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Edexcel A Level Further Mathematics

9FM0/4D - Decision Mathematics 2

October 2020

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2020 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Four workers A, B, C and D must each be assigned one of tasks 1–4. Profits (£) are

	1	2	3	4
A	29	20	–	23
B	32	30	28	–
C	35	32	34	25
D	29	31	27	30

.

(a) Use the Hungarian algorithm, reducing rows first, to maximise total profit. (b) Find the resulting total profit.

Model solution

(a) The Hungarian algorithm is a minimisation method, so convert profit to a cost by subtracting every permitted profit from 35, the largest profit. A very large cost (100) prevents an impossible assignment from being selected.

	1	2	3	4
A	6	15	100	12
B	3	5	7	100
C	0	3	1	10
D	6	4	8	5

Reduce each row by its smallest entry, then reduce each column by its smallest entry.

	1	2	3	4
A	0	9	94	6
B	0	2	4	97
C	0	3	1	10
D	2	0	4	1

$\longrightarrow$

	1	2	3	4
A	0	9	93	5
B	0	2	3	96
C	0	3	0	9
D	2	0	3	0

.

There are not yet four independent zeros. Cover the zeros with the minimum three lines: column 1, row C and row D. The least uncovered entry is 2. Subtract 2 from every uncovered entry and add 2 at each intersection of two lines.

	1	2	3	4
A	0	7	91	3
B	0	0	1	94
C	2	3	0	9
D	4	0	3	0

Now select independent zeros: A--1, B--2, C--3 and D--4. Each task and each worker occurs once, so this is the required allocation.

$$A \to 1, \quad B \to 2, \quad C \to 3, \quad D \to 4.$$

(7)

(b) Return to the original profit table when calculating the objective value.

$$29 + 30 + 34 + 30 = 123.$$

Maximum total profit $= \$123.$

(1)

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Question 2

Question 2

Jenny chooses A, B or C. Their outcomes (profit in £) are respectively

$(350, -140)$ with probabilities $(0.6, 0.4)$, $(260, -190)$ with $(0.75, 0.25)$, $(220, -230)$ with $(0.8, 0.2)$.

(a) Find the optimal EMV. (b) Using $u(x) = 1 - e^{-x/400}$, choose by expected utility.

Model solution

(a) EMV is appropriate when money itself is the criterion: multiply each possible profit by its probability and add.

$$\begin{aligned} \text{EMV}(A) &= 0.6(350) + 0.4(-140) = 154, \\ \text{EMV}(B) &= 0.75(260) + 0.25(-190) = 147.5, \\ \text{EMV}(C) &= 0.8(220) + 0.2(-230) = 130. \end{aligned}$$

The optimal EMV is $\$154$, so choose A by EMV.

(3)

(b) The stated utility function models Jenny's preferences, so replace each monetary payoff by its utility before taking the expectation.

option	*utilities of its two outcomes*	*expected utility*
A	$u(350) = 0.583138,\ u(-140) = -0.419068$	$0.6(0.583138) + 0.4(-0.419068) = 0.182256$
B	$u(260) = 0.477954,\ u(-190) = -0.608014$	$0.75(0.477954) + 0.25(-0.608014) = 0.206462$
C	$u(220) = 0.423050,\ u(-230) = -0.777131$	$0.8(0.423050) + 0.2(-0.777131) = 0.183014$

Option B has the largest expected utility, even though A had the largest EMV; the utility criterion therefore reflects a different attitude to risk.

Choose B; its expected utility is 0.206 utils (3 s.f.).

(4)

Question 3

Question 3

Transportation costs (£ per unit), supplies and demands are

	P	Q	R	supply
A	25	24	17	42
B	7	12	14	68
C	13	11	20	25
D	16	15	13	40
demand	59	72	44	

.

The north-west corner initial allocation is $x_{AP} = 42, x_{BP} = 17, x_{BQ} = 51, x_{CQ} = 21, x_{CR} = 4, x_{DR} = 40$. (a) Take AR as entering cell. (b) Make one further stepping-stone iteration. (c) Test optimality. (d) Formulate an LP. (e) Explain why Simplex cannot be used directly.

