Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/4D - Decision Mathematics 2

Monday 26 June 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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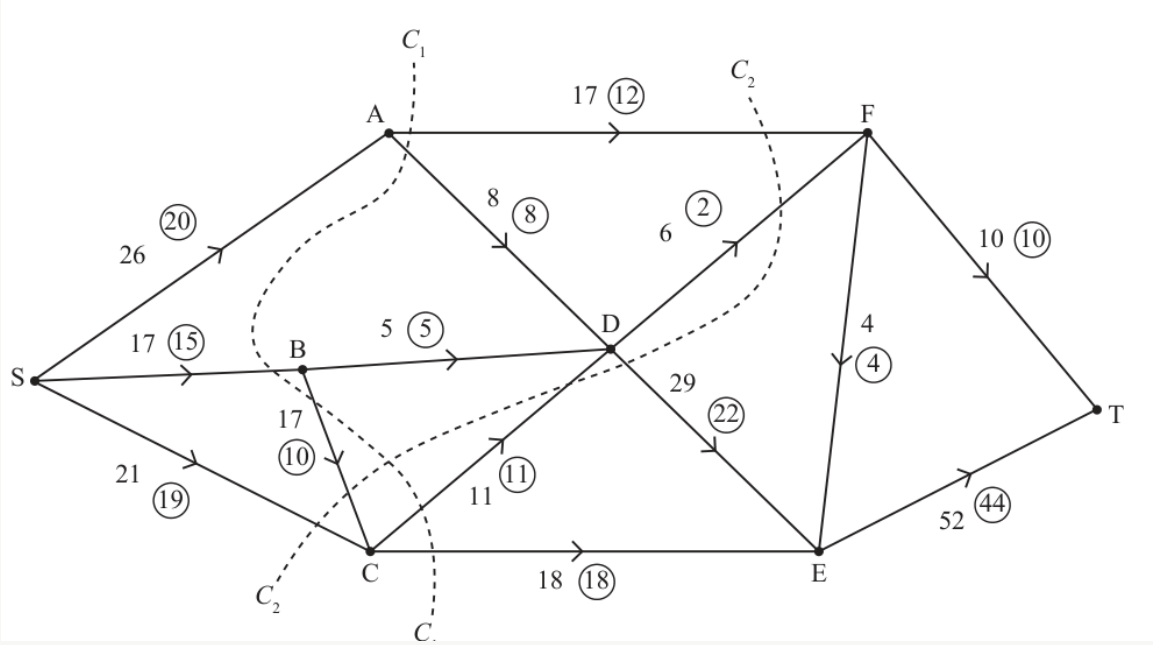
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Question 1

Question 1

Figure 1 gives a capacitated directed network and a feasible flow from S to T.



Model solution

(a) The value of a flow is the total leaving the source (equivalently, the total entering the sink).

$$20 + 15 + 19 = 54.$$

The feasible flow has value 54.

(1)

(b) The capacity of a cut is found by adding the capacities of all arcs crossing it in the forward direction.

$$\text{cap}(C_1) = 17 + 8 + 17 + 11 + 18 = 71, \qquad \text{cap}(C_2) = 17 + 6 + 29 + 17 + 21 = 90.$$

$\text{cap}(C_1) = 71$ and $\text{cap}(C_2) = 90$.

(2)

(c) An augmenting route must use residual capacity on forward arcs, and may use a reverse arc to reduce an existing flow. The route below has bottleneck residual capacity 2.

$$S \to A \to F \to D \to E \to T.$$

Here $F \to D$ *is a reverse residual arc: its use reduces the flow on* $D \to F$ *by 2.*

Use SAFDET, increasing the flow by 2.

