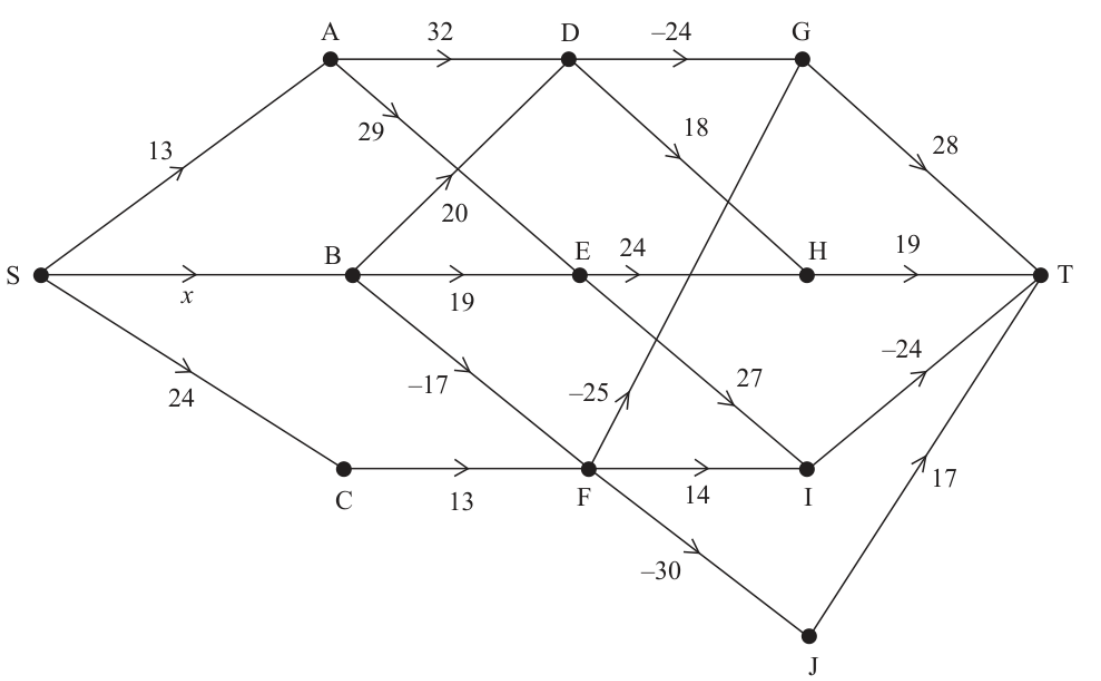


Figure 1

Model solution

(a) Dijkstra's algorithm relies on non-negative arc weights, so that a permanently labelled vertex cannot later be improved. This network has negative arc weights, for example FJ $= -30$.

Dijkstra's algorithm cannot be used because the network contains negative weights.

(1)

(b) Dynamic programming is appropriate because the network is staged from left to right. Work backwards: at each vertex choose the outgoing arc which gives the smallest total weight to T.

stage	*state*	*best action*	*value to T*
0	G	GT	28
	H	HT	19
	I	IT	-24
	J	JT	17
1	D	DG	$-24 + 28 = 4$
	E	EI	$27 - 24 = 3$
	F	FJ	$-30 + 17 = -13$
2	A	AE	$29 + 3 = 32$
	B	BF	$-17 - 13 = -30$
	C	CF	$13 - 13 = 0$

For completeness, the comparisons used are

$$\begin{array}{c|l} D & \min(4, 18+19) = 4 \\ E & \min(24+19, 27-24) = 3 \\ F & \min(-25+28, 14-24, -30+17) = -13 \\ A & \min(32+4, 29+3) = 32 \\ B & \min(20+4, 19+3, -17-13) = -30 \\ C & 13-13 = 0. \end{array}$$

At S the three possible route values are

$$SA : 13+32 = 45, \qquad SB : x-30, \qquad SC : 24+0 = 24.$$

The stated fact that the minimum route passes through B means $x-30 < 24$*, so*

$$0 \leqslant x < 54.$$

$$(b)(i) 0 \leqslant x < 54.$$

(b)(ii) Trace the selected minimum action at each stage, beginning with the required arc SB. Following the chosen actions from B gives $B \to F \to J \to T$.

$$(b)(ii) \text{Minimum route } S \to B \to F \to J \to T.$$

(12)

Question 4

Question 4

Analyse the given zero-sum game, correct its linear-programming model, and determine both players' optimal strategies.