Model solution

(a) For an entering empty cell, form a closed horizontal--vertical route through occupied cells, alternately adding and subtracting θ. Starting at AR gives

$$x_{AR} = \theta, \quad x_{AP} = 42 - \theta, \quad x_{BP} = 17 + \theta, \quad x_{BQ} = 51 - \theta, \quad x_{CQ} = 21 + \theta, \quad x_{CR} = 4 - \theta.$$

The smallest cell from which θ is subtracted is $x_{CR} = 4$, so $\theta = 4$.

	P	Q	R
A	38		4
B	21	47	
C		25	
D			40

The improved allocation is as displayed.

(2)

(b) Use shadow costs u_i, v_j, with $u_A = 0$, so that $u_i + v_j = c_{ij}$ in each occupied cell. Then an empty-cell improvement index is $c_{ij} - (u_i + v_j)$; a negative value can reduce cost.

	P	Q	R	u_i
A	25	24	17	0
B	7	12	14	−18
C	13	11	20	−19
D	16	15	13	−4
	$v_P = 25$	$v_Q = 30$	$v_R = 17$	

	P	Q	R
A	X	−6	X
B	X	X	15
C	7	X	22
D	−5	−11	X

DQ has the most negative index, so it enters. Its route is DQ(+)--DR(-)--AR(+)--AP(-)--BP(+)--BQ(-). The minus cells are DR, AP and BQ, so $\theta = \min(40, 38, 47) = 38$*, and AP exits.*

	P	Q	R
A			42
B	59	9	
C		25	
D		38	2

Enter DQ, exit AP; the resulting allocation is displayed.

(4)

(c) Repeat the shadow-cost test for the new basic cells. Optimality is reached exactly when every empty-cell improvement index is non-negative.

	P	Q	R	u_i
A	25	24	17	0
B	7	12	14	−7
C	13	11	20	−8
D	16	15	13	−4
	$v_P = 14$	$v_Q = 19$	$v_R = 17$	

	P	Q	R
A	11	5	X
B	X	X	4
C	7	X	11
D	6	X	X

This solution is optimal, since there are no negative improvement indices.

(3)

(d) Let x_{ij} *be the non-negative number of units sent from supply point* $i \in \{A, B, C, D\}$ *to sales point* $j \in \{P, Q, R\}$*. The total transport cost is therefore minimised subject to available stock and required demand.*

Minimise $C = 25x_{AP} + 24x_{AQ} + 17x_{AR} + 7x_{BP} + 12x_{BQ} + 14x_{BR}$
$+ 13x_{CP} + 11x_{CQ} + 20x_{CR} + 16x_{DP} + 15x_{DQ} + 13x_{DR}$,

$x_{AP} + x_{AQ} + x_{AR} \leqslant 42$, $x_{BP} + x_{BQ} + x_{BR} \leqslant$

$x_{CP} + x_{CQ} + x_{CR} \leqslant 25$, $x_{DP} + x_{DQ} + x_{DR} \leqslant$

$x_{AP} + x_{BP} + x_{CP} + x_{DP} \geqslant 59$,

$x_{AQ} + x_{BQ} + x_{CQ} + x_{DQ} \geqslant 72$,

$x_{AR} + x_{BR} + x_{CR} + x_{DR} \geqslant 44$, $\quad x_{ij} \geqslant 0$.

(6)

(e) The standard Simplex algorithm needs constraints in the form $\sum x \leqslant k$*, with positive* k*. Here the demand constraints are* $\geqslant$ *constraints, so this transportation LP is not in that required form.*

Simplex cannot be used directly because not all constraints are $\sum x \leqslant k$ *with* $k > 0$.

(1)

Question 4

Question 4

The complementary function of $u_{n+2} + \alpha u_{n+1} + \beta u_n = 20(-3)^n$ is $u_n = A2^n + B(-1)^n$. (a) Find α, β. Given $2u_0 = u_1$ and $u_4 = 164$, (b) find u_n.

