Maths with Matt

Physics and maths resources

Edexcel A Level Mathematics

9MA0/01 - Paper 1: Pure Mathematics 1

Wednesday 5 June 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

$$g(x) = 3x^3 - 20x^2 + (k+17)x + k,$$

where k is a constant.
Given that $(x-3)$ is a factor of $g(x)$, find the value of k. **(3)**

Model solution

Use the Factor Theorem: since $(x-3)$ *is a factor,* $g(3) = 0$.

$$3(3)^3 - 20(3)^2 + 3(k+17) + k = 0.$$

Therefore

$$81 - 180 + 3k + 51 + k = 0,$$

so

$$4k - 48 = 0.$$

$$k = 12$$

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Question 2

Question 2

(**a**) Find, in ascending powers of x, the first four terms of the binomial expansion of

$$(1-9x)^{1/2},$$

giving each term in simplest form. **(3)**

(**b**) Give a reason why $x = -\frac{2}{9}$ should not be used in the expansion to find an approximation to $\sqrt{3}$. **(1)**

Model solution

(a)

Use the binomial expansion $(1+u)^n = 1 + nu + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \cdots$, *with* $n = \frac{1}{2}$ *and* $u = -9x$.

$$\begin{aligned}(1-9x)^{1/2} &= 1 + \frac{1}{2}(-9x) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}(-9x)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}(-9x)^3 + \cdots \\ &= 1 - \frac{9}{2}x - \frac{81}{8}x^2 - \frac{729}{16}x^3 + \cdots .\end{aligned}$$

$$1 - \frac{9}{2}x - \frac{81}{8}x^2 - \frac{729}{16}x^3$$

(b)

The expansion is valid only when

$$|-9x| < 1, \qquad \textit{that is,} \qquad |x| < \frac{1}{9}.$$

For $x = -\frac{2}{9}$, *we have* $|x| = \frac{2}{9} > \frac{1}{9}$ *(equivalently,* $|-9x| = 2 > 1$*), so the series does not converge there and cannot be used for the approximation.*

Question 3

Question 3

$$f(x) = x + \tan\left(\frac{1}{2}x\right), \quad \pi < x < \frac{3\pi}{2}.$$

Given that the equation $f(x) = 0$ has a single root α,
(**a**) show that α lies in the interval $[3.6, 3.7]$, **(2)**
(**b**) find $f'(x)$, **(2)**
(**c**) using 3.7 as a first approximation for α, apply the Newton–Raphson method once to obtain a second approximation for α. Give your answer to 3 decimal places. **(2)**

Model solution

(a)

The trigonometric values must be evaluated in radians.

$$f(3.6) = 3.6 + \tan(1.8) = -0.68626\ldots < 0,$$

while

$$f(3.7) = 3.7 + \tan(1.85) = 0.21194\ldots > 0.$$

The function is continuous on $[3.6, 3.7]$ *and changes sign, so it has a root in this interval. Since the root is given to be unique,*

$$\boxed{3.6 \leqslant \alpha \leqslant 3.7}.$$

(b)

$$f'(x) = 1 + \frac{1}{2}\sec^2\left(\frac{1}{2}x\right).$$

$$\boxed{f'(x) = 1 + \frac{1}{2}\sec^2\left(\frac{x}{2}\right)}$$

(c)

Newton--Raphson uses

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

With $x_1 = 3.7$,

$$\begin{aligned} x_2 &= 3.7 - \frac{3.7 + \tan(1.85)}{1 + \frac{1}{2}\sec^2(1.85)} \\ &= 3.6720516\ldots. \end{aligned}$$

$$\boxed{\alpha \approx 3.672}$$

Question 4

Question 4

Given that $y = x^2$, use differentiation from first principles to show that

$$\frac{dy}{dx} = 2x.$$

(3)

Model solution

From first principles, the derivative is the limiting value of the gradient of a chord.

