



Edexcel AS Further Mathematics

8FM0/01 - Core Pure Mathematics

Monday 15 May 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

$\begin{pmatrix} x & 9 \\ y & z \end{pmatrix} - 3 \begin{pmatrix} z & y \\ z & y \end{pmatrix} = kI$, where x, y, z, k are constants. Determine x, y, z .

Model solution

Because the right-hand side is a scalar multiple of the identity, both off-diagonal entries on the left must be zero. Equating corresponding entries gives

$$9 - 3y = 0, \quad y - 3z = 0.$$

$$y = 3, \quad z = 1.$$

The two diagonal entries must both equal the same number k . The lower-right entry gives $k = z - 3y = 1 - 9 = -8$, so the upper-left entry gives

$$x - 3z = -8 \implies x - 3 = -8 \implies x = -5.$$

$$x = -5, y = 3, z = 1.$$

(4)

Question 2

Question 2

$f(z) = z^3 + az^2 + bz + 175$, where a, b are real. Given that $-3 + 4i$ is a root of $f(z) = 0$: (a) find a, b ; (b) show all roots on one Argand diagram; (c) write down the roots of $f(z + 2) = 0$.

Model solution

(a) Real coefficients are the clue that the conjugate $-3 - 4i$ is also a root. These two roots give the quadratic factor

$$(z + 3 - 4i)(z + 3 + 4i) = (z + 3)^2 + 16 = z^2 + 6z + 25.$$

Its constant term is 25. Since the constant term of the cubic is 175, the remaining linear factor is $z + 7$:

$$f(z) = (z^2 + 6z + 25)(z + 7) = z^3 + 13z^2 + 67z + 175.$$

$$a = 13, \quad b = 67.$$

(4) (b) The factorisation gives three roots. On an Argand diagram plot the points $(-3, 4)$, $(-3, -4)$ and $(-7, 0)$, labelled $-3 + 4i$, $-3 - 4i$ and -7 , respectively.

$$\text{Roots } -3 + 4i, \quad -3 - 4i, \quad -7.$$

(2) (c) Put $u = z + 2$. For $f(z + 2) = 0$, u must be a root of f , so subtract 2 from each root.

$$-5 + 4i, \quad -5 - 4i, \quad -9.$$

(1)

Question 3

Question 3

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 0 \\ \sqrt{3} & 0 & 5\sqrt{3} \\ 1 & 2 & 0 \end{pmatrix}$. (a) Describe A. Transformation A followed by B is C. (b) State the error in using AB. (c) Find C.

Model solution

(a) The first coordinate is unchanged and the y, z block is the standard rotation matrix with $\cos \theta = \sqrt{3}/2$ and $\sin \theta = 1/2$.

Rotation through 30° about the x -axis.

(2) (b) A column vector is first multiplied by A, then by B, so $Cx = B(Ax) = (BA)x$. Matrix multiplication must therefore be in the reverse order from the verbal order.

The matrices were multiplied in the wrong order: $C = BA$, not AB .

(1) (c) Multiply B by A, taking each row of B times each column of A:

$$C = BA = \begin{pmatrix} 1 & 3 & 0 \\ \sqrt{3} & 0 & 5\sqrt{3} \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 1 & 3\sqrt{3}/2 & -3/2 \\ \sqrt{3} & 5\sqrt{3}/2 & 15/2 \\ 1 & \sqrt{3} & -1 \end{pmatrix}.$$

$$C = \begin{pmatrix} 1 & 3\sqrt{3}/2 & -3/2 \\ \sqrt{3} & 5\sqrt{3}/2 & 15/2 \\ 1 & \sqrt{3} & -1 \end{pmatrix}.$$

(1)

Question 4

Question 4

(i)(a) Show that $\frac{2+3i}{5+i} = k(1+i)$. (b) Hence find the smallest positive integer n for which $\left(\frac{2+3i}{5+i}\right)^n$ is real. (ii) $z = a + bi$, where a, b are real, satisfies $|z^{10}| = 59049$ and $\arg(z^{10}) = -5\pi/3$. Determine a, b .

Model solution

(i)(a) To put a quotient of complex numbers into a usable form, multiply numerator and denominator by the conjugate of the denominator.

$$\frac{2+3i}{5+i} = \frac{(2+3i)(5-i)}{(5+i)(5-i)} = \frac{10-2i+15i-3i^2}{26} = \frac{13+13i}{26} = \frac{1}{2}(1+i).$$

$$k = \frac{1}{2}.$$

(3) (i)(b) From part (a), the expression is a positive real multiple of $1+i$, whose argument is $\pi/4$. Its n th power is real first when $n\pi/4 = \pi$.

$$n = 4.$$

(1) (ii) Moduli multiply under powers, so $|z|^{10} = 59049 = 3^{10}$, giving $|z| = 3$. Using the specified argument,

$$10 \arg z = -\frac{5\pi}{3} \implies \arg z = -\frac{\pi}{6}.$$

$$z = 3 \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}i.$$

$$a = \frac{3\sqrt{3}}{2}, \quad b = -\frac{3}{2}.$$

(4)

Question 5

Question 5

The region bounded by the axes and $y + x^2 = 2$, $0 \leq x \leq \sqrt{2}$, is rotated through 360° about the y -axis. The density is 900 kg m^{-3} . (a) Estimate its mass to 2 significant figures. (b) Give a limitation. (c) Evaluate the model if the actual mass is approximately 5500 kg.