Model solution

(a) A stable game has equal maximin and minimax values. Compare A's worst outcome in each row with A's best outcome in each column.

row minima $= -2, -4, -1, \quad \text{maximin} = -1;$ *column maxima* $= 3, 4, 0, \quad \text{minimax} = 0.$

Since $-1 \neq 0$*, there is no saddle point.*

There is no stable solution.

(2)

(b) Simplex requires a non-negative game value. Add 5 to every payoff (at least 4 is needed), which does not change either optimal strategy but increases the value by 5. Also, V *is the guaranteed minimum expected payoff, so it must be no greater than the expected payoff against each column.*

maximise V

$$V \leqslant 8p_1 + p_2 + 6p_3,$$
$$V \leqslant 3p_1 + 9p_2 + 7p_3,$$
$$V \leqslant 5p_1 + 3p_2 + 4p_3,$$
$$p_1 + p_2 + p_3 = 1, \qquad p_1, p_2, p_3, V \geqslant 0.$$

The corrected inequalities are those displayed; the original signs must be $V \leqslant \cdots$.

(3)

(c) Introduce non-negative slack variables r, s, t *for the three payoff constraints and* u *for the probability equation. This gives the initial Simplex tableau.*

b.v.	V	p_1	p_2	p_3	r	s	t	u	*Value*
r	1	−8	−1	−6	1	0	0	0	0
s	1	−3	−9	−7	0	1	0	0	0
t	1	−5	−3	−4	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

Initial tableau as displayed.

(3)

(d) The optimal probabilities are given. Substitute them into the shifted constraints; the smallest upper bound is the shifted value.

$$8(0.6) + 0 + 6(0.4) = 7.2, \quad 3(0.6) + 0 + 7(0.4) = 4.6, \quad 5(0.6) + 0 + 4(0.4) = 4.6.$$

Thus $V = 4.6$ *for the game with 5 added, so the original value is* $4.6 - 5 = -0.4$.

$$\text{Value to player A} = -0.4.$$

(2)

(e) Since A mixes only P and R, use B's columns to see which can be binding. B must make A indifferent between P and R using Y and Z. Let q *be B's probability of Y, so Z has probability* $1 - q$.

$$\text{payoff for P} = -2q, \qquad \text{payoff for R} = 2q - (1 - q) = 3q - 1.$$

At equilibrium both equal the value -0.4:

$$-2q = -0.4 \quad \Rightarrow \quad q = 0.2,$$

and $3(0.2) - 1 = -0.4$ *confirms it. Column X gives A a larger payoff and is therefore not used by the minimising player B.*

$$\text{B plays X with probability 0, Y with probability 0.2, and Z with probability 0.8.}$$

(4)

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Question 5

Question 5

Solve the recurrence relation defined by the stated mean condition and determine its asymptotic linear behaviour.

Model solution

(a) "The difference … is 6" translates directly into an equation involving the current term and the mean of the two preceding terms.

$$u_n - \frac{u_{n-1} + u_{n-2}}{2} = 6.$$

Multiplying by 2 and rearranging gives

$$2u_n - u_{n-1} - u_{n-2} = 12.$$

$$\boxed{2u_n - u_{n-1} - u_{n-2} = 12.}$$

(2)