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} (2x + h) \\
&= 2x.
\end{aligned}$$

$$\frac{dy}{dx} = 2x$$

Question 5

Question 5

The function f is defined by

$$f(x) = \frac{2x-3}{x^2+4}, \qquad x \in \mathbb{R}.$$

(**a**) Show that

$$f'(x) = \frac{ax^2+bx+c}{(x^2+4)^2},$$

where a, b and c are constants to be found. **(3)**

(**b**) Hence, using algebra, find the values of x for which f is decreasing. You must show each step in your working. **(3)**

Model solution

(a)

Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{2(x^2+4)-(2x-3)(2x)}{(x^2+4)^2} \\ &= \frac{2x^2+8-4x^2+6x}{(x^2+4)^2} \\ &= \frac{-2x^2+6x+8}{(x^2+4)^2}. \end{aligned}$$

Thus $a=-2$, $b=6$ *and* $c=8$.

(b)

The denominator $(x^2+4)^2$ *is positive for every real* x, *so the sign of* $f'(x)$ *is the sign of its numerator.*

The critical values satisfy

$$-2x^2+6x+8=0.$$

Factorising algebraically,

$$-2(x^2-3x-4)=0 \quad \Longrightarrow \quad -2(x+1)(x-4)=0,$$

so $x=-1$ *or* $x=4$.

The downward-opening quadratic $-2(x+1)(x-4)$ *is negative outside these roots. Hence* $f'(x)<0$ *when*

$$x<-1 \qquad \textit{or} \qquad x>4.$$

$$x<-1 \textit{ or } x>4$$

Question 6

Question 6

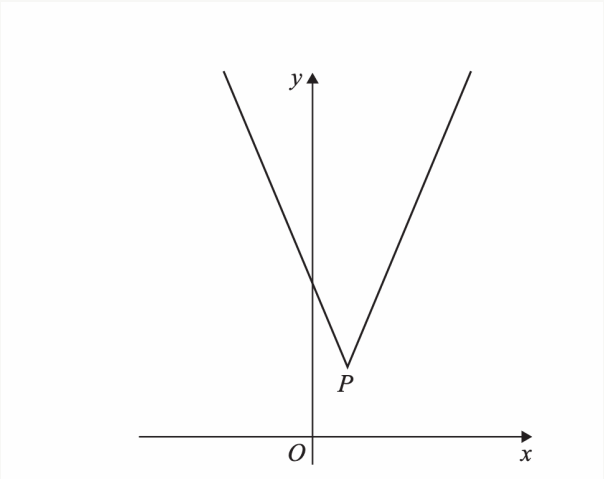


Figure 1 shows a sketch of the graph with equation

$$y = 3|x - 2| + 5.$$

The vertex of the graph is at the point P, shown in Figure 1.

(**a**) Find the coordinates of P. **(2)**

(**b**) Solve the equation

$$16 - 4x = 3|x - 2| + 5.$$

(2)

A line l has equation $y = kx + 4$, where k is a constant.
Given that l intersects $y = 3|x - 2| + 5$ at 2 distinct points,

(**c**) find the range of values of k. **(2)**

Model solution

(a)

The minimum of $|x - 2|$ *occurs at* $x = 2$*, where* $|x - 2| = 0$*. Therefore* $y = 5$*.*

$$P = (2, 5)$$

(b)

For $x \geqslant 2$*,* $|x - 2| = x - 2$*, so*

$$16 - 4x = 3(x - 2) + 5 = 3x - 1.$$

Hence

$$17 = 7x \quad \Longrightarrow \quad x = \frac{17}{7},$$

which satisfies $x \geqslant 2$*.*
For $x < 2$*,* $|x - 2| = 2 - x$*, giving*

$$16 - 4x = 3(2 - x) + 5 = 11 - 3x,$$

and hence $x = 5$*, which contradicts* $x < 2$ *and must be rejected.*

$$x = \frac{17}{7}$$

(c)

The two arms of the graph are

$$y = 3x - 1 \quad (x \geqslant 2), \qquad y = 11 - 3x \quad (x \leqslant 2).$$

On the right arm, intersection with $y = kx + 4$ *gives*

$$kx + 4 = 3x - 1 \quad \Longrightarrow \quad x = \frac{5}{3-k}.$$

For this to lie strictly to the right of the vertex, $x > 2$*, which gives*

$$k < 3 \quad \textit{and} \quad 5 > 2(3-k) \quad \Longrightarrow \quad k > \frac{1}{2}.$$

