



Edexcel AS Further Mathematics

8FM0/01 - Core Pure Mathematics

November 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Describe each transformation fully. (b) Find the matrix R for P followed by Q . (c) Find $\det R$ and explain its meaning.

Model solution

(a)(i) Compare P with the standard rotation matrices. A quarter-turn anticlockwise sends $(1, 0)$ to $(0, 1)$ and $(0, 1)$ to $(-1, 0)$, exactly as this matrix does.

P represents a rotation of 90° anticlockwise about the origin.

(2) (a)(ii) The x -coordinate is unchanged while every y -coordinate is multiplied by 3.

Q represents a stretch of scale factor 3, parallel to the y -axis.

(2) (b) Transformations are applied from right to left in a matrix product, so " P followed by Q " gives $R = QP$.

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}.$$

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(1) (c) For a two-by-two matrix, the determinant gives the signed area scale factor.

$$\det R = 0 \times 0 - (-1)(3) = 3.$$

$\det R = 3$, so areas are enlarged by scale factor 3.

(2)

Question 2

Question 2

The roots of $9x^3 - 5x^2 + 4x + 7 = 0$ are α, β, γ . Find the cubic, in w , whose roots are $3\alpha - 2, 3\beta - 2, 3\gamma - 2$.

Model solution

The new roots are obtained by a linear change of variable, so substitute $x = (w + 2)/3$ into the original equation. This avoids finding any individual root.

$$9\left(\frac{w+2}{3}\right)^3 - 5\left(\frac{w+2}{3}\right)^2 + 4\left(\frac{w+2}{3}\right) + 7 = 0.$$

Multiplying by 3 clears the denominators before expanding.

$$(w+2)^3 - \frac{5}{3}(w+2)^2 + 4(w+2) + 21 = 0,$$

$$3(w+2)^3 - 5(w+2)^2 + 12(w+2) + 63 = 0,$$

$$3w^3 + 13w^2 + 28w + 91 = 0.$$

$$3w^3 + 13w^2 + 28w + 91 = 0.$$

(5)

Question 3

Question 3

(a) Show that $\sum_{r=1}^n (5r - 2)^2 = \frac{1}{6}n(an^2 + bn + c)$. (b) Hence find k if this sum is $94k^2$.

Model solution

(a) Expand first, because the resulting powers of r can be summed using the standard formulae for $\sum r$ and $\sum r^2$.

$$(5r - 2)^2 = 25r^2 - 20r + 4,$$

$$\begin{aligned}\sum_{r=1}^n (5r - 2)^2 &= 25 \frac{n(n+1)(2n+1)}{6} - 20 \frac{n(n+1)}{2} + 4n. \\ &= \frac{n}{6} [25(n+1)(2n+1) - 60(n+1) + 24] \\ &= \frac{n}{6} (50n^2 + 15n - 11).\end{aligned}$$

$$a = 50, \quad b = 15, \quad c = -11.$$

(5) (b) Use the result just proved with $n = k$, then solve the resulting equation. Since k is a positive integer, the zero solution created by factorising is not admissible.

$$\frac{k}{6} (50k^2 + 15k - 11) = 94k^2,$$

$$k(50k^2 - 549k - 11) = 0, \quad (k - 11)(50k + 1) = 0.$$

$$k = 11.$$

(4)

Question 4

Question 4

$$M = \begin{pmatrix} 2 & 1 & 4 \\ k & 2 & -2 \\ 4 & 1 & -2 \end{pmatrix}, \quad N = \begin{pmatrix} k-7 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{pmatrix}.$$

(a) Find MN . Given $k = 5$, find (b)(i) MN , (b)(ii) M^{-1} , (c) the solution of the given three simultaneous equations, and (d) its geometric interpretation.

