Maths with Matt

Physics and maths resources

Edexcel A Level Mathematics

9MA0/01 - Pure Mathematics 1

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Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2025 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

The point $P(6, -4)$ lies on the curve with equation $y = f(x)$, $x \in \mathbb{R}$.
Find the point to which P is mapped when the curve with equation $y = f(x)$ is transformed to the curve with equation

(a) $y = f(x + 2)$ **(1)**

(b) $y = f^{-1}(x)$ **(1)**

(c) $y = 2|f(x)| - 3$ **(2)**

Model solution

(a)

Replacing x by $x+2$ translates the graph two units to the left; the y-coordinate is unchanged.

$$(6, -4) \longmapsto (6 - 2, -4).$$

$$(4, -4)$$

(b)

The graph of an inverse function is the reflection in $y = x$, so the two coordinates are interchanged.

$$(6, -4) \longmapsto (-4, 6).$$

$$(-4, 6)$$

(c)

The transformation changes the ordinate according to

$$y = 2|-4| - 3 = 8 - 3 = 5,$$

while the x-coordinate remains 6.

$$(6, 5)$$

Question 2

Question 2

The circle C has equation

$$(x+3)^2 + (y-4)^2 = 24.$$

(a) **(i)** State the coordinates of the centre of C.
(ii) Find the radius of C, writing your answer as a fully simplified surd. **(3)**
(b) Determine, giving a reason, whether or not the origin lies inside the circle C. **(2)**

Model solution

(a)(i)

Comparing with $(x-a)^2 + (y-b)^2 = r^2$*, the centre is*

$$(-3, 4)$$

(a)(ii)

$$r^2 = 24 \implies r = \sqrt{24} = 2\sqrt{6}.$$

$$2\sqrt{6}$$

(b)

Compare the squared distance from the centre to the origin with the squared radius.

$$d^2 = (0+3)^2 + (0-4)^2 = 9 + 16 = 25.$$

Since $25 > 24 = r^2$*, the origin is farther from the centre than the circumference.*

The origin lies outside C.

Question 3

Question 3

The first three terms of an arithmetic sequence are

$$6k, \quad 10, \quad 2k,$$

where k is a constant.

(**a**) Find the value of k. **(2)**

(**b**) Hence find the value of the sum of the first 50 terms of this sequence. **(3)**

Model solution

(a)

Consecutive terms of an arithmetic sequence have the same difference.

$$10 - 6k = 2k - 10.$$

Hence $20 = 8k$, *so*

$$k = \frac{5}{2}$$

(b)

The first term and common difference are

$$a = 6\left(\frac{5}{2}\right) = 15, \qquad d = 10 - 15 = -5.$$

Using $S_n = \frac{n}{2}\{2a + (n-1)d\}$,

$$S_{50} = \frac{50}{2}\{2(15) + 49(-5)\} = 25(30 - 245) = -5375.$$

$$S_{50} = -5375$$

Question 4

Question 4

Given that

$$f(x) = 2x^3 + 3x^2 - 16x + 16 \qquad \text{and} \qquad f(-4) = 0,$$

(a) write $f(x)$ in the form $(x + a)Q(x)$, where a is a constant and $Q(x)$ is a quadratic expression.
(3)

(b) Hence prove that -4 is the only real root of the equation $f(x) = 0$.
Solutions relying on calculator technology are not acceptable. **(2)**

Model solution

(a)

Since $f(-4) = 0$, *the Factor Theorem shows that* $x + 4$ *is a factor. Synthetic division by* $x + 4$ *gives*

$$\begin{array}{c|cccc} -4 & 2 & 3 & -16 & 16 \\ & & -8 & 20 & -16 \\ \hline & 2 & -5 & 4 & 0 \end{array}$$

so the quotient is $2x^2 - 5x + 4$ *and therefore*

$$f(x) = (x + 4)(2x^2 - 5x + 4).$$

$$a = 4, \quad Q(x) = 2x^2 - 5x + 4$$

(b)