For these values, the intersection with the left arm is at

$$x = \frac{7}{k+3} < 2,$$

so there is one intersection on each arm. The endpoints are excluded because they do not give two distinct intersections.

$$\frac{1}{2} < k < 3$$

Question 7

Question 7

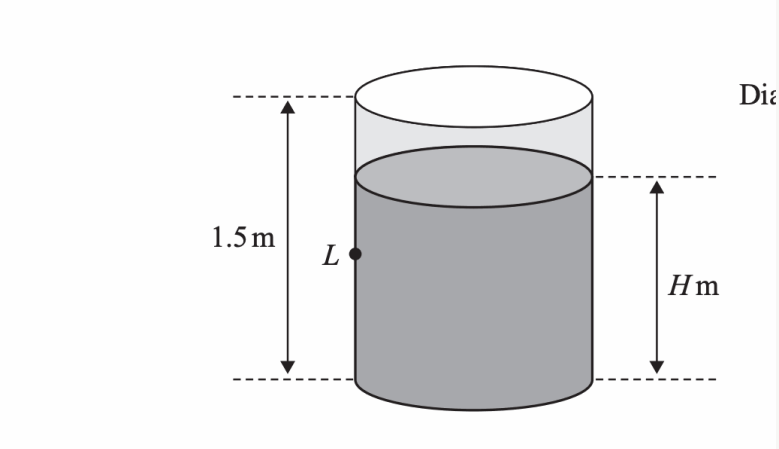


Diagram not drawn to scale.

Figure 2 shows a cylindrical tank of height 1.5 m. Initially the tank is full of water. The water starts to leak from a small hole, at a point L, in the side of the tank.
While the tank is leaking, the depth, H metres, of the water in the tank is modelled by the differential equation

$$\frac{dH}{dt} = -0.12e^{-0.2t},$$

where t hours is the time after the leak starts.
Using the model,
(**a**) show that

$$H = Ae^{-0.2t} + B,$$

where A and B are constants to be found, **(3)**
(**b**) find the time taken for the depth of the water to decrease to 1.2 m. Give your answer in hours and minutes, to the nearest minute. **(3)**
In the long term, the water level in the tank falls to the same height as the hole.
(**c**) Find, according to the model, the height of the hole from the bottom of the tank. **(2)**

Model solution

(a)
Integrating with respect to t,

$$H = \int -0.12e^{-0.2t}\,dt = 0.6e^{-0.2t} + C.$$

Initially $t = 0$ ***and the full tank has*** $H = 1.5$, ***so***

$$1.5 = 0.6e^{0} + C = 0.6 + C,$$

giving $C = 0.9$.

$$H = 0.6e^{-0.2t} + 0.9, \textit{ so } A = 0.6 \textit{ and } B = 0.9$$

(b)

Set $H = 1.2$*:*

$$1.2 = 0.6e^{-0.2t} + 0.9.$$

Therefore

$$e^{-0.2t} = \frac{0.3}{0.6} = \frac{1}{2}.$$

Taking natural logarithms,

$$-0.2t = \ln\left(\frac{1}{2}\right), \qquad t = -5\ln\left(\frac{1}{2}\right) = 5\ln 2 = 3.4657\ldots \text{ hours.}$$

The decimal part is $0.4657\ldots \times 60 = 27.94\ldots$ *minutes.*

3 *hours* 28 *minutes*

(c)

As $t \to \infty$, $e^{-0.2t} \to 0$, *so*

$$H \to 0.6(0) + 0.9 = 0.9.$$

This limiting water depth is the height of the hole.

0.9 *m (or* 90 *cm) from the bottom of the tank*

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Question 8

Question 8

The functions f and g are defined by

$$f(x) = 4 - 3x^2, \qquad x \in \mathbb{R},$$

$$g(x) = \frac{5}{2x-9}, \qquad x \in \mathbb{R}, \quad x \neq \frac{9}{2}.$$

(a) Find $fg(2)$. **(2)**

(b) Find g^{-1}. **(3)**

(c) **(i)** Find $gf(x)$, giving your answer as a simplified fraction.

