Maths with Matt

Physics and maths resources

Edexcel A Level Further Mathematics

9FM0/4D - Decision Mathematics 2

Monday 24 June 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Figure 1 is a capacitated directed network. Circled numbers are initial flows and the other numbers are capacities (litres s^{-1}).

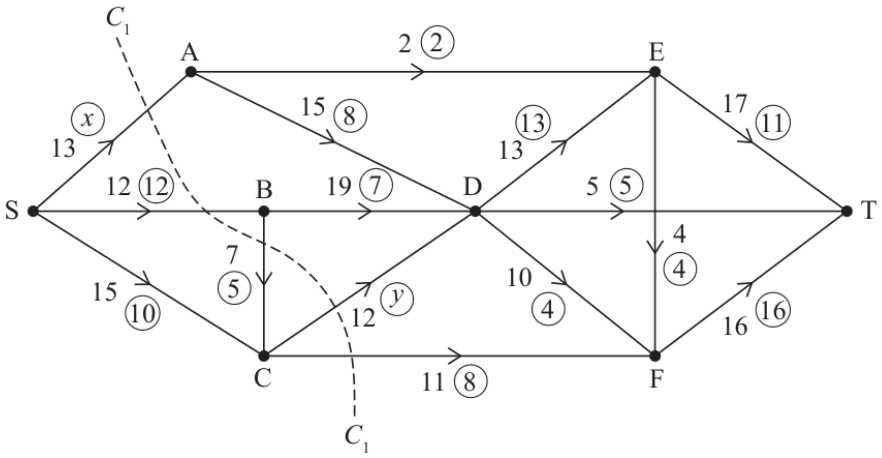


Figure 1

Figure 1 shows a capacitated, directed network of pipes. The numbers in circles

Find x, y, the initial flow and the capacity of C_1. Give an augmenting route of 4 units, prove the resulting flow maximal, then apply the vertex restriction $E \leqslant 12$.

Model solution

Use conservation of flow at each intermediate vertex. At A, $x = 2 + 8 = 10$. *At* C, *the inflow is* $10 + 5$ *and the outflow is* $8 + y$, *so* $y = 7$.

$$x = 10,\ y = 7.$$

(2) The flow value is total flow out of S: $10 + 12 + 10 = 32$.

$$\text{Initial flow} = 32 \text{litres} s^{-1}.$$

(1) The cut capacity is the sum of forward arcs crossing C_1*:* $13 + 12 + 12 + 11 = 48$.

$$\text{cap}(C_1) = 48 \text{litres} s^{-1}.$$

(1) A residual route of capacity at least 4 is needed. The step $F \to E$ *reverses flow on the full arc* $E \to F$.

$$S \to C \to D \to F \to E \to T.$$

$$\text{Augment along SCDFET by 4.}$$

(1) The new flow has value $32 + 4 = 36$. *To prove maximality, find a cut of this capacity. With* $\{S, A, B, C, D, F\}$ *on the source side, the crossing arcs are*

AE, DE, DT, FT, *of total capacity*

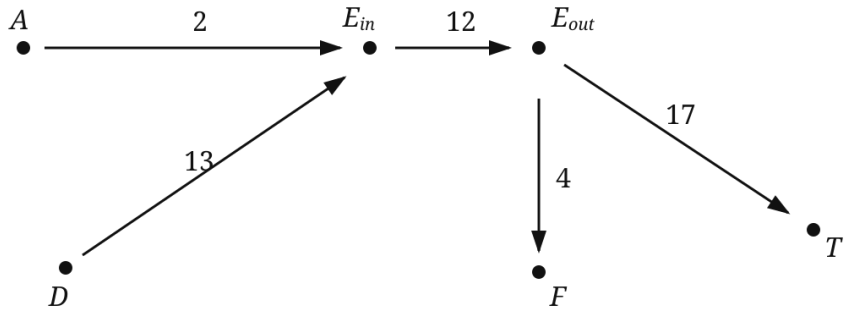
$$2 + 13 + 5 + 16 = 36.$$

They are saturated, so the max-flow min-cut theorem shows that no flow can exceed 36.

