Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/4D - Decision Mathematics 2

Tuesday 28 June 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Four workers (A,B,C,D) must be assigned one each to tasks (1,2,3,4), at minimum cost. The cost matrix is

	1	2	3	4
A	32	45	34	48
B	37	39	50	46
C	46	44	40	42
D	43	45	48	52

Model solution

a *The Hungarian algorithm is appropriate because this is a one-to-one minimum-cost assignment. First subtract the smallest entry in each row, then the smallest entry in each column; neither operation changes which allocation is optimal.*

	1	2	3	4
A	0	13	2	16
B	0	2	13	9
C	6	4	0	2
D	0	2	5	9

$\longrightarrow$

	1	2	3	4
A	0	11	2	14
B	0	0	13	7
C	6	2	0	0
D	0	0	5	7

The zeros cannot yet be covered with four independent lines. Cover the zeros in the usual way; the least uncovered entry is 2. Subtract 2 from every uncovered entry and add 2 at each intersection of two covering lines. This gives

	1	2	3	4
A	0	11	0	12
B	0	0	11	5
C	8	4	0	0
D	0	0	3	5

There are now four independent zeros. Choosing one in each row and column gives, for example,

$$A \to 3, \qquad B \to 1, \qquad C \to 4, \qquad D \to 2.$$

Allocate A to 3, B to 1, C to 4 and D to 2.

(5) b *Return to the original costs when finding the cost of the allocation.*

$$34 + 37 + 42 + 45 = 158.$$

Minimum total cost $= 158$.

(1)

Question 2

Question 2

The recurrence $u_{n+2} + k_1 u_{n+1} + k_2 u_n = 0$ has general solution $u_n = (A + Bn)(-3)^n$, where $A, B \neq 0$. Also $u_0 = u_1 = 1$.

Model solution

(a) The factor $A + Bn$ tells us that the auxiliary equation has a repeated root. Since the root is -3, its auxiliary equation is

$$(m+3)^2 = 0 \quad \Rightarrow \quad m^2 + 6m + 9 = 0.$$

Comparing this with $m^2 + k_1 m + k_2 = 0$ identifies the two coefficients.

$$k_1 = 6, \quad k_2 = 9.$$

(2) b Substitute the initial conditions into the given general solution, using $n = 0$ and $n = 1$.

$$u_0 = A = 1,$$

$$u_1 = (A + B)(-3) = 1 \quad \Rightarrow \quad 1 + B = -\frac{1}{3}.$$

$$B = -\frac{4}{3}.$$

$$(A = 1, \quad B = -\frac{4}{3}.)$$

(2)

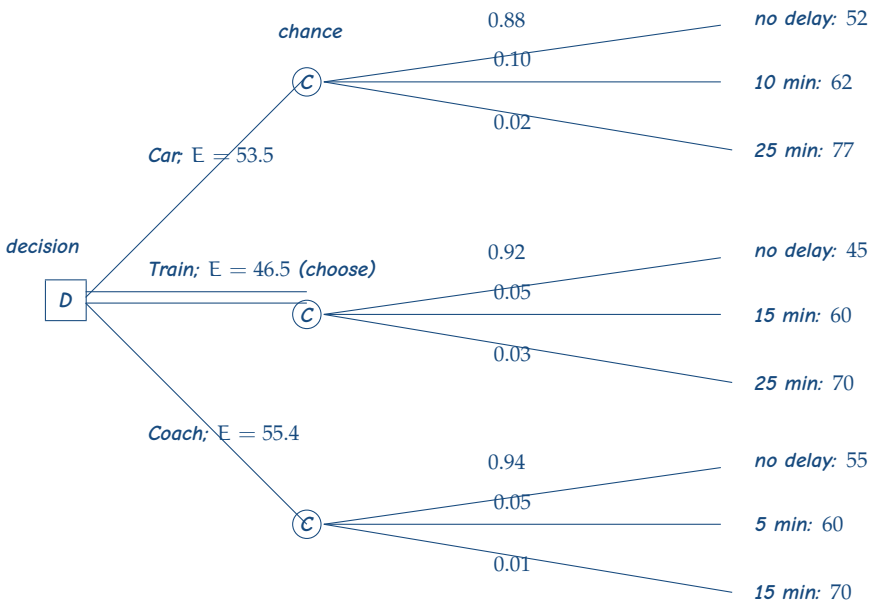
Question 3

Question 3

Usual journey times, possible delays and delay probabilities (in minutes) are: car 52, delays 10 with 0.10 and 25 with 0.02; train 45, delays 15 with 0.05 and 25 with 0.03; coach 55, delays 5 with 0.05 and 15 with 0.01.