(ii) Deduce the range of $gf(x)$. **(3)**

The function h is defined by

$$h(x) = 2x^2 - 6x + k, \qquad x \in \mathbb{R},$$

where k is a constant.

(d) Find the range of values of k for which the equation $f(x) = h(x)$ has no real solutions. **(3)**

Model solution

(a)

The notation $fg(2)$ *means* $f(g(2))$.

$$g(2) = \frac{5}{4-9} = -1, \qquad f(-1) = 4 - 3(-1)^2 = 1.$$

$$\boxed{fg(2) = 1}$$

(b)

Let

$$y = \frac{5}{2x-9}.$$

Then

$$y(2x-9) = 5 \implies 2xy = 5 + 9y \implies x = \frac{5+9y}{2y}.$$

Interchanging x *and* y *gives*

$$\boxed{g^{-1}(x) = \frac{5+9x}{2x}, \qquad x \in \mathbb{R},\ x \neq 0}$$

(c)(i)

$$gf(x) = g(f(x)) = \frac{5}{2(4-3x^2)-9} = -\frac{5}{1+6x^2}.$$

$$\boxed{gf(x) = -\frac{5}{1+6x^2}}$$

(c)(ii)

Since $x^2 \geqslant 0$, the denominator satisfies $1+6x^2 \geqslant 1$. The value -5 occurs when $x = 0$, while the expression approaches 0 from below as $|x| \to \infty$ but never equals 0.

$$-5 \leqslant gf(x) < 0$$

(d)

Setting $f(x) = h(x)$ gives

$$4 - 3x^2 = 2x^2 - 6x + k,$$

or

$$5x^2 - 6x + k - 4 = 0.$$

For no real solutions, the discriminant must be negative:

$$(-6)^2 - 4(5)(k-4) < 0.$$

Thus

$$36 - 20k + 80 < 0 \implies 116 < 20k \implies k > \frac{29}{5}.$$

$$k > \frac{29}{5}$$

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Question 9

Question 9

The first 3 terms of a geometric sequence are

$$3^{4k-5}, \qquad 9^{7-2k}, \qquad 3^{2(k-1)},$$

where k is a constant.

(**a**) Using algebra and making your reasoning clear, prove that $k = \frac{5}{2}$. **(3)**

(**b**) Hence find the sum to infinity of the geometric sequence. **(3)**

Model solution

(a)

For three consecutive terms of a geometric sequence, the square of the middle term equals the product of the terms on either side.

$$\left(9^{7-2k}\right)^2 = 3^{4k-5} \cdot 3^{2(k-1)}.$$

Writing $9 = 3^2$ and using the laws of indices,

$$3^{28-8k} = 3^{(4k-5)+(2k-2)} = 3^{6k-7}.$$

The bases are equal, so the powers are equal:

$$28 - 8k = 6k - 7 \quad \Longrightarrow \quad 35 = 14k \quad \Longrightarrow \quad \boxed{k = \frac{5}{2}}.$$

(b)