(1)

(d) After this augmentation the flow value is $54+2=56$. *Consider the cut with* S, A, B, C, F *on the source side and* D, E, T *on the sink side.*

Cut arcs: FT, FE, DF, AD, BD, CD, CE.

$$\text{cap}(\textit{cut}) = 10+4+8+5+11+18 = 56.$$

The arc $D \to F$ *is directed from the sink side to the source side, so it crosses the cut backwards and its capacity* 6 *is not included. Every flow is at most the capacity of every cut. This flow has the same value as this cut, so by the max-flow min-cut theorem it is a maximum flow and this is a minimum cut.*

The augmented flow is maximal, with value 56.

(3)

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Question 5

Question 5

The sequence satisfies $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n) + 3$, with $u_0 = 15$, $u_1 = 20$. Given $u_n = v_n + 2n$, find u_n and describe its long-term behaviour.

Model solution

(a) The supplied substitution is chosen to remove the non-homogeneous constant term and leave a homogeneous recurrence for v_n. Substitute $u_r = v_r + 2r$ throughout.

$$v_{n+2} + 2n + 4 = \tfrac{1}{2}\big(v_{n+1} + 2n + 2 + v_n + 2n\big) + 3 = \tfrac{1}{2}(v_{n+1} + v_n) + 2n + 4,$$

so

$$v_{n+2} = \tfrac{1}{2}(v_{n+1} + v_n).$$

For this constant-coefficient recurrence, try $v_n = \lambda^n$.

$$\lambda^2 = \tfrac{1}{2}(\lambda + 1) \quad \Longrightarrow \quad 2\lambda^2 - \lambda - 1 = 0 \quad \Longrightarrow \quad (2\lambda + 1)(\lambda - 1) = 0.$$

$$\lambda = 1 \quad \textit{or} \quad \lambda = -\tfrac{1}{2}, \qquad v_n = A + B\left(-\tfrac{1}{2}\right)^n.$$

Convert the two initial values before finding the constants: $v_0 = 15$ and $v_1 = 20 - 2 = 18$.

$$A + B = 15, \qquad A - \tfrac{1}{2}B = 18.$$

$$B = -2, \qquad A = 17.$$

$$u_n = v_n + 2n = 17 - 2\left(-\tfrac{1}{2}\right)^n + 2n.$$

$$\boxed{u_n = 17 + 2n - 2\left(-\frac{1}{2}\right)^n.}$$

(7)

(b) Since $\left(-\tfrac{1}{2}\right)^n \to 0$, the small alternating term disappears as n becomes large.

$$\boxed{\text{For large } n,\ u_n \approx 17 + 2n, \text{ so the terms approach an arithmetic progression with common difference } 2.}$$

(1)

Question 6

Question 6

Polly chooses one country in each of four weeks, travelling S $\rightarrow$ (A or B) $\rightarrow$ (C, D or E) $\rightarrow$ (F or G) $\rightarrow$ (H, I or J) $\rightarrow$ S. Fees and journey costs are in hundreds of pounds. Use dynamic programming to maximise expected income.

Model solution

Work backwards from the return journey. At each state, compare every permitted next destination, using fee at that destination minus travel cost plus the best future value. An asterisk marks the best choice for that state.

stage	*state*	*action*	*value (hundreds of pounds)*
1	H	HS	$47-7=40^*$
	I	IS	$49-8=41^*$
	J	JS	$48-8=40^*$
2	F	FH	$44-6+40=78^*$
		FI	$44-7+41=78^*$
		FJ	$48-8+40=76$
	G	GH	$45-7+40=78$
		GI	$45-8+41=78$
		GJ	$48-6+40=79^*$
3	C	CF	$48-7+78=119$
		CG	$48-5+79=122^*$
	D	DF	$47-6+78=119^*$
		DG	$45-7+79=119^*$
	E	EF	$49-7+78=120$
		EG	$49-6+79=122^*$
4	A	AC	$47-3+122=166^*$
		AD	$47-4+119=162$
		AE	$47-5+122=164$
	B	BC	$45-5+122=162^*$
		BD	$45-4+119=160$
		BE	$45-6+122=161$
5	S	SA	$-5+166=161^*$
		SB	$-2+162=160$

Trace the starred choices from S: S $\rightarrow$ A $\rightarrow$ C $\rightarrow$ G $\rightarrow$ J $\rightarrow$ S. *The final value is in hundreds of pounds.*

The optimal schedule is SACGJS, with maximum expected income $161 \times \$100 = \$16\,100$.