Model solution

a *The decision node branches by transport choice. At each chance node, the probability of no stated delay is the remainder to* 1. *The completed decision tree is:*



The double line marks the selected train branch. Work back through the tree by multiplying each terminal time by its branch probability.

$$\mathrm{E}(\textit{car}) = 0.8852 + 0.1062 + 0.0277 = 53.5,$$
$$\mathrm{E}(\textit{train}) = 0.9245 + 0.0560 + 0.0370 = 46.5,$$
$$\mathrm{E}(\textit{coach}) = 0.9455 + 0.0560 + 0.0170 = 55.4.$$

The completed decision tree is as displayed; train has the smallest expected travel time.

(5) b *At a decision node involving travel time, choose the smallest expected value.*

Minimumexpectedtraveltime = 46.5minutes, usingthetrain.

(2)

Question 4

Question 4

Figure 1 is the given capacitated directed pipe network from source S to sink T, with circled initial flows.

Model solution

a *An arc is saturated when its flow equals its capacity. Comparing the labelled flows and capacities gives*

$$\textit{(AE,,BE,,BF,,BG,,DB,,DT,,SB) are saturated.}$$

(1) b *The value of a flow is the total leaving the source (equivalently, the total entering the sink).*

$$29 + 41 + 25 = 95.$$

$$\text{Initialflowvalue} = 95.$$

(1) c *Conservation of flow at* F *means that its outgoing flow cannot exceed what can enter it.*

$$\textit{maximum into } \mathrm{F} = 22 \ (\textit{from } \mathrm{BF} \textit{ and } \mathrm{DF}), \qquad \textit{capacity of } \mathrm{FT} = 24.$$

$$\text{Sinceatmost22} < 24\text{unitscanreachF, (FT)cannotbefulltocapacity.}$$

(1) (d) The capacity of a cut is the sum of capacities of all arcs crossing it from the source side to the sink side.

$$C_1 = 33 + 41 + 30 = 104, \qquad C_2 = 53 + 30 + 14 + 0 + 17 = 114.$$

$$(C_1 = 104, \quad C_2 = 114.)$$

(2) (e) For an augmenting route, every forward arc must have at least three units of spare capacity. The route

$$S \to A \to B \to D \to F \to T$$

has residual capacities at least 3 *throughout (the step* B → D *uses the residual reverse arc of* DB*).*

$$\text{Augmentby3on(SABDFT), increasingtheflowto98.}$$

(1) (f) To prove maximality, use the max-flow min-cut theorem. After this augmentation, the cut through the saturated arcs

$$\mathrm{AE},\ \mathrm{BE},\ \mathrm{BG},\ \mathrm{BF},\ \mathrm{DF},\ \mathrm{DT}$$

has capacity

$$20 + 22 + 30 + 22 + 8 + 4 = 98.$$

The augmented flow also has value $95 + 3 = 98$. *A feasible flow equal in value to a cut capacity is a maximum flow, because that cut is a minimum cut.*

The flow of value 98 is maximal.

(3)

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Question 5

Question 5

A balanced transportation problem has supplies (34,27,41,18) at (A,B,C,D), demands (44,37,k) at (R,S,T), and the stated cost matrix.

Model solution

a *In a balanced transportation problem, total supply equals total demand.*

$$34 + 27 + 41 + 18 = 44 + 37 + k \quad \Rightarrow \quad 120 = 81 + k.$$

$$k = 39.$$

(1) b *The index* i *ranges over the four supply points. Thus the constraint says*

$$\text{Thetotaltransportedtodestination R from } (A, B, C, D) \text{ must be at least its demand}$$

(2) c *The north-west corner method starts at the top-left cell and allocates as much as possible, moving right when a demand is met and down when a supply is exhausted.*

	R	S	T	*supply*
A	34			34
B	10	17		27
C		20	21	41
D			18	18
demand	44	37	39	

$$P = 3423 + 1015 + 1729 + 2025 + 2127 + 1825 = 2942.$$

$$\text{Initial north} - \text{west} - \text{corner cost} = 2942.$$

(2) (d) Use the occupied cells to find shadow costs u_i, v_j*, where* $u_i + v_j = c_{ij}$*. Taking* $u_A = 0$ *gives*

$$u_A = 0, \quad u_B = -8, \quad u_C = -12, \quad u_D = -14; \qquad v_R = 23, \; v_S = 37, \; v_T = 39.$$

For each unoccupied cell, its improvement index is $c_{ij} - (u_i + v_j)$.