With $k = \frac{5}{2}$, the terms begin

$$3^5 = 243, \qquad 9^2 = 81, \qquad 3^3 = 27.$$

Thus the first term is $a = 243$ and the common ratio is

$$r = \frac{81}{243} = \frac{1}{3}.$$

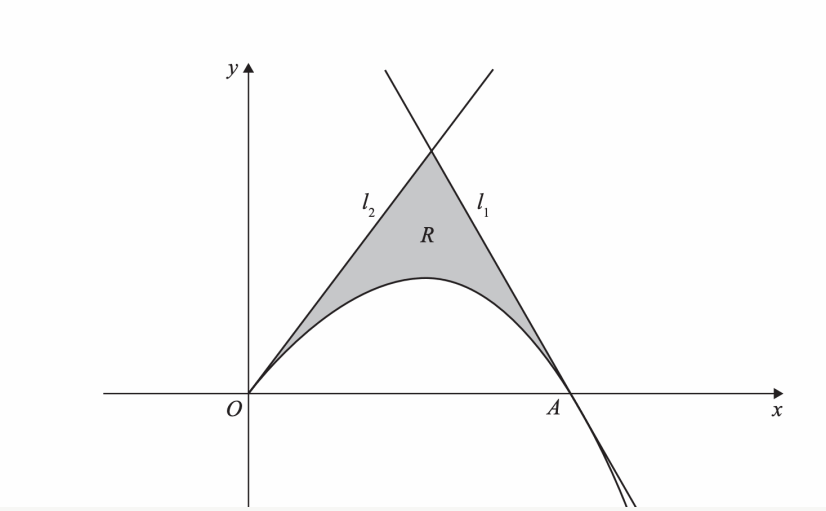
Since $|r| < 1$, the sum to infinity is

$$S_\infty = \frac{a}{1-r} = \frac{243}{1-\frac{1}{3}} = \frac{729}{2}.$$

$$\boxed{S_\infty = \frac{729}{2}}$$

Question 10

Question 10



In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.
Figure 3 shows a sketch of part of the curve with equation

$$y = 8x - x^{5/2}, \qquad x \geqslant 0.$$

The curve crosses the x-axis at the point A.
(a) Verify that the x coordinate of A is 4. **(1)**
The line l_1 is the tangent to the curve at A.
(b) Use calculus to show that an equation of line l_1 is

$$12x + y = 48.$$

(3)
The line l_2 has equation $y = 8x$. The region R, shown shaded in Figure 3, is bounded by the curve, the line l_1 and the line l_2.
(c) Use algebraic integration to find the exact area of R. **(5)**

Model solution

(a)

$$y = 8(4) - 4^{5/2} = 32 - (\sqrt{4})^5 = 32 - 32 = 0.$$

Therefore the curve meets the x-axis at $x = 4$, as required.
(b)
Differentiating,

$$\frac{dy}{dx} = 8 - \frac{5}{2}x^{3/2}.$$

At $A = (4, 0)$,

$$\frac{dy}{dx} = 8 - \frac{5}{2}(4^{3/2}) = 8 - \frac{5}{2}(8) = -12.$$

The tangent through $(4, 0)$ is therefore

$$y = -12(x - 4) = -12x + 48,$$

so

$$\boxed{12x + y = 48}.$$

(c)

The two lines intersect where

$$8x = 48 - 12x,$$

so

$$x = \frac{12}{5}, \qquad y = 8\left(\frac{12}{5}\right) = \frac{96}{5}.$$

The area enclosed by the two lines and the x-axis is the triangle

$$\frac{1}{2} \times 4 \times \frac{96}{5} = \frac{192}{5}.$$

The area below the curve from $x = 0$ to $x = 4$ is

$$\begin{aligned}\int_0^4 (8x - x^{5/2})\,dx &= \left[4x^2 - \frac{2}{7}x^{7/2}\right]_0^4 \\ &= 64 - \frac{2}{7}(128) = \frac{192}{7}.\end{aligned}$$

Hence

$$\textit{area}(R) = \frac{192}{5} - \frac{192}{7} = \frac{384}{35}.$$

$$\frac{384}{35} \textit{ square units}$$

Question 11

Question 11

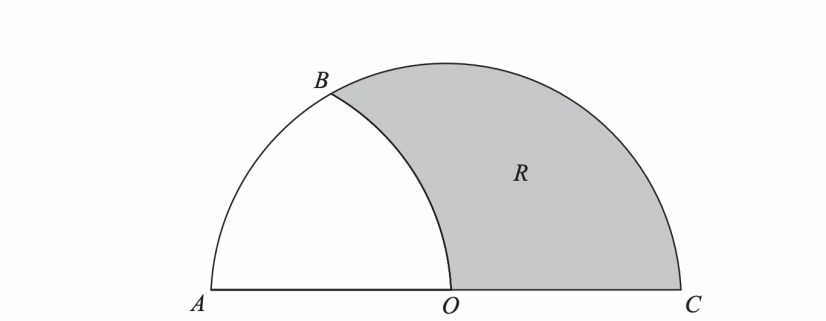


Figure 4 shows the design of a badge.
The shape ABCOA is a semicircle with centre O and diameter 10 cm. OB is the arc of a circle with centre A and radius 5 cm.
The region R, shown shaded in Figure 4, is bounded by the arc OB, the arc BC and the line OC.
Find the exact area of R. Give your answer in the form $(a\sqrt{3} + b\pi)\,\text{cm}^2$, where a and b are rational numbers. **(4)**