Model solution

(a) Matrix multiplication uses a row of M dotted with a column of N .

$$MN = \begin{pmatrix} 2k-24 & 0 & 0 \\ k^2-7k+10 & 6k-44 & 50-10k \\ 4k-20 & 0 & -14 \end{pmatrix}.$$

$$MN = \begin{pmatrix} 2k-24 & 0 & 0 \\ k^2-7k+10 & 6k-44 & 50-10k \\ 4k-20 & 0 & -14 \end{pmatrix}.$$

(2) (b)(i) Substitute $k = 5$ into the product found in part (a).

$$MN = -14I.$$

$$MN = \begin{pmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{pmatrix}.$$

(1) (b)(ii) Since $MN = -14I$, multiplying by $-1/14$ shows that $(-1/14)N$ is the inverse of M .

$$M^{-1} = -\frac{1}{14} \begin{pmatrix} -2 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{pmatrix}.$$

(1) (c) The coefficient matrix is M when $k = 5$, so premultiply the equation $Mx = b$ by the inverse.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = -\frac{1}{14} \begin{pmatrix} 24 \\ -80 \\ 1 \end{pmatrix} = \begin{pmatrix} -12/7 \\ 40/7 \\ -1/14 \end{pmatrix}.$$

$$(x, y, z) = \left(-\frac{12}{7}, \frac{40}{7}, -\frac{1}{14} \right).$$

(2) (d) Each linear equation in three variables represents a plane. A unique simultaneous solution is their common point.

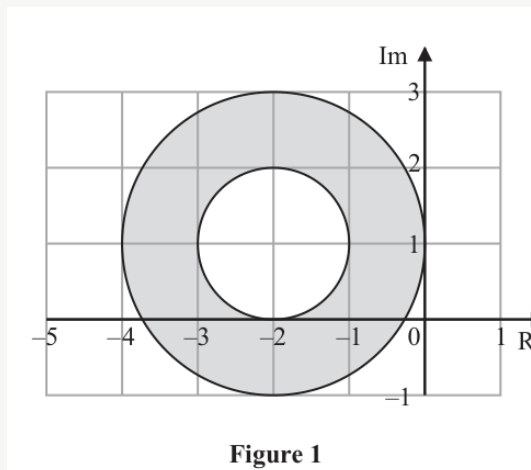
The three planes meet at the single point $\left(-\frac{12}{7}, \frac{40}{7}, -\frac{1}{14}\right)$.

(1)

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Question 5

Question 5



The shaded set is $P = \{z \in \mathbb{C} : a \leq |z + b + ci| \leq d\}$. **(a)** Find a, b, c, d . **(b)** Find the exact area also satisfying $|z - i| \leq |z - 3i|$.

Model solution

(a) The diagram is an annulus centred at $-2 + i$, so the modulus must measure distance from that point. Its inner and outer radii are 1 and 2.

$$a = 1, \quad b = 2, \quad c = -1, \quad d = 2.$$

(3) (b) Write $z = x + iy$. The second condition identifies which side of a perpendicular-bisector line is required.

$$|z - i| \leq |z - 3i| \iff x^2 + (y - 1)^2 \leq x^2 + (y - 3)^2 \iff y \leq 2.$$

The required area is the annulus area less the minor segment above $y = 2$. The centre is $(-2, 1)$, so the chord is distance 1 from the centre of the radius-2 circle.

$$A_{\text{annulus}} = \pi(2^2) - \pi(1^2) = 3\pi.$$

$$\theta = 2 \cos^{-1}\left(\frac{1}{2}\right) = \frac{2\pi}{3}.$$

$$A_{\text{segment}} = \frac{1}{2}(2)^2\theta - \frac{1}{2}(2)^2 \sin \theta = \frac{4\pi}{3} - \sqrt{3}.$$

$$A_Q = 3\pi - \left(\frac{4\pi}{3} - \sqrt{3}\right).$$

$$\boxed{\frac{5\pi}{3} + \sqrt{3}}.$$

(7)

Question 6

Question 6

The ground is $z = 0$. The mineral plane contains $A(10, 5, -50)$, $B(15, 30, -45)$, $C(-5, 20, -60)$, in metres. Find (a) its equation $\mathbf{r} \cdot \mathbf{n} = d$, (b) its acute angle with the ground, (c) the vertical drilling distance from $(5, 12, 0)$, and (d) one limitation.