	R	S	T
A	×	−20	−15
B	×	×	1
C	14	×	×
D	10	−3	×

The most negative index is −20*, so* AS *enters. Its closed stepping-stone route is*

$$AS+, \quad AR-, \quad BR+, \quad BS-, \quad AS.$$

The negative cells contain 34 *and* 17*, hence* $\theta = 17$*. Subtracting and adding* θ *alternately gives*

	R	S	T
A	17	17	
B	27		
C		20	21
D			18

Entering cell (AS); exiting cell (BS); the improved solution is as displayed.

(4)

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Question 6

Question 6

Shed demands from January to May are (1,3,3,5,2). At most four sheds can be made each month and at most three can be stored; storage costs (£80) per shed per month, production overhead is (£35) per shed, and making more than two sheds costs a further (£250).

Model solution

Dynamic programming is appropriate because a decision this month determines the stock state for the next. Work backwards. A state is stock at the start of the month, an action is the number made, and the destination is next month's starting stock. Each value is the current production/storage cost plus the optimal value at the destination. A star marks the least value for that state. For May, no stock may remain after delivery of 2, so the only feasible actions are

state	*action*	*destination*	*value* ($)
2	0	0	160^*
1	1	0	$80 + 35 = 115^*$
0	2	0	70^*

For April (demand 5), use the May values just found.

3	2	0	$240 + 70 + 70 = 380^*$
3	3	1	$240 + 105 + 250 + 115 = 710$
3	4	2	$240 + 140 + 250 + 160 = 790$
2	3	0	$160 + 105 + 250 + 70 = 585^*$
2	4	1	$160 + 140 + 250 + 115 = 665$
1	4	0	$80 + 140 + 250 + 70 = 540^*$

For March (demand 3), use the optimum April value for each destination.

3	1	1	$240 + 35 + 540 = 815^*$
3	2	2	$240 + 70 + 585 = 895$
3	3	3	$240 + 105 + 250 + 380 = 975$
2	2	1	$160 + 70 + 540 = 770^*$
2	3	2	$160 + 105 + 250 + 585 = 1100$
2	4	3	$160 + 140 + 250 + 380 = 930$
1	3	1	$80 + 105 + 250 + 540 = 975^*$
1	4	2	$80 + 140 + 250 + 585 = 1055$
0	4	1	$140 + 250 + 540 = 930^*$

For February (demand 3), the same state-action-destination rule gives

3	0	0	$240 + 930 = 1170$
3	1	1	$240 + 35 + 975 = 1250$
3	2	2	$240 + 70 + 770 = 1080^*$
3	3	3	$240 + 105 + 250 + 815 = 1410$
2	1	0	$160 + 35 + 930 = 1125^*$
2	2	1	$160 + 70 + 975 = 1205$
2	3	2	$160 + 105 + 250 + 770 = 1285$
2	4	3	$160 + 140 + 250 + 815 = 1365$
1	2	0	$80 + 70 + 930 = 1080^*$
1	3	1	$80 + 105 + 250 + 975 = 1410$
1	4	2	$80 + 140 + 250 + 770 = 1240$
0	3	0	$105 + 250 + 930 = 1285^*$
0	4	1	$140 + 250 + 975 = 1365$

Finally January begins with state 0 and has demand 1.

0	1	0	$35 + 1285 = 1320$
0	2	1	$70 + 1080 = 1150^*$
0	3	2	$105 + 250 + 1125 = 1480$
0	4	3	$140 + 250 + 1080 = 1470$

Trace the starred choices forward: January makes 2 and leaves 1; February makes 2 and leaves 0; March makes 4 and leaves 1; April makes 4 and leaves 0; May makes 2.

month	*Jan*	*Feb*	*Mar*	*Apr*	*May*
number made	2	2	4	4	2

$\text{Optimal production schedule} (= (2, 2, 4, 4, 2)); \text{minimum cost} (= \$1150).$

(14)

Question 7

Question 7

Player A's payoff matrix has rows (Q,R,S), columns (W,X,Y,Z), and entries $\begin{pmatrix} 4 & 3 & -1 & -2 \\ -3 & 5 & -4 & k \\ -1 & 6 & 3 & -3 \end{pmatrix}$, where k is an integer.