Model solution

The large semicircle has radius 5 *cm. Also*

$$\text{AO} = \text{AB} = \text{OB} = 5,$$

so triangle AOB *is equilateral. Therefore*

$$\angle\text{BAO} = \angle\text{AOB} = \frac{\pi}{3}, \qquad \angle\text{BOC} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

The sector BOC *of the large circle has area*

$$\frac{1}{2}(5^2)\left(\frac{2\pi}{3}\right) = \frac{25\pi}{3}.$$

The small circular segment between chord OB *and arc* OB *has area*

$$\begin{aligned}\textit{sector AOB} - \triangle\text{AOB} &= \frac{1}{2}(5^2)\left(\frac{\pi}{3}\right) - \frac{1}{2}(5^2)\sin\left(\frac{\pi}{3}\right)\\ &= \frac{25\pi}{6} - \frac{25\sqrt{3}}{4}.\end{aligned}$$

The required region is the large sector minus this small segment:

$$\begin{aligned}\textit{area}(\text{R}) &= \frac{25\pi}{3} - \left(\frac{25\pi}{6} - \frac{25\sqrt{3}}{4}\right)\\ &= \frac{25\sqrt{3}}{4} + \frac{25\pi}{6}.\end{aligned}$$

$$\left(\frac{25}{4}\sqrt{3} + \frac{25}{6}\pi\right)\text{cm}^2$$

Question 12

Question 12

(**a**) Express $140\cos\theta - 480\sin\theta$ in the form $K\cos(\theta + \alpha)$, where $K > 0$ and $0 < \alpha < 90°$. State the value of K and give the value of α, in degrees, to 2 decimal places. **(3)**

A scientist studies the number of rabbits and the number of foxes in a wood for one year.

The number of rabbits, R, is modelled by the equation

$$R = A + 140\cos(30t)° - 480\sin(30t)°,$$

where t months is the time after the start of the year and A is a constant.

Given that, during the year, the maximum number of rabbits in the wood is 1500,

(**b**) (**i**) find a complete equation for this model,

(**ii**) hence write down the minimum number of rabbits in the wood during the year according to the model. **(2)**

The actual number of rabbits in the wood is at its minimum value in the middle of April.

(**c**) Use this information to comment on the model for the number of rabbits. **(2)**

The number of foxes, F, in the wood during the same year is modelled by the equation

$$F = 100 + 70\sin(30t + 70)°.$$

The number of foxes is at its minimum value after T months.

(**d**) Find, according to the models, the number of rabbits in the wood at time T months. **(4)**

Model solution

(a)

Expanding the required form,

$$K\cos(\theta + \alpha) = K\cos\alpha\cos\theta - K\sin\alpha\sin\theta.$$

Comparing coefficients gives

$$K\cos\alpha = 140, \qquad K\sin\alpha = 480.$$

Thus

$$K = \sqrt{140^2 + 480^2} = 500$$

and

$$\tan\alpha = \frac{480}{140} = \frac{24}{7}, \qquad \alpha = 73.7397\ldots° = 73.74°.$$

$$140\cos\theta - 480\sin\theta = 500\cos(\theta + 73.74°)$$

(b)(i)

The maximum of $A + 500\cos(30t + 73.74)°$ *is* $A + 500$. *Therefore*

$$A + 500 = 1500 \implies A = 1000.$$

$$R = 1000 + 500\cos(30t + 73.74)°$$

(b)(ii)
The minimum occurs when the cosine is -1*:*

$$R_{\min} = 1000 - 500 = 500.$$

500 ***rabbits***

(c)
The model reaches its minimum when

$$30t + 73.74 = 180,$$

so

$$t = \frac{106.26}{30} = 3.542 \text{ } \textit{months}.$$

This is approximately the middle of April, agreeing closely with the observed timing. Therefore the model is reliable in respect of the time at which the minimum occurs.
(d)
The number of foxes is a minimum when the sine term is -1*. During the year,*

$$30T + 70 = 270 \quad \Longrightarrow \quad T = \frac{200}{30} = \frac{20}{3}.$$

At this time, $30T = 200°$*, so using the unrounded rabbit model,*

$$\begin{aligned} R &= 1000 + 140\cos 200° - 480\sin 200° \\ &= 1032.6\ldots. \end{aligned}$$

The number of rabbits must be a whole number.