Model solution

(a) Rotation about the y -axis is most direct with cylindrical shells. A strip at distance x has radius x , height $y = 2 - x^2$, and thickness dx , so

$$V = 2\pi \int_0^{\sqrt{2}} x(2 - x^2) dx = 2\pi \left[x^2 - \frac{x^4}{4} \right]_0^{\sqrt{2}} = 2\pi(2 - 1) = 2\pi \text{ m}^3.$$

$$\text{mass} = 900(2\pi) = 1800\pi = 5654 \dots \text{ kg}.$$

$$5.7 \times 10^3 \text{ kg (2 s.f.)}$$

(6) (b) The model assumes that the pile is exactly a solid of revolution with the stated smooth parabolic cross-section. A real pile need not have this rotationally symmetric shape.

For example, the pile is unlikely to be perfectly rotationally symmetric.

(1) (c) The predicted mass is about 5655 kg, only about 155 kg (roughly 3%) above 5500 kg.

The model is reasonably suitable because its estimate is close to the actual mass.

(1)

Question 6

Question 6

$l_1 : \mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$. l_2 is parallel to $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$. (a) Show the lines are perpendicular. (b) Find the plane Π containing l_1 and perpendicular to l_2 . (c) Verify $A(3, 1, 1)$ lies on Π . The intersection of Π and l_2 is $(2, 3, 2)$; $B(p, q, r)$ lies on l_2 , $AB = 2\sqrt{5}$, and p, q, r are positive integers. Find B .

Model solution

(a) Perpendicular directions have scalar product zero.

$$\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 3 + 0 - 3 = 0.$$

$$l_1 \perp l_2.$$

(2) (b) Since Π is perpendicular to l_2 , the direction of l_2 is a normal to the plane. Using the point $(-2, 2, 0)$ on l_1 ,

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} x + 2 \\ y - 2 \\ z \end{pmatrix} = 0 \implies x + 2y - 3z = 2.$$

$$\Pi : x + 2y - 3z = 2.$$

(2) (c) Substitute the coordinates of A :

$$3 + 2(1) - 3(1) = 2.$$

Hence $A(3, 1, 1)$ lies on Π .

(1) (d) Starting from the stated intersection point, write a general point of l_2 as

$$B = (2, 3, 2) + t(1, 2, -3) = (2 + t, 3 + 2t, 2 - 3t).$$

The distance condition determines the parameter:

$$(2 + t - 3)^2 + (3 + 2t - 1)^2 + (2 - 3t - 1)^2 = 20,$$

$$(t - 1)^2 + (2t + 2)^2 + (1 - 3t)^2 = 20 \implies 14t^2 + 6 = 20,$$

$$t = \pm 1.$$

These give $(3, 5, -1)$ and $(1, 1, 5)$. Only the latter has all coordinates positive.

$$B = (1, 1, 5).$$

(6)

Question 7

Question 7

(i) Shade the set $|z - 3| \leq |z + 6i|$ on an Argand diagram. (ii) Find the exact complex number w satisfying $\arg(w - 2) = \pi/3$ and $\arg(w + 1) = \pi/6$.

Model solution

(i) Write $z = x + iy$. The condition compares distances from $(3, 0)$ and $(0, -6)$, so its boundary is their perpendicular bisector. Algebra identifies both the boundary and the correct side:

$$(x - 3)^2 + y^2 \leq x^2 + (y + 6)^2 \implies 2x + 4y + 9 \geq 0.$$

Draw the solid line $y = -\frac{1}{2}x - \frac{9}{4}$ (through $(-9/2, 0)$ and $(0, -9/4)$) and shade $y \geq -\frac{1}{2}x - \frac{9}{4}$, the side containing $(3, 0)$.

(3) (ii) An argument condition fixes a ray. The first ray starts at $(2, 0)$, with gradient $\tan(\pi/3) = \sqrt{3}$; the second starts at $(-1, 0)$, with gradient $\tan(\pi/6) = 1/\sqrt{3}$. Their intersection is $w = x + iy$:

$$y = \sqrt{3}(x - 2), \quad y = \frac{x + 1}{\sqrt{3}}.$$

$$3x - 6 = x + 1 \implies x = \frac{7}{2}, \quad y = \frac{3\sqrt{3}}{2}.$$

$$w = \frac{7}{2} + \frac{3\sqrt{3}}{2}i.$$

(6)

Question 8

Question 8

- (a) Use the standard results for $\sum_{r=1}^n r^2$ and $\sum_{r=1}^n r$ to show $\sum_{r=1}^n (2r-1)^2 = \frac{n}{3}(4n^2-1)$, finding a. (b) Hence find the sum of the squares of all positive odd three-digit integers.