Model solution

(a) For player A, the play-safe rule is to choose the row with the largest row minimum.

$$\min(Q) = -2, \qquad \min(R) = \min\{k, -4\}, \qquad \min(S) = -3.$$

Whether the middle minimum is -4 or $k < -4$, it is below -2.

$$\text{Qisplay} - \text{safe, sinceitsrowminimum} - 2\text{exceedsbothotherrowminima.}$$

(2) (b) For player B, calculate column maxima and choose the smallest.

$$\max(W) = 4, \quad \max(X) = 6, \quad \max(Y) = 3, \quad \max(Z) = \max\{k, -2\}.$$

For Z to be play-safe its column maximum must be less than 3. As k is an integer, this means $k \leqslant 2$.

$$(k \leqslant 2).$$

(2) c Player B minimises A's payoff. Column Y dominates column X because every entry in Y is less than the corresponding entry in X:

$$-1 < 3, \qquad -4 < 5, \qquad 3 < 6.$$

$$\text{BshouldneverplayX : itisdominatedbyY.}$$

(2) di Discard the dominated column X, then add 4 to every remaining payoff so all entries are non-negative:

$$\begin{pmatrix} 4 & -1 & -2 \\ -3 & -4 & k \\ -1 & 3 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 8 & 3 & 2 \\ 1 & 0 & k+4 \\ 3 & 7 & 1 \end{pmatrix}.$$

If A uses probabilities p_1, p_2, p_3, the expected shifted payoff against each remaining column must be at least V. Introducing slack variables gives the equations

$$\begin{aligned} V - 8p_1 - p_2 - 3p_3 + r &= 0, \\ V - 3p_1 - 7p_3 + s &= 0, \\ V - 2p_1 - k + 4p_2 - p_3 + t &= 0, \\ p_1 + p_2 + p_3 + u &= 1. \end{aligned}$$

Here $r, s, t, u \geqslant 0$, as are the probabilities. *d(ii) Taking (r,s,t,u) as the initial basic variables gives the following simplex tableau for maximising $P = V$.*

	V	p_1	p_2	p_3	r	s	t	u	*Value*
r	1	−8	−1	−3	1	0	0	0	0
s	1	−3	0	−7	0	1	0	0	0
t	1	−2	$-k+4$	−1	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

The constraints and initial tableau are as displayed.

(7) *e The final tableau gives $p_1 = \frac{7}{37}$, $p_2 = \frac{17}{37}$, and all slacks zero. The probability equation first gives*

$$p_3 = 1 - \frac{7}{37} - \frac{17}{37} = \frac{13}{37}.$$

Use either equation not involving k to find V, for example the first:

$$V - 8\left(\frac{7}{37}\right) - \frac{17}{37} - 3\left(\frac{13}{37}\right) = 0 \quad \Rightarrow \quad V = \frac{112}{37}.$$

Now use the third equality, whose slack is zero.

$$\frac{112}{37} - 2\left(\frac{7}{37}\right) - k + 4\left(\frac{17}{37}\right) - \frac{13}{37} = 0.$$

$$112 - 14 - 17k - 68 - 13 = 0 \quad \Rightarrow \quad 17 - 17k = 0.$$

$$k = 1.$$

(4)

Question 8

Question 8

A proportion p, $0 < p < 1$, of customers is retained each month and k new customers are added. $u_1 = 5000$.

Model solution

a *Retaining a proportion of the previous total and then adding a fixed number gives the first-order recurrence*

$$u_{n+1} = pu_n + k.$$

Solve it as a linear recurrence. The complementary solution comes from $m - p = 0$, *so it is* Ap^n. *For a constant particular solution* λ,

$$\lambda = p\lambda + k \quad \Rightarrow \quad \lambda = \frac{k}{1-p}.$$

Therefore

$$u_n = Ap^n + \frac{k}{1-p}.$$

Use $u_1 = 5000$ *to determine* A:

$$5000 = Ap + \frac{k}{1-p} \quad \Rightarrow \quad A = \frac{1}{p}\left(5000 - \frac{k}{1-p}\right).$$

$$u_n = \left(5000 - \frac{k}{1-p}\right)p^{n-1} + \frac{k}{1-p}.$$

$$u_n = 5000p^{n-1} + \frac{k1-p^{n-1}}{1-p}.$$

(6) b *Substitute* $p = 0.95$, $k = 10000$, *and set the model's value equal to the threshold to locate the crossing.*

$$135000 = 50000.95^{m-1} + \frac{10000\left(1-0.95^{m-1}\right)}{0.05}.$$

$$135000 = 200000 - 1950000.95^{m-1} \quad \Rightarrow \quad 0.95^{m-1} = \frac{1}{3}.$$

Logarithms are used because the unknown is an exponent.

$$m - 1\log 0.95 = \log\left(\frac{1}{3}\right) \quad \Rightarrow \quad m = 1 + \frac{\log 1/3}{\log 0.95} = 22.418\ldots$$

The number of customers first reaches at least the threshold at the next whole month.

$$m = 23.$$

(3)