Model solution

(a) The bases in the complementary function are the roots of the auxiliary equation. Thus the roots are 2 *and* -1.

$$(m-2)(m+1) = m^2 - m - 2 = 0.$$

Comparing this with $m^2 + \alpha m + \beta = 0$ *gives*

$$\alpha = -1, \qquad \beta = -2.$$

(2)

(b) Since the forcing term is a multiple of $(-3)^n$*, and* -3 *is not an auxiliary root, try a particular solution* $u_n = \lambda(-3)^n$*. Substitute it into* $u_{n+2} - u_{n+1} - 2u_n = 20(-3)^n$.

$$9\lambda(-3)^n + 3\lambda(-3)^n - 2\lambda(-3)^n = 20(-3)^n, \qquad 10\lambda = 20,$$

so $\lambda = 2$*. Hence*

$$u_n = A2^n + B(-1)^n + 2(-3)^n.$$

Use the two given conditions to determine the two arbitrary constants.

$$2u_0 = u_1: \quad 2(A + B + 2) = 2A - B - 6 \quad \Rightarrow \quad B = -\frac{10}{3},$$

$$u_4 = 164: \quad 16A + B + 2(81) = 164 \quad \Rightarrow \quad 16A + B = 2,$$

so $A = \frac{1}{3}$*. Therefore*

$$u_n = \tfrac{1}{3}2^n - \tfrac{10}{3}(-1)^n + 2(-3)^n.$$

(6)

Question 5

Question 5

A capacitated directed pipe network has source/sink candidates A–J. Arc capacities are written (lower, upper). In particular the arcs are BA(1,6), CA(4,8), CB(1,1), BF(0,4), CD(4,6), CF(3,6), AD(5,8), DE(1,2), DG(5,7), CE(2,3), FE(4,6), FJ(2,5), EJ(
Answer (a)–(f), including a feasible flow of 18 L s^{-1} and a proof of maximality.

Model solution

(a) A source has no arc entering it.

The source node is C.

(1)

(b) A sink has no arc leaving it. Every arc incident with G is directed into G.

The sink is G.

(1)

(c) For a cut, add upper capacities of arcs crossing from the source side to the sink side and subtract lower capacities of arcs crossing in the reverse direction. For C_1, this gives

$$10 + 2 - 1 + 6 + 8 + 1 - 0 = 26.$$

$$\text{cap}(C_1) = 26 \text{ L } s^{-1}.$$

(1)

(d)(i) The lower flows into J are 2 on FJ and 3 on EJ, totalling 5. Flow conservation at J means its only outgoing arc JH must carry at least 5; its upper capacity is 5.

JH must be at its upper capacity, 5 L s^{-1}.

(1)

(d)(ii) AD and CD have lower capacities 5 and 4, totalling 9. The only arcs leaving D are DG and DE, whose upper capacities total $7 + 2 = 9$. Thus conservation forces the inflow to D to be exactly 9.

AD and CD must be at their lower capacities.

(1)

(e) Write one flow on each arc. The following values respect every lower and upper bound; at each internal node, total inflow equals total outflow.

arc	BA	CA	CB	BF	CD	CF	AD	DE	DG
flow	1	4	1	0	4	6	5	2	7
arc	CE	FE	FJ	EJ	EH	EG	JH	HG	
flow	3	4	2	3	5	1	5	10	

For example, C sends $4+1+4+6+3=18$, while G receives $7+1+10=18$; so the value of the flow is 18 L s^{-1}.

A feasible flow of 18 L s^{-1} is given in the table.

(2)

(f) To prove a feasible flow is maximum, use the max-flow min-cut theorem. Consider the cut crossing DG, DE, CE, CF, CB and BA. Its capacity is

$$7+2+3+6+1-1=18.$$

No flow can exceed the capacity of any cut, but part (e) achieves 18. Hence the achieved flow equals the capacity of this cut.

By the max-flow min-cut theorem, the maximum flow is 18 L s^{-1}.

(3)

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Question 6

Question 6

The pay-off matrix for player A is

	X	Y	Z
Q	1	5	3
R	4	−3	1
S	2	−4	−2
T	3	−2	0

.

Answer (a)–(f), where after deleting S, A uses probabilities p_1, p_2, p_3 for Q,R,T and the augmented game value is V, original value +3. Given $p_1 = \frac{7}{11}$, $p_3 = 0$, find the game value and B's optimal strategy.

Model solution

(a) A maximising player will never choose a row that is worse in every column than another available row. Row R dominates S because

$$4 > 2, \qquad -3 > -4, \qquad 1 > -2.$$

Remove S: option R dominates option S.

(2)

(b) Test for a saddle point using the reduced 3×3 matrix with rows Q, R, T. The maximin is the largest row minimum; the minimax is the smallest column maximum.