The flow is maximal, with value 36 litres^{-1}.

(3) Represent a vertex capacity by splitting the vertex: direct all incoming arcs to E_{in}*, all outgoing arcs from* E_{out}*, and join them by a capacity-12 arc. The completed restricted part of the network is*

Restricted network at vertex E



The restricted maximum flow is 33 litres^{-1}.

(2)

Question 2

Question 2

The recurrence $u_{n+1} = au_n + bn^2 + cn + d$ has general solution $u_n = A3^n + 5n^2 + 1$. Find a, b, c, d.

Model solution

Express u_{n+1} in terms of u_n, so that the arbitrary term $A3^n$ cancels.

$$u_{n+1} = 3A3^n + 5n^2 + 10n + 6 = 3u_n - 10n^2 + 10n + 3.$$

$$a = 3,\ b = -10,\ c = 10,\ d = 3.$$

(3)

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Question 3

Question 3

A transportation problem has costs

	A	B	C	Supply
E	23	28	22	21
F	26	19	29	32
G	29	24	20	29
H	24	26	19	23
Demand	45	19	23	

.

Explain and complete the dummy demand column. From $EA = 21, FA = 19, FB = 13, GB = 6, GC = 23, HA = 5, HD = 18$, take GD as entering cell, perform two stepping-stone iterations, find the cost, and test optimality.

Model solution

The table must balance. Total supply is 105, *but total demand is* 87, *leaving 18 surplus units.*

Add dummy destination D, with demand 18.

(1) Shipment to the dummy point represents unused stock, so every dummy cost is zero.

	A	B	C	D	*Supply*
E	23	28	22	0	21
F	26	19	29	0	32
G	29	24	20	0	29
H	24	26	19	0	23
Demand	45	19	23	18	

The dummy column has costs 0, 0, 0, 0 and demand 18.

(1) For GD, *form a closed route and alternate plus and minus signs. The least allocation at a minus corner is the transferable amount.*

$$GD(+), HD(-), HA(+), FA(-), FB(+), GB(-), GD; \qquad \theta = \min(18, 19, 6) = 6.$$

Thus GB *exits, giving* $EA = 21, FA = 13, FB = 19, GC = 23, GD = 6, HA = 11, HD = 12$. *Shadow costs* $r_i + s_j = c_{ij}$ *for occupied cells can be*

$$(r_E, r_F, r_G, r_H) = (0, 3, 1, 1), \quad (s_A, s_B, s_C, s_D) = (23, 16, 19, -1).$$

The improvement index $c_{ij} - (r_i + s_j)$ *is* -2 *at* FD, *so* FD *enters.*

$$FD(+), FA(-), HA(+), HD(-), FD; \qquad \theta = \min(13, 12) = 12.$$

Hence HD *exits and the allocations are* $EA = 21, FA = 1, FB = 19, FD = 12, GC = 23, GD = 6, HA = 23$. *(6) Cost is found using the original unit costs:*

$$21(23) + 1(26) + 19(19) + 23(20) + 23(24) = 1882.$$

$1882.

(1) For the final basis take $\mathrm{r} = (0, 3, 3, 1)$, $\mathrm{s} = (23, 16, 17, -3)$. *The unused-cell indices are* $12, 5, 3, 9, 3, 5, 9, 1, 2$, *all non-negative. This is the optimality condition for minimisation.*

The solution is optimal.

(3)

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Question 4

Question 4

Workers A, B, C, D are assigned to tasks P, Q, R, S, with earnings

	P	Q	R	S
A	65	72	69	75
B	71	–	68	65
C	70	69	73	77
D	73	70	–	71

.

Modify the table for the Hungarian algorithm and formulate a linear programme.