(13)

Question 7

Question 7

Martina pays £k into an account each month and spends one quarter of the previous month-end balance. $u_1 = k$. Find u_n, then the minimum k which makes $u_8 \geqslant \$1750$.

Model solution

(a) Three quarters of the old balance remains after the spending, then the new monthly payment is added. Therefore

$$u_{n+1} = \tfrac{3}{4}u_n + k.$$

Solve the homogeneous recurrence first: $u_{n+1} = \frac{3}{4}u_n$ *gives complementary solution* $A(\frac{3}{4})^{n-1}$. *Because the added amount is constant, use a constant particular solution* $u_n = C$.

$$C = \tfrac{3}{4}C + k \quad \Longrightarrow \quad C = 4k.$$

$$u_n = A\left(\tfrac{3}{4}\right)^{n-1} + 4k.$$

Use $u_1 = k$ *to fix* A.

$$k = A + 4k \quad \Longrightarrow \quad A = -3k.$$

$$u_n = 4k - 3k\left(\frac{3}{4}\right)^{n-1}.$$

(6)

(b) Substitute $n = 8$ *and make the balance at least* £1750.

$$4k - 3k\left(\tfrac{3}{4}\right)^7 \geqslant 1750, \qquad k \geqslant \frac{1750}{4 - 3(3/4)^7} = 486.177\ldots$$

The payment must be rounded up to the nearest penny so that the required balance is still reached.

Theminimummonthlypaymentis$486.18.

(2)

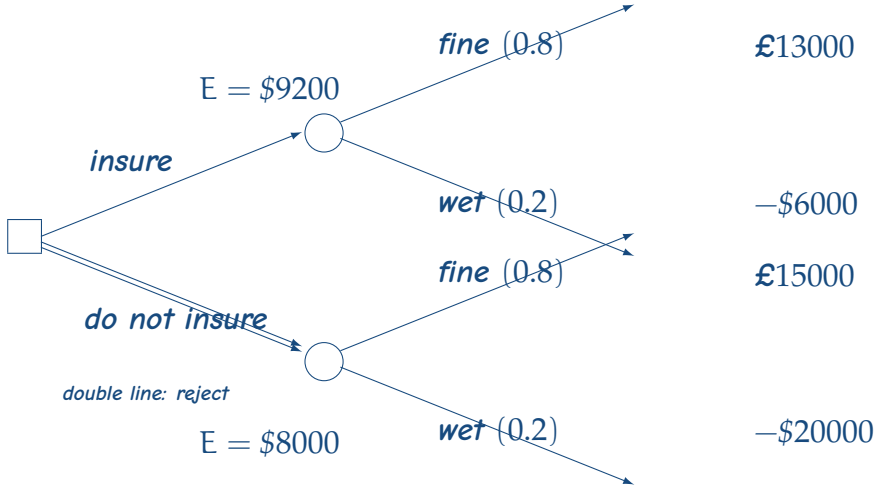
Question 2

Question 2

Decide whether an outdoor theatre should buy rain insurance costing £2000. Rain has probability 0.2; without insurance the fine-day profit is £15000 and wet-day loss is £20000. A cancelled wet performance receives £16000 from the insurer.

Model solution

Use a decision tree because the decision is made before the uncertain weather is known. The insurance premium is paid in either weather outcome. The square is a decision node and each circle is a chance node.



decision	*weather*	*probability*	*pay-off (£)*
insure	*fine*	0.8	$15000 - 2000 = 13000$
	wet	0.2	$-20000 - 2000 + 16000 = -6000$
do not insure	*fine*	0.8	15000
	wet	0.2	−20000

$$\mathrm{E}(\textit{insure}) = 0.8(13000) + 0.2(-6000) = \$9200,$$

$$\mathrm{E}(\textit{do not insure}) = 0.8(15000) + 0.2(-20000) = \$8000.$$

The insured branch has the larger expected profit, by £1200.