For the quadratic factor, the discriminant is

$$(-5)^2 - 4(2)(4) = 25 - 32 = -7 < 0.$$

Thus $2x^2 - 5x + 4$ *has no real roots. The linear factor supplies* $x = -4$, *so*

-4 *is the only real root.*

Question 5

Question 5

(**a**) Express

$$\lim_{\delta x \to 0} \sum_{x=1.44}^{2.89} \frac{2}{\sqrt{x}} \delta x$$

as an integral. **(1)**

(**b**) Hence show that

$$\lim_{\delta x \to 0} \sum_{x=1.44}^{2.89} \frac{2}{\sqrt{x}} \delta x = k,$$

where k is an integer to be found.
Solutions relying on calculator technology are not acceptable. **(2)**

Model solution

(a)

The limiting sum is a Riemann sum: the summand gives the integrand and the displayed endpoints give the limits.

$$\int_{1.44}^{2.89} \frac{2}{\sqrt{x}} \, dx$$

(b)

Since $2x^{-1/2}$ *integrates to* $4x^{1/2}$,

$$\int_{1.44}^{2.89} \frac{2}{\sqrt{x}} \, dx = \left[4\sqrt{x}\right]_{1.44}^{2.89} = 4(1.7) - 4(1.2) = 2.$$

$$k = 2$$

Question 6

Question 6

A scientist is monitoring the flight of sea birds after they leave their nests on a cliff.
The height above the sea, h metres, of one of the birds is modelled by

$$h = A - Bt^{1.5}, \qquad h \geqslant 0, \qquad 0 \leqslant t \leqslant T,$$

where t seconds is the time after the bird leaves its nest and A, B and T are positive constants.
Given that the bird was 17.6 m above the sea exactly 4 seconds after leaving its nest and 11.9 m above the sea exactly 9 seconds after leaving its nest,
(**a**) find a complete equation for the model. **(4)**
Find, according to the model,
(**b**) the height of the bird's nest above the sea, **(1)**
(**c**) the limitation on the value of T. **(2)**

Model solution

(a)
Substituting the two observations gives

$$17.6 = A - B(4^{1.5}) = A - 8B,$$

$$11.9 = A - B(9^{1.5}) = A - 27B.$$

Subtracting the second equation from the first,

$$5.7 = 19B \implies B = 0.3.$$

Then $A = 17.6 + 8(0.3) = 20$.

$$h = 20 - 0.3t^{1.5}$$

(b)
At the nest, $t = 0$, *so* $h = 20$.

$$20 \text{ } m$$

(c)
The model requires $h \geqslant 0$, *so its endpoint cannot be later than the time at which the model reaches sea level.*

$$0 = 20 - 0.3T^{3/2} \implies T^{3/2} = \frac{200}{3} \implies T = \left(\frac{200}{3}\right)^{2/3} = 16.4\ldots$$

$$0 < T \leqslant 16.4 \text{ } s \text{ (to 3 s.f.)}$$

Question 7

Question 7

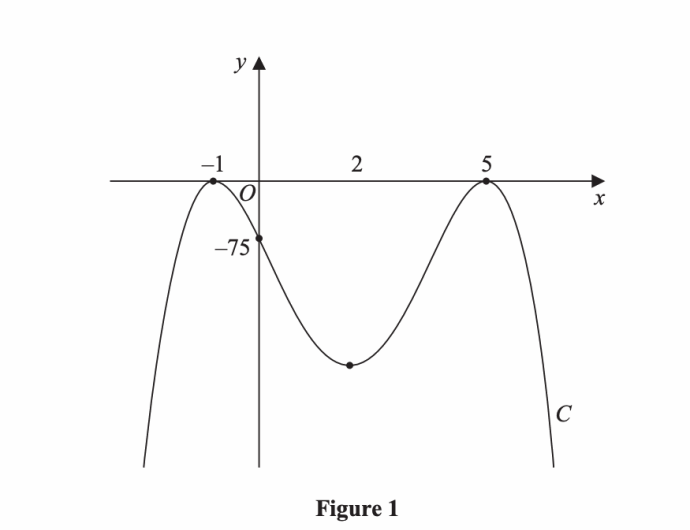


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = f(x)$, where $f(x)$ is a quartic expression in x.