	X	Y	Z	*row minimum*
Q	1	5	3	1
R	4	−3	1	−3
T	3	−2	0	−2

$\Rightarrow \quad \max\{1, -3, -2\} = 1,$

$$\textit{column maxima} = 4, 5, 3 \quad \Rightarrow \quad \min\{4, 5, 3\} = 3.$$

There is no stable solution, because maximin $1 \neq 3$ minimax.

(3)

(c) Against each pure option of B, A obtains one of the three displayed augmented expected pay-offs. B can choose the least favourable column, so A can guarantee only their minimum.

$$V \leqslant 4p_1 + 7p_2 + 6p_3, \quad V \leqslant 8p_1 + p_3, \quad V \leqslant 6p_1 + 4p_2 + 3p_3.$$

(1)

(d) The probabilities must add to 1 at the optimum, but Simplex requires an inequality constraint so that a pivot can increase the objective value from its initial solution.

$p_1 + p_2 + p_3 \leqslant 1$ *is needed to put the problem in Simplex form.*

(1)

(e) Since probabilities sum to 1, $p_2 = 1 - \frac{7}{11} - 0 = \frac{4}{11}$. *Substitute into all three bounds for* V.

$$4p_1 + 7p_2 + 6p_3 = \frac{56}{11}, \qquad 8p_1 + p_3 = \frac{56}{11}, \qquad 6p_1 + 4p_2 + 3p_3 = \frac{58}{11}.$$

The minimum bound is the augmented value $V = \frac{56}{11}$. *Undo the augmentation by subtracting 3.*

Value to A $= \frac{56}{11} - 3 = \frac{23}{11}$.

(3)

(f) Let B choose X, Y, Z with probabilities q_1, q_2, q_3. *Every row used by A with positive probability must give A the value* $\frac{23}{11}$; *Q and R are used, and the probabilities sum to 1.*

$$q_1 + 5q_2 + 3q_3 = \frac{23}{11}, \qquad 4q_1 - 3q_2 + q_3 = \frac{23}{11}, \qquad q_1 + q_2 + q_3 = 1.$$

$$q_1 = \frac{8}{11}, \qquad q_2 = \frac{3}{11}, \qquad q_3 = 0.$$

B plays X with probability $\frac{8}{11}$, *Y with* $\frac{3}{11}$, *and Z with* 0.

(4)

Question 7

Question 7

A manufacturer exports five batches of one type of footwear per batch. Profits (£000s) for 0–5 batches are

batches	0	1	2	3	4	5
trainers	0	50	90	170	225	295
sandals	0	70	110	165	245	300
high heels	0	75	115	x	235	305

(a) Use dynamic programming to find the two possible expressions for the maximum annual profit.
(b) If it is £320 000, advise on allocations.

Model solution

(a) Dynamic programming is suitable because the five batches are a limited resource that must be allocated between three stages. At each stage, keep the best profit for every number of batches still available. All values below are in £000s.
At the trainers stage, allocating all available batches to trainers gives the initial value function.

n	0	1	2	3	4	5
$T(n)$	0	50	90	170	225	295

At the sandals stage, for each total n, compare $S(k)+T(n-k)$ for $k=0,\ldots,n$. The table records every comparison, so the selected maximum is clear.

n	values $S(k)+T(n-k)$ for $k=0,\ldots,n$	$ST(n)$
0	0	0
1	50, 70	70
2	90, 120, 110	120
3	170, 160, 160, 165	170
4	225, 240, 200, 215, 245	245
5	295, 295, 280, 255, 295, 300	300

For example, $ST(2)=120$ arises from one sandal batch and one trainer batch; this is the best use of two batches before high heels are considered.
At the final stage, compare $H(k)+ST(5-k)$ for each possible number k of high-heel batches.

k	$H(k)+ST(5-k)$	total profit (£000s)
0	$0+300$	300
1	$75+245$	320
2	$115+170$	285
3	$x+120$	$x+120$
4	$235+70$	305
5	$305+0$	305

The only candidates exceeding all other totals are 320 and $x+120$.

Maximum annual profit is either $\$320\,000$ or $\$(x+120)\,000$.

(10)

(b) If the maximum is £320 000, trace back the two rows in the final comparison that equal 320. For $k = 1$*, the earlier value* $ST(4) = 245$ *is obtained with four sandal batches and no trainer batches. For* $k = 3$*,* $ST(2) = 120$ *is obtained with one trainer and one sandal batch.*

	trainers	*sandals*	*high heels*
allocation 1	0	4	1
allocation 2	1	1	3

(2)

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