The theatre owner should take out the insurance.

(5)

Question 3

Question 3

For the transportation problem with supplies 45, 27, 34, 50 at A, B, C, D, demands 75, 37, 44 at Q, R, S, and the stated costs, use the north-west corner and stepping-stone methods.

Model solution

(a) Each supply may not be exceeded, and each demand must be met. These are the constraints required by the transportation model.

$$\begin{aligned} x_{AQ}+x_{AR}+x_{AS} &\leqslant 45, & x_{BQ}+x_{BR}+x_{BS} &\leqslant 27,\\ x_{CQ}+x_{CR}+x_{CS} &\leqslant 34, & x_{DQ}+x_{DR}+x_{DS} &\leqslant 50,\\ x_{AQ}+x_{BQ}+x_{CQ}+x_{DQ} &\geqslant 75,\\ x_{AR}+x_{BR}+x_{CR}+x_{DR} &\geqslant 37,\\ x_{AS}+x_{BS}+x_{CS}+x_{DS} &\geqslant 44. \end{aligned}$$

(2)

(b) Start at the north-west corner AQ*, allocate as much as possible, then move right when a demand is met and down when a supply is exhausted.*

	Q	R	S
A	45		
B	27		
C	3	31	
D		6	44

(1)

(c) Test the requested empty cell AS*. The closed stepping-stone route is* $AS(+), AQ(-), CQ(+), CR(-), DR(+), DS(-), AS$*. The minus allocations are* $45, 31, 44$*, so* $\theta = 31$*.*

	Q	R	S
A	$45-31=14$		$0+31=31$
B	27		
C	$3+31=34$	$31-31=0$	
D		$6+31=37$	$44-31=13$

The zero at CR *is omitted from the new basic solution.*

$$AQ = 14,\ AS = 31,\ BQ = 27,\ CQ = 34,\ DR = 37,\ DS = 13.$$

(2)

(d) First calculate shadow costs a_i, b_j *from* $a_i + b_j = c_{ij}$ *at occupied cells. Taking* $a_A = 0$ *gives the following valid set. The improvement index is* $c_{ij} - (a_i + b_j)$*; a negative index identifies an improving entering cell.*

	a_A	a_B	a_C	a_D	b_Q	b_R	b_S
shadow cost	0	−15	−12	−1	23	16	12

	Q	R	S
A	–	2	–
B	–	9	17
C	–	10	21
D	-3	–	–

The most negative improvement index is -3 *at* DQ, *so enter* DQ. *Its route is* $DQ(+), AQ(-), AS(+), DS(-), DQ$. *Thus* $\theta = \min(14, 13) = 13$; DS *exits.*

	Q	R	S
A	1		44
B	27		
C	34		
D	13	37	0

Enter DQ and exit DS; the improved allocation is $AQ = 1, AS = 44, BQ = 27, CQ = 34, DQ = 13, DR = 37$.

(4)

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Question 4

Question 4

Allocate students A, B, C, D to rounds 1, 2, 3, 4 to maximise the total estimated score.

Model solution

(a) The Hungarian algorithm is a minimisation method, so first convert a score s *to a cost* $52 - s$*, using the largest score* 52*. Reduce rows first, then reduce columns.*

	1	2	3	4
A	18	32	34	37
B	3	21	40	18
C	4	25	29	26
D	0	7	10	10

$\xrightarrow{\text{row reduction}}$

	1	2	3	4
A	0	14	16	19
B	0	18	37	15
C	0	21	25	22
D	0	7	10	10

$\xrightarrow{\text{column reduction}}$

	1	2	3	4
A	0	7	6	9
B	0	11	27	5
C	0	14	15	12
D	0	0	0	0

Cover column 1 and row D*. The smallest uncovered entry is* 5*; subtract it from every uncovered entry and add it at the double-covered entry* D1*. Entries covered once are unchanged.*