The curve has maximum turning points at $(-1, 0)$ and $(5, 0)$, crosses the y-axis at $(0, -75)$, and has a minimum turning point at $x = 2$.

(**a**) Find the set of values of x for which $f'(x) \geqslant 0$, writing your answer in set notation. **(2)**

(**b**) Find the equation of C. You may leave your answer in factorised form. **(3)**

The curve C_1 has equation $y = f(x) + k$, where k is a constant. Given that the graph of C_1 intersects the x-axis at exactly four places,

(**c**) find the range of possible values for k. **(2)**

Model solution

(a)

$f'(x) \geqslant 0$ *where the graph is increasing or horizontal. Reading the turning points from left to right gives*

$$x \leqslant -1 \quad \textit{or} \quad 2 \leqslant x \leqslant 5.$$

$$\{x \in \mathbb{R} : x \leqslant -1\} \cup \{x \in \mathbb{R} : 2 \leqslant x \leqslant 5\}$$

(b)

The graph touches, rather than crosses, the x*-axis at* $x = -1$ *and* $x = 5$*. These are therefore repeated roots, so*

$$f(x) = a(x+1)^2(x-5)^2.$$

Using $(0, -75)$*,*

$$-75 = a(1)^2(-5)^2 = 25a,$$

so $a = -3$*.*

$$y = -3(x+1)^2(x-5)^2$$

(c)
At the minimum, where $x = 2$*,*

$$f(2) = -3(3)^2(-3)^2 = -243.$$

For four distinct x*-intercepts, the upward vertical shift must move both maxima above the axis while leaving the minimum below it. Thus* $k > 0$ *and* $-243 + k < 0$*; equality at either end would merge roots.*

$$0 < k < 243$$

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Question 8

Question 8

A student was asked to prove that the square of any number can be expressed in the form $5n$ or $5n \pm 1$, where $n \in \mathbb{N}$.
The start of the student's proof is shown below.

$$\begin{aligned} m = 5k: \quad & m^2 = (5k)^2 = 25k^2 = 5(5k^2) = 5n. \\ m = 5k+1: \quad & m^2 = (5k+1)^2 = 25k^2 + 10k + 1 \\ & = 5(5k^2 + 2k) + 1 = 5n + 1. \\ m = 5k+2: \quad & m^2 = (5k+2)^2 = 25k^2 + 10k + 4 \\ & = 5(5k^2 + 2k + 1) - 1 = 5n - 1. \end{aligned}$$

(**a**) Identify and correct an algebraic error in the work above. **(1)**
(**b**) Show the calculations and statements that are required to complete the proof. **(4)**

Model solution

(a)
The expansion in the case $m = 5k + 2$ *is incorrect:*

$$(5k + 2)^2 = 25k^2 + 20k + 4,$$

not $25k^2 + 10k + 4$. *In full,*

$$25k^2 + 20k + 4 = 5(5k^2 + 4k + 1) - 1.$$

Replace $10k$ *by* $20k$ *(and hence* $2k$ *by* $4k$*).*

(b)
Every integer has exactly one of the five forms $5k$, $5k + 1$, $5k + 2$, $5k + 3$ *and* $5k + 4$. *The first three cases have been handled, so it remains to check the last two.*
If $m = 5k + 3$, *then*

$$m^2 = (5k + 3)^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 2) - 1,$$

which has the form $5n - 1$.
If $m = 5k + 4$, *then*

$$m^2 = (5k + 4)^2 = 25k^2 + 40k + 16 = 5(5k^2 + 8k + 3) + 1,$$

which has the form $5n + 1$.
All five possible residue classes have now produced $5n$, $5n - 1$ *or* $5n + 1$. *Therefore the result is proved.*