Model solution

Hungarian is a minimisation algorithm. Subtract each earning from 77, so minimising transformed cost maximises earnings. A value 13, larger than all valid transformed costs, prevents forbidden assignments.

	P	Q	R	S
A	12	5	8	2
B	6	13	9	12
C	7	8	4	0
D	4	7	13	6

Usethisminimisationtable;BQandDRhaveprohibitivecost13.

(3) Let $x_{ij} = 1$ when worker i is assigned task j, and 0 otherwise. One assignment is required in each worker row and task column.

$$\min C = 12x_{AP} + 5x_{AQ} + 8x_{AR} + 2x_{AS} + 6x_{BP} + 13x_{BQ} + 9x_{BR} + 12x_{BS} + 7x_{CP} + 8x_{CQ} + 4x_{CR} + 0x_{CS} + 4x_{DP} + 7x_{DQ} + 13x_{DR} + 6x_{DS},$$

$$\sum_j x_{ij} = 1 \ (i = A, B, C, D), \qquad \sum_i x_{ij} = 1 \ (j = P, Q, R, S), \qquad x_{ij} \in \{0, 1\}.$$

The displayed binary programme models the assignment.

(6)

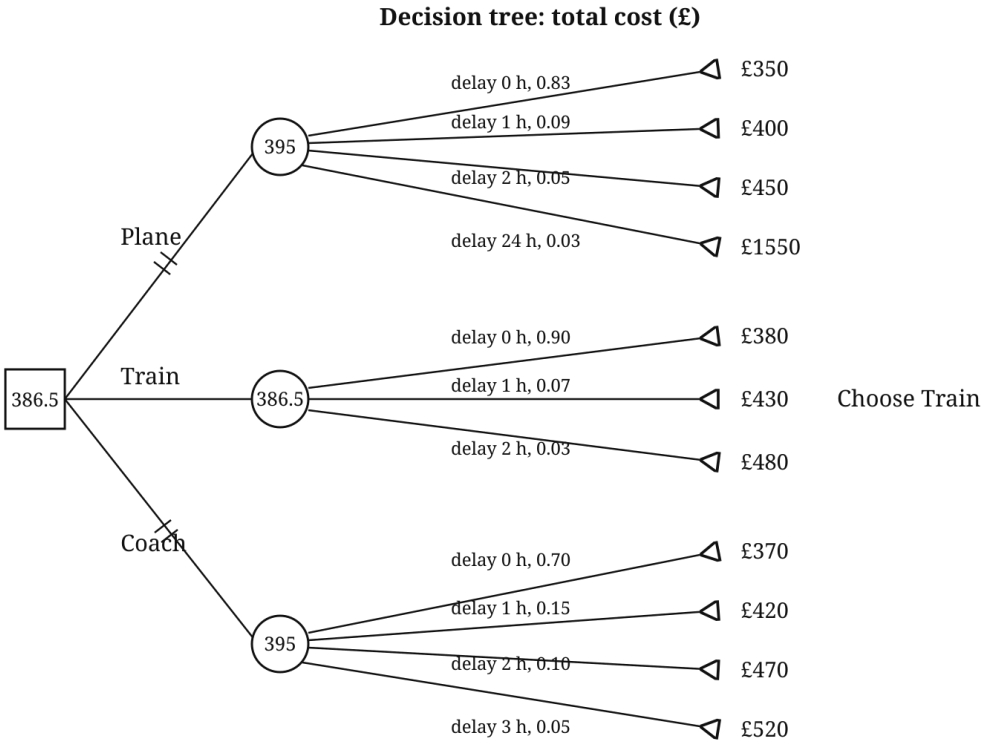
Question 5

Question 5

Plane, train and coach costs are $\$200, \$130, \$70$, ordinary times are $3, 5, 6$ hours, and time costs $\$50$ per hour. Delay probabilities for $1, 2, 3, 24$ hours are plane $(.09, .05, 0, .03)$, train $(.07, .03, 0, 0)$, coach $(.15, .10, .05, 0)$. Find EMVs, the EMV choice, and the best expected cube-root utility.