Model solution

(a) Two non-parallel vectors in the plane give a normal vector through their cross product.

$$\overrightarrow{AB} = (5, 25, 5), \quad \overrightarrow{AC} = (-15, 15, -10),$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = (-325, -25, 450) = 25(-13, -1, 18).$$

Use point A in $\mathbf{r} \cdot \mathbf{n} = d$ to find the constant.

$$(-13, -1, 18) \cdot (10, 5, -50) = -1035.$$

$$\mathbf{r} \cdot (-13\mathbf{i} - \mathbf{j} + 18\mathbf{k}) = -1035.$$

(5) (b) The normal to the horizontal ground is $\mathbf{k}$. For the acute angle between the planes, take the absolute value of the scalar product of their normals.

$$\cos \theta = \frac{|(-13, -1, 18) \cdot \mathbf{k}|}{\sqrt{13^2 + 1^2 + 18^2}} = \frac{18}{\sqrt{494}}, \quad \theta = 35.9 \dots^\circ.$$

The acute angle is 36° .

(3) (c) A vertical drill keeps $x = 5$ and $y = 12$. Substitute these values into the plane to find its z -coordinate.

$$-13(5) - 12 + 18z = -1035 \implies z = -\frac{958}{18} = -53.222 \dots$$

The required vertical distance is 53.2 m (to 1 d.p.).

(2) (d) A plane is perfectly flat, whereas a real mineral layer need not be.

For example, the mineral layer may be curved or have uneven thickness, so a planar model is only approximate.

(1)

Question 7

Question 7

$f(z) = z^4 - 6z^3 + pz^2 + qz + r$ has real coefficients and roots $3, 2 + i, \gamma, \delta$, with γ complex. Find (a) γ and why δ is real, (b) δ , (c) p, q, r , and (d) the roots of $f(-2z) = 0$.

Model solution

(a) Non-real roots of a polynomial with real coefficients occur in conjugate pairs.

$\gamma = 2 - i$; this pairs with $2 + i$, leaving δ real because 3 is already the only other real root.

(2) (b) The coefficient of z^3 is minus the sum of the roots.

$$3 + (2 + i) + (2 - i) + \delta = 6 \implies \delta = -1.$$

$$\delta = -1.$$

(2) (c) Form the polynomial from its roots; pairing the conjugates first keeps the algebra real.

$$\begin{aligned} f(z) &= (z - 3)(z + 1)(z - (2 + i))(z - (2 - i)) \\ &= (z^2 - 2z - 3)((z - 2)^2 + 1) = (z^2 - 2z - 3)(z^2 - 4z + 5) \\ &= z^4 - 6z^3 + 10z^2 + 2z - 15. \end{aligned}$$

$$p = 10, q = 2, r = -15.$$

(3) (d) For $f(-2z) = 0$, $-2z$ must equal each original root, so divide every root by -2 .

$$-\frac{3}{2}, -\frac{1}{2} - \frac{i}{2}, -\frac{1}{2} + \frac{i}{2}, \frac{1}{2}.$$

(2)

Question 8

Question 8

(a) Prove by induction that $\sum_{r=1}^n r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$. (b) Hence show that $\sum_{r=n}^{2n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)(an+b)(cn+d)$.