Question 9

Question 9

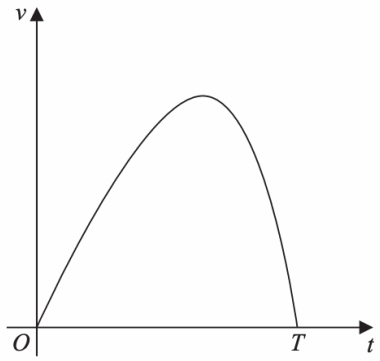


Figure 2

A racing car is driven along a straight road. Figure 2 shows a graph of its speed. The car starts from rest and is driven for T seconds before stopping. Its speed is modelled by

$$v = 15t - te^{0.2t}, \qquad 0 \leqslant t \leqslant T,$$

where t seconds is the time after the car starts to move.
According to the model,
(**a**) find T, giving your answer to one decimal place, **(2)**
(**b**) show that the maximum speed occurs when

$$t = 5\ln\left(\frac{75}{t+5}\right).$$

(4)

Using

$$t_{n+1} = 5\ln\left(\frac{75}{t_n+5}\right), \qquad t_1 = 8,$$

(**c**) (**i**) find t_3 to 3 decimal places,
(**ii**) find, by repeated iteration, the time taken for the car to reach maximum speed. **(3)**

Model solution

(a)

At the stopping time $T > 0$, $v = 0$. *Therefore*

$$0 = 15T - Te^{0.2T} \quad \Longrightarrow \quad e^{0.2T} = 15.$$

Hence

$$T = \frac{\ln 15}{0.2} = 5\ln 15 = 13.540\ldots$$

$$T = 13.5 \text{ s (1 d.p.)}$$

(b)

Using the product rule on $te^{0.2t}$,

$$\frac{dv}{dt} = 15 - \left(e^{0.2t} + 0.2te^{0.2t}\right) = 15 - e^{0.2t}(1 + 0.2t).$$

At an interior maximum, $\frac{dv}{dt} = 0$, *so*

$$e^{0.2t}(1 + 0.2t) = 15 \quad \Longrightarrow \quad e^{0.2t} = \frac{15}{1 + 0.2t} = \frac{75}{t + 5}.$$

Taking natural logarithms,

$$0.2t = \ln\left(\frac{75}{t + 5}\right),$$

and hence

$$t = 5\ln\left(\frac{75}{t + 5}\right)$$

(c)(i)

$$t_2 = 5\ln\left(\frac{75}{8 + 5}\right) = 8.762693780\ldots,$$

$$t_3 = 5\ln\left(\frac{75}{8.762693780\ldots + 5}\right) = 8.477632656\ldots$$

$$t_3 = 8.478 \textit{ (3 d.p.)}$$

(c)(ii)

Continuing the iteration gives

$$8.477633,\ 8.582283,\ 8.543609,\ 8.557867,\ 8.552606,\ 8.554546,\ldots$$

which converges to $8.554023\ldots$ *seconds.*

$$8.55 \textit{ s (3 s.f.)}$$

Question 10

Question 10

Given that

$$\overrightarrow{PQ} = 2\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}, \qquad \overrightarrow{QR} = 6\mathbf{i} + 6\mathbf{k},$$

(**a**) find $\overrightarrow{PR}$. **(2)**

(**b**) Hence show that triangle PQR is both right-angled and isosceles. **(4)**

Model solution

(a)

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = (2+6)\mathbf{i} + (8+0)\mathbf{j} + (-2+6)\mathbf{k}.$$

$$\overrightarrow{PR} = 8\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$$

(b)