1033 ***rabbits***

Question 13

Question 13

(**a**) Given that a is a positive constant, use the substitution $x = a\sin^2\theta$ to show that

$$\int_0^a x^{1/2}\sqrt{a-x}\,dx = \frac{1}{2}a^2\int_0^{\pi/2}\sin^2(2\theta)\,d\theta.$$

(4)

(**b**) Hence use algebraic integration to show that

$$\int_0^a x^{1/2}\sqrt{a-x}\,dx = k\pi a^2,$$

where k is a constant to be found. **(4)**

Model solution

(a)

With $x = a\sin^2\theta$,

$$\frac{dx}{d\theta} = 2a\sin\theta\cos\theta, \qquad dx = 2a\sin\theta\cos\theta\,d\theta.$$

For $0 \leqslant \theta \leqslant \frac{\pi}{2}$,

$$x^{1/2} = \sqrt{a}\sin\theta$$

and

$$\sqrt{a-x} = \sqrt{a-a\sin^2\theta} = \sqrt{a}\cos\theta.$$

The limits transform as

$$x = 0 \Rightarrow \theta = 0, \qquad x = a \Rightarrow \theta = \frac{\pi}{2}.$$

Therefore

$$\begin{aligned}\int_0^a x^{1/2}\sqrt{a-x}\,dx &= \int_0^{\pi/2}(\sqrt{a}\sin\theta)(\sqrt{a}\cos\theta)(2a\sin\theta\cos\theta)\,d\theta \\ &= 2a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta.\end{aligned}$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$2\sin^2\theta\cos^2\theta = \frac{1}{2}\sin^2(2\theta).$$

Hence

$$\boxed{\int_0^a x^{1/2}\sqrt{a-x}\,dx = \frac{1}{2}a^2\int_0^{\pi/2}\sin^2(2\theta)\,d\theta}.$$

(b)

Using $\sin^2(2\theta) = \dfrac{1-\cos 4\theta}{2}$,

$$\begin{aligned}\int_0^a x^{1/2}\sqrt{a-x}\,dx &= \frac{1}{2}a^2\int_0^{\pi/2}\frac{1-\cos 4\theta}{2}\,d\theta \\ &= \frac{a^2}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} \\ &= \frac{a^2}{4}\left(\frac{\pi}{2}\right) \\ &= \frac{1}{8}\pi a^2.\end{aligned}$$

$$k = \frac{1}{8}$$

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Question 14

Question 14

A balloon is being inflated. In a simple model,

- the balloon is modelled as a sphere,
- the rate of increase of the radius of the balloon is inversely proportional to the square root of the radius of the balloon.

At time t seconds, the radius of the balloon is r cm.
(**a**) Write down a differential equation to model this situation. **(1)**
At the instant when $t = 10$,

- the radius is 16 cm,
- the radius is increasing at a rate of $0.9\,\text{cm}\,\text{s}^{-1}$.

(**b**) Solve the differential equation to show that

$$r^{3/2} = 5.4t + 10.$$

(5)
(**c**) Hence find the radius of the balloon when $t = 20$. Give your answer to the nearest millimetre.
(2)
(**d**) Suggest a limitation of the model. **(1)**

Model solution

(a)

Inverse proportionality to $\sqrt{r}$ *gives*

$$\frac{dr}{dt} = \frac{k}{\sqrt{r}}, \text{ where } k > 0$$

(b)