(b) First solve the homogeneous recurrence. Its auxiliary equation is

$$2m^2 - m - 1 = 0 \quad \Rightarrow \quad m = 1,\ -\frac{1}{2},$$

so $u_n^{(h)} = A + B(-\frac{1}{2})^n$. *The non-zero constant right-hand side requires a particular solution of the form* $u_n = \lambda n$, *because* $m = 1$ *is already a homogeneous root. Substitution gives*

$$2\lambda n - \lambda(n-1) - \lambda(n-2) = 3\lambda = 12,$$

so $\lambda = 4$. *Hence*

$$u_n = A + B\left(-\frac{1}{2}\right)^n + 4n.$$

Use the initial values:

$$u_1 = 2: \quad 2A - B = -4, \qquad u_2 = 8: \quad 4A + B = 0.$$

Solving, $A = -\frac{2}{3}$ *and* $B = \frac{8}{3}$. *Therefore*

$$\boxed{u_n = -\tfrac{2}{3} + \tfrac{8}{3}\left(-\tfrac{1}{2}\right)^n + 4n.}$$

(7)

(c) As $n \to \infty$, $(-\frac{1}{2})^n \to 0$, *and the constant* $-\frac{2}{3}$ *is negligible compared with the linear term* $4n$.

$$u_n \to 4n.$$

$$\boxed{k = 4.}$$

(2)

Question 6

Question 6

For the capacitated network in Figure 2, find cut capacities and use lower and upper bounds to construct minimum and maximum feasible flows.

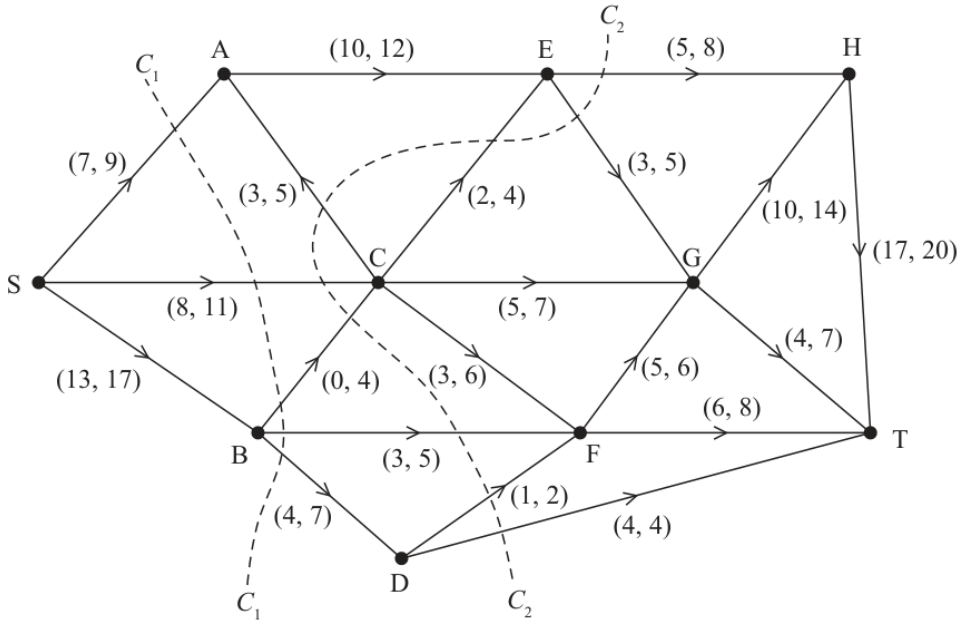


Figure 2

Figure 2 shows a capacitated, directed network. The network represents a system of pipes through which fluid flows from a source, S, to a sink, T.

Model solution

(a)(i) A cut capacity is the sum of upper capacities on forward arcs minus the sum of lower capacities on backward arcs. For C_1, all five crossings are forward.

$$\text{cap}(C_1) = 9 + 11 + 4 + 5 + 7 = 36.$$

$$\text{cap}(C_1) = 36 \text{ L s}^{-1}.$$

(1)

(a)(ii) For C_2, include the appropriate sign for each crossing direction.

$$\text{cap}(C_2) = 8 + 5 - 2 - 3 + 11 + 4 + 5 + 2 + 4 = 34.$$

$$\text{cap}(C_2) = 34 \text{ L s}^{-1}.$$

(1)

(b) Compare the possible flow into and out of E. If AE and CE were both at upper capacity, E would receive $12 + 4 = 16$ L s^{-1}, but EH and EG can carry at most $8 + 5 = 13$ L s^{-1}. Conservation of flow would fail.