First compare the two shorter sides:

$$|\overrightarrow{PQ}|^2 = 2^2 + 8^2 + (-2)^2 = 72,$$

$$|\overrightarrow{QR}|^2 = 6^2 + 0^2 + 6^2 = 72.$$

Thus $PQ = QR = 6\sqrt{2}$, *so the triangle is isosceles.*

Also,

$$\overrightarrow{PQ} \cdot \overrightarrow{QR} = 2(6) + 8(0) + (-2)(6) = 0.$$

Hence $PQ \perp QR$, *so* $\angle PQR = 90°$. *Therefore*

Triangle PQR *is right-angled at* Q *and isosceles.*

Question 11

Question 11

In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.
The value of a car, $V, is modelled by

$$V = 1500 + Ae^{-kt},$$

where A and k are positive constants and t is the age of the car in years.
Given that the initial value was $20 000 and the value was $12 000 when the car was 2.5 years old,
(**a**) find a complete equation for the model, giving the exact value of A and the value of k to 3 significant figures. **(4)**
(**b**) Show that the rate of change in the value of the car can be expressed as

$$-k(V - 1500).$$

(3)
(**c**) State a limitation of this model. **(1)**

Model solution

(a)
At $t = 0$, $V = 20000$, *so*

$$20000 = 1500 + Ae^{0} \quad \Longrightarrow \quad A = 18500.$$

At $t = 2.5$, $V = 12000$, *so*

$$12000 = 1500 + 18500e^{-2.5k}$$

and therefore

$$e^{-2.5k} = \frac{10500}{18500} = \frac{21}{37}.$$

Taking logarithms,

$$-2.5k = \ln\left(\frac{21}{37}\right) \quad \Longrightarrow \quad k = \frac{2}{5}\ln\left(\frac{37}{21}\right) = 0.2265581899\ldots$$

Thus, to the requested accuracy,

$$V = 1500 + 18500e^{-0.227t}$$

(b)
Differentiating the general model,

$$\frac{dV}{dt} = -kAe^{-kt}.$$

From the original equation, $Ae^{-kt} = V - 1500$. *Substituting this into the derivative gives*

$$\frac{dV}{dt} = -k(V - 1500).$$

$$\textit{Rate of change} = -k(V - 1500)$$

(c)

As t increases, the model approaches $V = 1500$ and never falls below it. In reality, a sufficiently old car could be worth less than $\$1500$ (or have only scrap value), so the model is unsuitable indefinitely.

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Question 12

Question 12

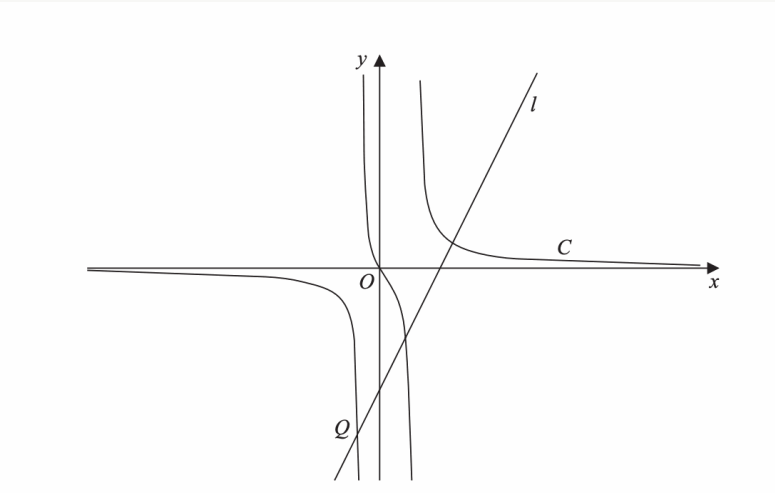


Figure 3 shows a sketch of the curve C with equation

$$y = \frac{15x}{(2x+3)(x-3)}, \qquad x \neq -\frac{3}{2}, \quad x \neq 3,$$

and the straight line l with equation $y = 2x - 10$.