Model solution

At each chance node, add the travel cost to £50 times the actual time. The missing probability at each node is the no-delay probability.



option	*terminal total costs (probability)*	*EMV*
Plane	$350(.83), 400(.09), 450(.05), 1550(.03)$	395
Train	$380(.90), 430(.07), 480(.03)$	386.50
Coach	$370(.70), 420(.15), 470(.10), 520(.05)$	395

$$\text{Plane} = \$395, \text{train} = \$386.50, \text{coach} = \$395.$$

(6) Choose the smallest expected cost.

Train.

(1) Apply $U(C) = \sqrt[3]{C}$ to each terminal cost before calculating the expectation. Lower

utility is preferred because it is a utility of cost.

$$E(U_P) = .83\sqrt[3]{350} + .09\sqrt[3]{400} + .05\sqrt[3]{450} + .03\sqrt[3]{1550} = 7.2427\ldots,$$
$$E(U_T) = .90\sqrt[3]{380} + .07\sqrt[3]{430} + .03\sqrt[3]{480} = 7.2820\ldots,$$
$$E(U_C) = .70\sqrt[3]{370} + .15\sqrt[3]{420} + .10\sqrt[3]{470} + .05\sqrt[3]{520} = 7.3282\ldots.$$

Plane, with expected utility 7.2427....

(3)

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Question 6

Question 6

The staged directed network gives road times in hours. Elena must arrive at T by 9 pm. Use dynamic programming to find her latest start and route.

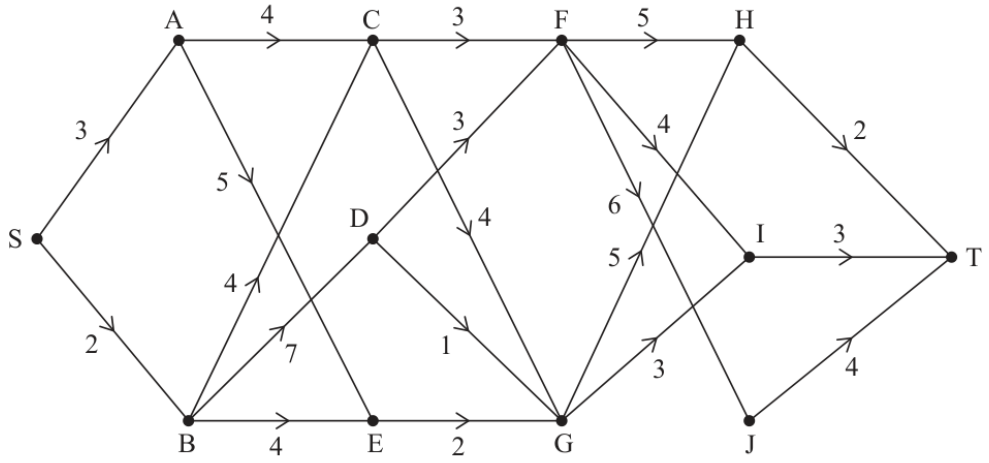


Figure 2

Model solution

Work backwards from T*, choosing at each state the smallest remaining journey time.*

stage	*calculations*	*minimum*
1	H : 2, I : 3, J : 4	2, 3, 4
2	F : FH = 5 + 2 = 7, FI = 4 + 3 = 7, FJ = 6 + 4 = 10	7
	G : GH = 5 + 2 = 7, GI = 3 + 3 = 6	6
3	C : CF = 3 + 7 = 10, CG = 4 + 6 = 10	10
	D : DF = 3 + 7 = 10, DG = 1 + 6 = 7	7
	E : EG = 2 + 6 = 8	8
4	A : AC = 4 + 10 = 14, AE = 5 + 8 = 13	13
	B : BC = 4 + 10 = 14, BD = 7 + 7 = 14, BE = 4 + 8 = 12	12
5	S : SA = 3 + 13 = 16, SB = 2 + 12 = 14	14

The minimum duration is 14 hours, so the latest start is 14 hours before 9 pm.