AE and CE cannot both be at upper capacity.

(1)

(c) The arcs AE, CE, CG, FG, FT and DT form a cut whose lower capacities must all be supplied.

$$10 + 2 + 5 + 5 + 6 + 4 = 32.$$

Thus every feasible flow is at least 32 L s^{-1}, so a flow of 31 L s^{-1} is impossible.

$31\text{Ls}^{-1}\text{isnotfeasible.}$

(1)

(d) The lower-bound cut in part (c) suggests trying 32. The following assignments list the flow on every arc and satisfy conservation at every intermediate vertex.

arc	SA	SC	SB	CA	AE	BC	BF	BD	DF	CE	CG	CF	EH	EG	FG	FT	GH	GT	H
32 *flow*	7	10	15	3	10	4	5	6	2	2	5	4	7	5	5	6	10	5	1
34 *flow*	8	11	15	3	11	4	5	6	2	2	5	5	8	5	5	7	10	5	1

For example, in the 32-flow, node C receives $10 + 4 = 14$ and sends $3 + 2 + 5 + 4 = 14$; the other nodes check similarly. Thus 32 is feasible and, by part (c), is the minimum. The 34-flow is also feasible. Since C_2 has capacity 34, no larger flow is possible, making it the maximum.

$\text{Minimumfeasibleflow} = 32\text{Ls}^{-1}; \text{maximumfeasibleflow} = 34\text{Ls}^{-1}.$

(4)

Question 7

Question 7

Use a decision tree and expected utility to decide whether Aisha should play the three-dice game.

Model solution

(a)(i) The favourable totals are 16, 17 and 18. Counting ordered dice outcomes gives $6+3+1=10$ *favourable outcomes from* $6^3=216$*, so* $\mathrm{P}(\textit{win})=10/216=5/108$*. The decision tree is therefore*

$$\textit{Decision:}\quad \begin{cases} \textit{Do not play} \longrightarrow \$0, \\ \textit{Play} \longrightarrow \begin{cases} \textit{win},\ 10/216 \longrightarrow \$15, \\ \textit{lose},\ 206/216 \longrightarrow -\$3. \end{cases} \end{cases}$$

The expected monetary value at the chance node is

$$\frac{10}{216}(15)+\frac{206}{216}(-3)=\frac{150-618}{216}=-\frac{13}{6}.$$

The play branch is inferior to 0, so it is rejected at the decision node.

(a)(i) Tree as displayed.

(a)(ii) At the decision node compare the EMV of playing, $-13/6$*, with the certain outcome 0 from not playing.*

$$\mathrm{(a)(ii)OptimalEMV}=\$0;\mathrm{Aishashouldnotplay.}$$

(6)

(b) If she plays, a loss leaves Aisha with $-3+3=0$*, whose utility is* $u(0)=0$*. A win leaves her with* $x+3$*, so only the winning branch contributes to expected utility.*

$$E(U)=\frac{10}{216}\left(1-e^{-(x+3)/500}\right)+\frac{206}{216}(1-e^{0})=\frac{10}{216}\left(1-e^{-(x+3)/500}\right).$$

$$E(U)=\frac{10}{216}\left(1-e^{-(x+3)/500}\right).$$

(2)

(c) If she does not play, she keeps £3, giving utility $1-e^{-3/500}$*. Playing is worthwhile when its expected utility is at least this value.*

$$\frac{10}{216}\left(1-e^{-(x+3)/500}\right)>1-e^{-3/500}.$$

Rearranging before taking logs,

$$e^{-(x+3)/500}<1-\frac{216}{10}\left(1-e^{-3/500}\right),$$

and hence

$$x > -500 \ln\left[1 - \frac{216}{10}\left(1 - e^{-3/500}\right)\right] - 3 = 66.178\ldots$$

Money is paid to the nearest penny, so the smallest prize exceeding this threshold is £66.18.

Minimumprize : $66.18.

(4)

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