(**a**) Verify that C and l intersect where $x = 6$. **(2)**

The curve and line also intersect at the point Q shown in Figure 3.

(**b**) Show that the x coordinate of Q is a solution of

$$4x^3 - 26x^2 - 3x + 90 = 0.$$

(2)

(**c**) Using algebra and showing all stages of working, find the exact x coordinate of Q. **(4)**

Model solution

(a)

On C, *when* $x = 6$,

$$y = \frac{15(6)}{(2(6)+3)(6-3)} = \frac{90}{15 \cdot 3} = 2.$$

On l, $y = 2(6) - 10 = 2$. *Both equations give the same point* $(6, 2)$, *so they intersect where* $x = 6$.

(b)

At an intersection,

$$\frac{15x}{(2x+3)(x-3)} = 2x - 10.$$

Cross-multiplying,

$$15x = (2x-10)(2x+3)(x-3).$$

Since

$$(2x-10)(2x+3) = 4x^2 - 14x - 30,$$

we obtain

$$15x = (4x^2 - 14x - 30)(x - 3) = 4x^3 - 26x^2 + 12x + 90.$$

Therefore

$$4x^3 - 26x^2 - 3x + 90 = 0$$

(c)

From part (a), $x = 6$ *is a root, so* $x - 6$ *is a factor. Division gives*

$$4x^3 - 26x^2 - 3x + 90 = (x - 6)(4x^2 - 2x - 15).$$

The remaining roots satisfy

$$4x^2 - 2x - 15 = 0,$$

so the quadratic formula gives

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(4)(-15)}}{8} = \frac{2 \pm 2\sqrt{61}}{8} = \frac{1 \pm \sqrt{61}}{4}.$$

The diagram places Q *on the negative-*x *branch, so the positive root is not the coordinate of* Q*.*

$$x_Q = \frac{1 - \sqrt{61}}{4}$$

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Question 13

Question 13

In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.
Show that

$$\int_0^2 \frac{x}{(2x+1)^3}\,dx = \frac{2}{25}.$$

(5)

Model solution

Use $u = 2x+1$ to turn the denominator into a simple power of u.
Let

$$u = 2x+1, \qquad x = \frac{u-1}{2}, \qquad dx = \frac{1}{2}\,du.$$

The limits change from $x = 0, 2$ to $u = 1, 5$. Hence

$$\int_0^2 \frac{x}{(2x+1)^3}\,dx = \frac{1}{4}\int_1^5 \frac{u-1}{u^3}\,du = \frac{1}{4}\int_1^5 \left(u^{-2} - u^{-3}\right)\,du.$$

Integrating,

$$= \left[-\frac{1}{4u} + \frac{1}{8u^2}\right]_1^5 = \left(-\frac{1}{20} + \frac{1}{200}\right) - \left(-\frac{1}{4} + \frac{1}{8}\right).$$

Therefore

$$-\frac{9}{200} + \frac{1}{8} = -\frac{9}{200} + \frac{25}{200} = \frac{16}{200} = \frac{2}{25}.$$

$$\int_0^2 \frac{x}{(2x+1)^3}\,dx = \frac{2}{25}, \text{ as required.}$$

Question 14

Question 14

In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

(**a**) Show that

$$\sin(x + 30°) + \sqrt{3}\cos(x + 30°) \equiv 2\cos x.$$

(3)

(**b**) Hence solve, for $0 \leqslant \theta < 180°$,

$$\sin(\theta + 30°) + \sqrt{3}\cos(\theta + 30°) = 3\sin 2\theta,$$

giving your answers to one decimal place, where appropriate. **(4)**

Model solution

(a)

Using the compound-angle formulae,

$$\begin{aligned}&\sin(x + 30°) + \sqrt{3}\cos(x + 30°)\\ &= \sin x\cos 30° + \cos x\sin 30° + \sqrt{3}\big(\cos x\cos 30° - \sin x\sin 30°\big).\end{aligned}$$

Since $\cos 30° = \dfrac{\sqrt{3}}{2}$ *and* $\sin 30° = \dfrac{1}{2}$,

$$= \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x + \sqrt{3}\left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right) = 2\cos x.$$

This proves the identity.