Model solution

(a) Expand the square so that the two supplied standard sums can be used:

$$\begin{aligned}\sum_{r=1}^n (2r-1)^2 &= \sum_{r=1}^n (4r^2 - 4r + 1) = 4 \frac{n(n+1)(2n+1)}{6} - 4 \frac{n(n+1)}{2} + n. \\ &= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n = \frac{n}{3}(4n^2-1).\end{aligned}$$

$$a = 4.$$

(5) (b) The positive odd three-digit numbers are 101, 103, ..., 999, namely $2r-1$ for $51 \leq r \leq 500$. Subtract the first 50 odd squares from the first 500:

$$\begin{aligned}\sum_{r=51}^{500} (2r-1)^2 &= \frac{500}{3}(4 \cdot 500^2 - 1) - \frac{50}{3}(4 \cdot 50^2 - 1). \\ &= \frac{500(999999) - 50(9999)}{3} = 166499850.\end{aligned}$$

$$166\,499\,850.$$

(3)

Question 9

Question 9

(i) $P = \begin{pmatrix} k & -2 & 7 \\ -3 & -5 & 2 \\ k & k & 4 \end{pmatrix}$. Show P is non-singular for all real k . (ii) $Q = \begin{pmatrix} 2 & -1 \\ -3 & 0 \end{pmatrix}$ maps $A(a, 2)$ and $B(4, -a)$ to A' and B' . Given $A'B' = \sqrt{58}$, find a .

Model solution

(i) A matrix is non-singular precisely when its determinant is non-zero. Expand the determinant along the first row:

$$|P| = k \begin{vmatrix} -5 & 2 \\ k & 4 \end{vmatrix} + 2 \begin{vmatrix} -3 & 2 \\ k & 4 \end{vmatrix} + 7 \begin{vmatrix} -3 & -5 \\ k & k \end{vmatrix} = -2k^2 - 10k - 24 = -2(k^2 + 5k + 12).$$

The quadratic $k^2 + 5k + 12$ has discriminant $25 - 48 = -23 < 0$, so it has no real zero.

$|P| \neq 0$ for every real k ; therefore P is non-singular.

(4) (ii) Apply the transformation to both column vectors, then use the distance formula:

$$A' = \begin{pmatrix} 2 & -1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} a \\ 2 \end{pmatrix} = \begin{pmatrix} 2a - 2 \\ -3a \end{pmatrix}, \quad B' = \begin{pmatrix} 2 & -1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ -a \end{pmatrix} = \begin{pmatrix} 8 + a \\ -12 \end{pmatrix}.$$

$$(a - 10)^2 + (12 - 3a)^2 = 58 \implies 5a^2 - 46a + 93 = 0.$$

$$(5a - 31)(a - 3) = 0.$$

$$a = 3 \text{ or } a = \frac{31}{5}.$$

(5)

Question 10

Question 10

(i) $z^4 + 5z^2 - 30 = 0$ has roots p, q, r, s . Without solving it, find the monic quartic whose roots are $3p - 1, 3q - 1, 3r - 1, 3s - 1$. (ii) The roots of $4x^3 + nx + 81 = 0$ are $\alpha, 2\alpha, \alpha - \beta$. Find (a) the roots and (b) n .

Model solution

(i) The new roots are obtained from old roots by $w = 3z - 1$, so substitute $z = (w+1)/3$ into the original equation. This creates an equation directly in the required new root variable:

$$\left(\frac{w+1}{3}\right)^4 + 5\left(\frac{w+1}{3}\right)^2 - 30 = 0.$$

$$(w+1)^4 + 45(w+1)^2 - 2430 = 0.$$

$$w^4 + 4w^3 + 51w^2 + 94w - 2384 = 0.$$

$$w^4 + 4w^3 + 51w^2 + 94w - 2384 = 0.$$

(5) (ii)(a) Vieta's relations are appropriate because the roots are expressed in terms of two unknown constants. The coefficient of x^2 is zero, so

$$\alpha + 2\alpha + (\alpha - \beta) = 0 \implies \beta = 4\alpha.$$

The product of the roots is $-81/4$:

$$\alpha(2\alpha)(\alpha - \beta) = 2\alpha^2(-3\alpha) = -6\alpha^3 = -\frac{81}{4}.$$

$$\alpha = \frac{3}{2}, \quad \beta = 6.$$

The roots are $\frac{3}{2}, 3, -\frac{9}{2}$.

(5) (ii)(b) The coefficient of x , divided by the leading coefficient, is the sum of the pairwise products:

$$\frac{n}{4} = \left(\frac{3}{2}\right)(3) + \left(\frac{3}{2}\right)\left(-\frac{9}{2}\right) + 3\left(-\frac{9}{2}\right) = \frac{9}{2} - \frac{27}{4} - \frac{27}{2} = -\frac{63}{4}.$$

$$n = -63.$$

(2)