	1	2	3	4
A	0	2	1	4
B	0	6	22	0
C	0	9	10	7
D	5	0	0	0

Now cover column 1 and rows B *and* D*. The smallest uncovered entry is* 1*; subtract it from every uncovered entry and add it at the double-covered entries* B1 *and* D1*. Entries covered once are unchanged.*

	1	2	3	4
A	0	1	0	3
B	1	6	22	0
C	0	8	9	6
D	6	0	0	0

There are independent zeros in $A3, B4, C1, D2$*, so this is an optimal allocation.*

$$A \to 3,\ B \to 4,\ C \to 1,\ D \to 2.$$

(7)

(b) Return to the original score table to evaluate the allocation.

$$18 + 34 + 48 + 45 = 145.$$

Themaximumtotalestimatedscoreis145.

(1)

Question 8

Question 8

Player A's pay-off matrix, with A's options Q, R, S, T and player B's options X, Y, Z, is

	X	Y	Z
Q	−3	2	5
R	2	−1	0
S	4	−2	−1
T	−4	0	2

.

Model solution

(a) A stable solution requires A's maximin to equal B's minimax.

$$\textit{row minima} = -3, -1, -2, -4, \qquad \textit{column maxima} = 4, 2, 5.$$

$$\max(-3, -1, -2, -4) = -1 \neq 2 = \min(4, 2, 5).$$

There is no stable solution.

(2)

(b) A seeks the larger pay-off. Option Q *dominates option* T*, since it is better against every option B can play.*

$$-3 > -4, \qquad 2 > 0, \qquad 5 > 2.$$

PlayerAshouldneverplayT.

(2)

(c)(i) Delete the dominated column T *and add* 3 *to every remaining payoff, making every entry non-negative. This translation leaves the optimal probabilities unchanged. Let* p_1, p_2, p_3 *be A's probabilities for* Q, R, S*, and let* V *be the augmented game value.*

	Q	R	S
X	0	5	7
Y	5	2	1
Z	8	3	2

For each choice by B, A's expected payoff must be at least V*. Introduce slack variables to write the constraints as Simplex equations.*

$$\begin{aligned} V - 5p_2 - 7p_3 + r &= 0, \\ V - 5p_1 - 2p_2 - p_3 + s &= 0, \\ V - 8p_1 - 3p_2 - 2p_3 + t &= 0, \\ p_1 + p_2 + p_3 + u &= 1, \end{aligned} \qquad p_1, p_2, p_3, r, s, t, u \geqslant 0,$$

$$\textit{maximise } P - V = 0.$$

(c)(ii) Taking r, s, t, u *as the initial basic variables gives:*

b.v.	V	p_1	p_2	p_3	r	s	t	u	*Value*
r	1	0	−5	−7	1	0	0	0	0
s	1	−5	−2	−1	0	1	0	0	0
t	1	−8	−3	−2	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

(7)

(d) Given $p_1 = 6/11$ *and* $p_2 = 0$, $p_3 = 5/11$. *The expected augmented payoffs against* X, Y, Z *are respectively* $35/11, 35/11, 58/11$, *so* $V = 35/11$. *Let B use probabilities* x, y, z *for* X, Y, Z. *A uses* Q *and* S, *so B makes their payoffs equal to* V, *together with* $x+y+z = 1$.

$$5y + 8z = \frac{35}{11}, \qquad 7x + y + 2z = \frac{35}{11}, \qquad x + y + z = 1.$$

$$x = \frac{4}{11}, \qquad y = \frac{7}{11}, \qquad z = 0.$$

The unused option R *gives A* $5x + 2y + 3z = 34/11 < 35/11$ *after augmentation, so A has no reason to use it.*

Player B should play X with probability $\frac{4}{11}$, Y with probability $\frac{7}{11}$, and never play Z.

(6)

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