(b)

By part (a), the equation becomes

$$2\cos\theta = 3\sin 2\theta = 6\sin\theta\cos\theta.$$

Bringing all terms to one side and factorising avoids losing the case $\cos\theta = 0$*:*

$$2\cos\theta(1 - 3\sin\theta) = 0.$$

Thus either $\cos\theta = 0$*, giving* $\theta = 90°$*, or*

$$\sin\theta = \frac{1}{3}.$$

In the stated interval this gives

$$\theta = \sin^{-1}\left(\frac{1}{3}\right) = 19.471\ldots°$$

or

$$\theta = 180° - 19.471\ldots° = 160.528\ldots°.$$

$$\theta = 19.5°,\ 90°,\ 160.5°$$

Question 15

Question 15

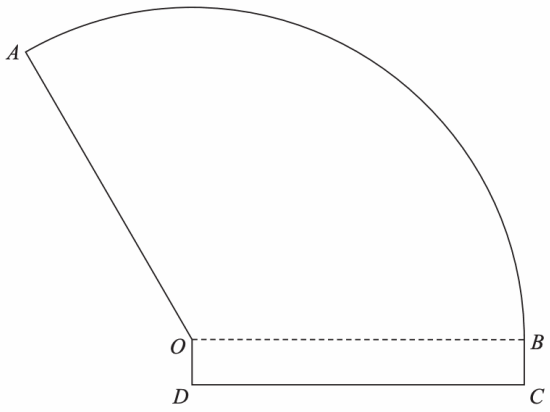


Figure 4

Figure 4 shows the plan view for the design of a stage. The shape consists of a sector of a circle AOB joined to a rectangle OBCD.

Given that the radius of the sector is r metres and angle AOB is θ radians, the length and width of the rectangle are r metres and $\frac{1}{10}r$ metres respectively, and the total area is 240 m^2,

(**a**) show that the perimeter, P metres, is

$$P = 2r + \frac{480}{r}.$$

You must make your method clear. **(4)**

Using algebraic differentiation,

(**b**) find the value of r for which P has a stationary value. **(3)**

(**c**) Prove, by further differentiation, that this value of r gives the minimum perimeter of the stage. **(2)**

Model solution

(a)

The total area is the sector area plus the rectangle area:

$$\frac{1}{2}r^2\theta + r\left(\frac{r}{10}\right) = 240.$$

Multiplying by $2/r$ *gives*

$$r\theta + \frac{r}{5} = \frac{480}{r}.$$

The outside perimeter consists of the arc $r\theta$*, the radius* OA*, the sides* BC *and* DO*, and the base* DC*:*

$$P = r\theta + r + \frac{r}{10} + \frac{r}{10} + r = r\theta + 2r + \frac{r}{5}.$$

Substituting the area relation,

$$P = 2r + \frac{480}{r}$$

(b)

$$\frac{dP}{dr} = 2 - 480r^{-2} = 2 - \frac{480}{r^2}.$$

At a stationary value,

$$2 - \frac{480}{r^2} = 0 \quad \Longrightarrow \quad r^2 = 240.$$

Since $r > 0$*,*

$$r = \sqrt{240} = 4\sqrt{15}\ m \approx 15.5\ m$$

(c)
Differentiating again,

$$\frac{d^2P}{dr^2} = 960r^{-3} = \frac{960}{r^3}.$$

At $r = 4\sqrt{15} > 0$*,*

$$\frac{d^2P}{dr^2} = \frac{960}{(4\sqrt{15})^3} = 0.258\ldots > 0.$$

Therefore the stationary value is a minimum, so $r = 4\sqrt{15}$ *m gives the minimum perimeter.*