Model solution

(a) Induction is requested, so establish a base case, assume the formula at k , and add the next term to prove it at $k+1$.

$$n = 1: \quad 1(2)(3) = 6 = \frac{1}{2}(1)(2)^2(3).$$

Assume $\sum_{r=1}^k r(r+1)(2r+1) = \frac{1}{2}k(k+1)^2(k+2)$. Then

$$\begin{aligned} \sum_{r=1}^{k+1} r(r+1)(2r+1) &= \frac{1}{2}k(k+1)^2(k+2) + (k+1)(k+2)(2k+3) \\ &= \frac{1}{2}(k+1)(k+2)[k(k+1) + 2(2k+3)] \\ &= \frac{1}{2}(k+1)(k+2)(k^2 + 5k + 6) \\ &= \frac{1}{2}(k+1)(k+2)^2(k+3). \end{aligned}$$

This is exactly the stated expression with $n = k+1$. Since the base case is true, the result is true for all positive integers n .

$$\sum_{r=1}^n r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2) \text{ for all positive integers } n.$$

(6) (b) The required range is obtained by subtracting the sum up to $n-1$ from the sum up to $2n$. This is the clue to use part (a) twice.

$$\begin{aligned} \sum_{r=n}^{2n} r(r+1)(2r+1) &= \frac{1}{2}(2n)(2n+1)^2(2n+2) - \frac{1}{2}(n-1)n^2(n+1) \\ &= \frac{1}{2}n(n+1)[4(2n+1)^2 - n(n-1)] \\ &= \frac{1}{2}n(n+1)(15n^2 + 17n + 4) \\ &= \frac{1}{2}n(n+1)(3n+1)(5n+4). \end{aligned}$$

$$a = 3, \quad b = 1, \quad c = 5, \quad d = 4.$$

(3)

Question 9

Question 9

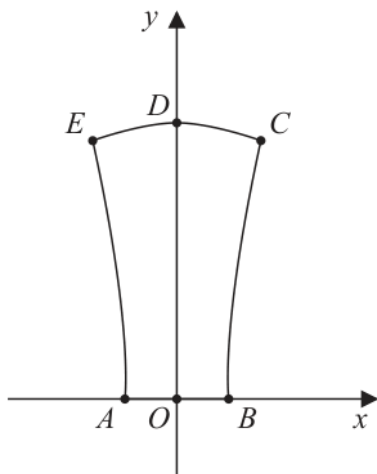


Figure 2

For BC, $y = \sqrt{225x^2 - 2025}/a$, $3 \leq x < 5$; for CD, $y = 16 - 0.04x^2$, $0 \leq x < 5$. The region is rotated about the y -axis. Find (a) a , (b) the exact wax volume, (c) a limitation, and (d) evaluate the model if the actual wax volume is 700 cm^3 .

Model solution

(a) Point $C(5, 15)$ lies on BC, so substituting it determines the unknown scale constant.

$$15 = \frac{\sqrt{225(5)^2 - 2025}}{a} = \frac{60}{a} \implies a = 4.$$

$$a = 4.$$

(2) (b) Rotation about the y -axis suggests washers, so rearrange each curve as x^2 in terms of y and use $V = \pi \int x^2 dy$. Below $y = 15$, BC supplies the radius; from 15 to 16, it is supplied by CD.

$$y = \frac{\sqrt{225x^2 - 2025}}{4} \implies x^2 = \frac{16y^2 + 2025}{225},$$

$$y = 16 - 0.04x^2 \implies x^2 = 400 - 25y.$$

$$V = \pi \int_0^{15} \frac{16y^2 + 2025}{225} dy + \pi \int_{15}^{16} (400 - 25y) dy.$$

Evaluating the two elementary integrals directly,

$$V = \pi \left\{ \left[\frac{16}{675}y^3 + 9y \right]_0^{15} + \left[400y - \frac{25}{2}y^2 \right]_{15}^{16} \right\} = \frac{455\pi}{2}.$$

$$\frac{455\pi}{2} \text{ cm}^3.$$

(9) (c) *The idealised curves and perfect rotational symmetry need not describe a manufactured candle exactly.*

For example, the candle has a wick hole, which the model omits.

(1) (d) *Compare like with like: the model predicts $455\pi/2 \approx 715 \text{ cm}^3$, only about 15 cm^3 above the actual amount.*

The model is a good estimate, although it slightly overestimates the wax required.

(1)

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