7 am.

(9) Trace the minimum choices from S*.*

S → B → E → G → I → T.

(1)

Question 7

Question 7

Player A's payoff matrix is

	X	Y	Z
R	3	2	−3
S	4	−2	1
T	−1	3	6

.

Test for a stable solution. Using $P = V$, where V is the value shifted by 3, form an initial Simplex tableau. A final tableau is supplied; find the optimal strategies and the value to B.

Model solution

Compare row maximin with column minimax.

row minima $= (-3,-2,-1)$, $\quad \text{maximin} = -1$; $\qquad$ *column maxima* $= (4,3,6)$, $\quad \text{minimax} = 3$.

$$\text{No stable solution, since } -1 \neq 3.$$

(2) After adding 3 to every payoff, A's three guaranteed-payoff constraints are

$$6p_1 + 7p_2 + 2p_3 \geqslant V, \quad 5p_1 + p_2 + 6p_3 \geqslant V, \quad 4p_2 + 9p_3 \geqslant V, \quad p_1 + p_2 + p_3 = 1.$$

With surplus variables r, s, t, *and* u *for the probability equation, one initial tableau is*

b.v.	V	p_1	p_2	p_3	r	s	t	u	*Value*
r	1	−6	−7	−2	1	0	0	0	0
s	1	−5	−1	−6	0	1	0	0	0
t	1	0	−4	−9	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

This tableau gives the required initial Simplex formulation.

(5) The final tableau gives $p_1 = 3/8$, $p_2 = 17/80$, $p_3 = 33/80$, $V = 73/16$. *Undo the shift to find the game value.*

$$v_A = \frac{73}{16} - 3 = \frac{25}{16}, \qquad v_B = -\frac{25}{16}.$$

$$\text{A plays } (R, S, T) = (3/8, 17/80, 33/80); \text{ B's value is } -25/16.$$

(3) Since all A strategies occur, B makes their expected payoffs equal to $25/16$. *Let B use* (x, y, z) *for* (X, Y, Z).

$$3x + 2y - 3z = \frac{25}{16}, \quad 4x - 2y + z = \frac{25}{16}, \quad x + y + z = 1.$$

Solving gives $x = 1/2$, $y = 5/16$, $z = 3/16$, *which also makes the T payoff* $25/16$.

$$\text{Bplays}(X, Y, Z) = (1/2, 5/16, 3/16).$$

(3)

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Question 8

Question 8

For $n \geqslant 0$, $2u_{n+2} + 5u_{n+1} = 3u_n + 8n + 2$. Find the general solution. Given $u_0 = 1$, $u_1 = k > 0$, and all terms positive, determine k.

Model solution

The homogeneous auxiliary equation is $2m^2 + 5m - 3 = 0$, *so*

$$u_n^{(c)} = A\left(\frac{1}{2}\right)^n + B(-3)^n.$$

As the non-homogeneous term is linear, try $u_n^{(p)} = an + b$. *Substitution into* $2u_{n+2} + 5u_{n+1} - 3u_n = 8n + 2$ *gives*

$$4an + (9a + 4b) = 8n + 2,$$

so $a = 2$, $b = -4$.

$$u_n = A(1/2)^n + B(-3)^n + 2n - 4.$$

(5) The condition $u_0 = 1$ *gives* $A + B = 5$. *If* $B \neq 0$, *the alternating, exponentially growing term* $B(-3)^n$ *eventually makes terms negative, so positivity requires* $B = 0$. *Thus* $A = 5$.

$$k = u_1 = \frac{5}{2} + 2 - 4 = \frac{1}{2}.$$

$$k = \tfrac{1}{2}.$$

(3)