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Question 16

Question 16

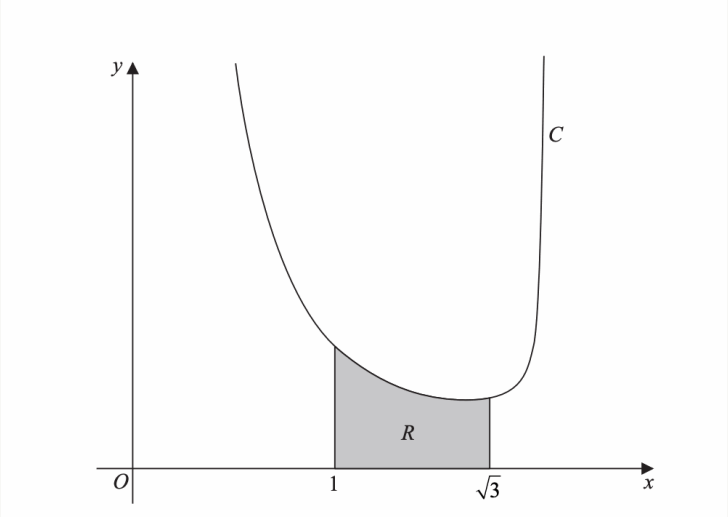


Figure 5 shows a sketch of the curve C with equation

$$y = \frac{1}{x^2\sqrt{4-x^2}}, \qquad 0 < x < 2.$$

The region R, shown shaded, is bounded by C, the line $x = 1$, the x-axis and the line $x = \sqrt{3}$.

(a) Use the substitution $x = 2\sin u$ to show that the area of R is given by

$$\int_a^b k\,\mathrm{cosec}^2 u\,du,$$

where a, b and k are constants to be found. **(4)**

(b) Hence, using algebraic integration, find the exact area of R. Give your answer in simplest form. **(3)**

Model solution

(a)

The required area is

$$R = \int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,dx.$$

Let $x = 2\sin u$. *Then*

$$dx = 2\cos u\,du, \qquad \sqrt{4-x^2} = \sqrt{4-4\sin^2 u} = 2\cos u,$$

where $\cos u > 0$ *for the relevant limits. Therefore*

$$\frac{1}{x^2\sqrt{4-x^2}}\,dx = \frac{2\cos u}{(4\sin^2 u)(2\cos u)}\,du = \frac{1}{4}\,\mathrm{cosec}^2 u\,du.$$

The new limits are

$$x = 1 \Rightarrow \sin u = \frac{1}{2} \Rightarrow u = \frac{\pi}{6},$$

$$x = \sqrt{3} \Rightarrow \sin u = \frac{\sqrt{3}}{2} \Rightarrow u = \frac{\pi}{3}.$$

Hence

$$R = \int_{\pi/6}^{\pi/3} \frac{1}{4} \operatorname{cosec}^2 u \, du, \text{ so } a = \tfrac{\pi}{6},\ b = \tfrac{\pi}{3},\ k = \tfrac{1}{4}.$$

(b)

Using $\int \operatorname{cosec}^2 u \, du = -\cot u + C$,

$$R = \left[-\frac{1}{4} \cot u \right]_{\pi/6}^{\pi/3} = -\frac{1}{4} \cot \frac{\pi}{3} + \frac{1}{4} \cot \frac{\pi}{6}.$$

Since $\cot(\pi/3) = 1/\sqrt{3}$ *and* $\cot(\pi/6) = \sqrt{3}$,

$$R = \frac{1}{4} \left(\sqrt{3} - \frac{1}{\sqrt{3}} \right) = \frac{1}{4} \left(\frac{2}{\sqrt{3}} \right) = \frac{\sqrt{3}}{6}.$$

$$\frac{\sqrt{3}}{6} \text{ square units}$$