When $r = 16$ *and* $\frac{dr}{dt} = 0.9$,

$$0.9 = \frac{k}{\sqrt{16}} = \frac{k}{4},$$

so $k = 3.6$. *Hence*

$$\frac{dr}{dt} = \frac{3.6}{\sqrt{r}}.$$

Separating variables and integrating,

$$\int r^{1/2}\,dr = \int 3.6\,dt,$$

which gives

$$\frac{2}{3}r^{3/2} = 3.6t + C.$$

Using $t = 10$ *and* $r = 16$,

$$\frac{2}{3}(16^{3/2}) = 36 + C.$$

Since $16^{3/2} = 64$,

$$C = \frac{128}{3} - 36 = \frac{20}{3}.$$

Thus

$$\frac{2}{3}r^{3/2} = 3.6t + \frac{20}{3}.$$

Multiplying by $\frac{3}{2}$ *gives*

$$\boxed{r^{3/2} = 5.4t + 10},$$

as required.

(c)

When $t = 20$,

$$r^{3/2} = 5.4(20) + 10 = 118,$$

so

$$r = 118^{2/3} = 24.0577\ldots \text{ cm}.$$

To the nearest millimetre, this is 24.1 *cm, or* 241 *mm.*

24.1 cm (or 241 mm)

(d)

The model predicts that the radius will continue to increase without limit, but a real balloon will eventually burst (and may cease to remain spherical).

Question 15

Question 15

(i) Show that $k^2 - 4k + 5$ is positive for all real values of k. **(2)**

(ii) A student was asked to prove by contradiction that

"There are no positive integers x and y such that

$$(3x + 2y)(2x - 5y) = 28.$$

"

The start of the student's proof is shown below.

Assume that positive integers x and y exist such that

$$(3x + 2y)(2x - 5y) = 28.$$

If $3x + 2y = 14$ and $2x - 5y = 2$, then

$$\left.\begin{aligned} 3x + 2y &= 14 \\ 2x - 5y &= 2 \end{aligned}\right\} \implies x = \frac{74}{19}, \quad y = \frac{22}{19},$$

which are not integers.

Show the calculations and statements needed to complete the proof. **(4)**

Model solution

(i)

Completing the square,

$$k^2 - 4k + 5 = (k - 2)^2 + 1.$$

For every real k, $(k - 2)^2 \geqslant 0$, *and therefore*

$$(k - 2)^2 + 1 \geqslant 1 > 0.$$

Hence $k^2 - 4k + 5$ *is positive for every real value of* k.

(ii)

Continue from the assumed existence of positive integers x *and* y. *We must exhaust the possible integer factor pairs of* 28.

Let

$$u = 3x + 2y, \qquad v = 2x - 5y.$$

Since x *and* y *are positive integers,*

$$u = 3x + 2y \geqslant 5 > 0.$$

As $uv = 28 > 0$, *it follows that* $v > 0$. *Also*

$$u - v = (3x + 2y) - (2x - 5y) = x + 7y > 0,$$

so $u > v$. *The only positive factor pairs of* 28 *with* $u > v$ *are*

$$(u, v) = (28, 1), \ (14, 2), \ (7, 4).$$

The pair $(14, 2)$ *has already been rejected in the question.*

For $(u, v) = (28, 1)$,

$$\left.\begin{aligned} 3x + 2y &= 28 \\ 2x - 5y &= 1 \end{aligned}\right\}$$

Multiplying the first equation by 2 *and the second by* 3 *gives*

$$6x + 4y = 56, \qquad 6x - 15y = 3.$$

Subtracting gives $19y = 53$, *so*

$$y = \frac{53}{19}, \qquad x = \frac{142}{19}.$$

These are not integers.
For $(u, v) = (7, 4)$,

$$\left.\begin{aligned} 3x + 2y &= 7 \\ 2x - 5y &= 4 \end{aligned}\right\}$$

Multiplying the first equation by 2 *and the second by* 3 *gives*

$$6x + 4y = 14, \qquad 6x - 15y = 12.$$

Subtracting gives $19y = 2$, *so*

$$y = \frac{2}{19}, \qquad x = \frac{43}{19}.$$

Again these are not integers.
Every possible factor pair has produced a contradiction. Therefore there are no positive integers x *and* y *satisfying* $(3x + 2y)(2x - 5y) = 28$, *which completes the proof.*