



Edexcel AS Further Mathematics

8FM0/21 - Further Pure Mathematics 1

Thursday 17 May 2018

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2018 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$5 \sin x + 12 \cos x = 2$$

can be written in the form $7t^2 - 5t - 5 = 0$.

(b) Hence solve, for $-180^\circ < x < 180^\circ$, the equation

$$5 \sin x + 12 \cos x = 2,$$

giving the answers to one decimal place.

Model solution

(a) With $t = \tan\left(\frac{x}{2}\right)$,

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}.$$

Hence

$$5 \left(\frac{2t}{1+t^2} \right) + 12 \left(\frac{1-t^2}{1+t^2} \right) = 2.$$

Multiplying by $1+t^2$ gives

$$10t + 12 - 12t^2 = 2 + 2t^2,$$

so

$$14t^2 - 10t - 10 = 0 \implies 7t^2 - 5t - 5 = 0. \quad (3)$$

(b) Solving this quadratic,

$$t = \frac{5 \pm \sqrt{(-5)^2 - 4(7)(-5)}}{14} = \frac{5 \pm \sqrt{165}}{14}.$$

Thus $t \approx 1.2746$ or $t \approx -0.5603$. Since $x = 2 \tan^{-1} t$,

$$x \approx 2 \tan^{-1}(1.2746) = 103.8^\circ, \quad x \approx 2 \tan^{-1}(-0.5603) = -58.5^\circ.$$

Both values lie in the required interval.

$$x = 103.8^\circ \text{ or } x = -58.5^\circ. \quad (4)$$

The half-angle substitution has a one-to-one range $-180^\circ < x < 180^\circ$, so no additional periodic solutions are required.

Question 2

Question 2

The temperature θ °C of coffee, t minutes after it is put in a room, is modelled by

$$\frac{d\theta}{dt} = -k(\theta - 20),$$

where k is a constant and initially $\theta = 80$. Use $k = 0.1$.

(a) Use two iterations of $y_1 = y_0 + h \left(\frac{dy}{dx} \right)_0$ to estimate the temperature after 3 minutes.

The coffee in a different cup also starts at 80°C and cools more slowly.

(b) State how k should be changed in the model.

Model solution

(a) Two equal iterations take t from 0 to 3, so $h = \frac{3}{2} = 1.5$ minutes. Initially,

$$\left. \frac{d\theta}{dt} \right|_{t=0} = -0.1(80 - 20) = -6.$$

Therefore the first Euler estimate is

$$\theta_1 = 80 + 1.5(-6) = 71.$$

At this estimate,

$$\frac{d\theta}{dt} = -0.1(71 - 20) = -5.1.$$

The second iteration gives

$$\theta_2 = 71 + 1.5(-5.1) = 63.35.$$

The estimated temperature after 3 minutes is 63.35°C.

(6)

(b) For the coffee to cool more slowly, the magnitude of its negative rate of change must be smaller. Thus k should be reduced, while remaining positive.

$$0 < k < 0.1.$$

(1)

The Euler step length must be 1.5 minutes because exactly two iterations are specified.

Question 3

Question 3

Use algebra to find the values of x for which

$$\frac{x}{x^2 - 2x - 3} \leq \frac{1}{x + 3}.$$

Model solution

First factor the quadratic denominator:

$$x^2 - 2x - 3 = (x - 3)(x + 1).$$

Bringing the two fractions to one side and using the common denominator gives

$$\frac{x}{(x - 3)(x + 1)} - \frac{1}{x + 3} \leq 0$$

$$\frac{x(x + 3) - (x - 3)(x + 1)}{(x - 3)(x + 1)(x + 3)} \leq 0.$$

The numerator simplifies as

$$x^2 + 3x - (x^2 - 2x - 3) = 5x + 3,$$

so the inequality is

$$\frac{5x + 3}{(x - 3)(x + 1)(x + 3)} \leq 0.$$

The critical values, in increasing order, are

$$-3, \quad -1, \quad -\frac{3}{5}, \quad 3.$$

Testing the sign in the resulting intervals gives

x	$(-\infty, -3)$	$(-3, -1)$	$(-1, -\frac{3}{5})$	$(-\frac{3}{5}, 3)$	$(3, \infty)$
$\frac{5x + 3}{(x - 3)(x + 1)(x + 3)}$	+	-	+	-	+

The denominator is zero at $x = -3, -1, 3$, so those values cannot be included. The numerator is zero at $x = -\frac{3}{5}$, so that value is included.

$$-3 < x < -1 \quad \text{or} \quad -\frac{3}{5} \leq x < 3.$$

(7)

A sign analysis is essential: multiplying an inequality by a denominator with unknown sign is not valid without splitting into cases.

Question 4

Question 4

A crystal is modelled as a tetrahedron with vertices

$$A(12, 4, -1), \quad B(10, 15, -3), \quad C(5, 8, 5), \quad D(2, 2, -6),$$

where one unit of length is one millimetre. Its mass is 0.5 grams.

(a) Show that the area of triangular face ABC, to one decimal place, is 52.2 mm².

(b) Find the density of the crystal in g cm⁻³.

Model solution

(a) Two edges of the triangular face are

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 11 \\ -2 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} -7 \\ 4 \\ 6 \end{pmatrix}.$$

Their vector product is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 11 & -2 \\ -7 & 4 & 6 \end{vmatrix} = 74\mathbf{i} + 26\mathbf{j} + 69\mathbf{k}.$$

Hence the area of ABC is half the magnitude of this vector:

$$\frac{1}{2} \sqrt{74^2 + 26^2 + 69^2} = 52.23265 \dots \text{ mm}^2.$$

$$\text{finalansArea(ABC)} = 52.2 \text{ mm}^2 \text{ (to 1 d.p.)}. \quad (3)$$

(b) Take the third edge from A:

$$\overrightarrow{AD} = \begin{pmatrix} -10 \\ -2 \\ -5 \end{pmatrix}.$$

The volume of the tetrahedron is one sixth of the absolute scalar triple product:

$$V = \frac{1}{6} \left| \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) \right| = \frac{1}{6} |(-10)(74) + (-2)(26) + (-5)(69)| = \frac{1137}{6} = 189.5 \text{ mm}^3.$$

Since 1 cm³ = 1000 mm³,

$$V = 0.1895 \text{ cm}^3.$$

Therefore

$$\text{density} = \frac{0.5}{0.1895} = 2.6385 \dots \text{ g cm}^{-3}.$$

$$\text{finalansDensity} = 2.6 \text{ g cm}^{-3} \text{ (to 1 d.p.)}. \quad (6)$$

The scalar triple product produces the volume of the associated parallelepiped; division by 6 is required for a tetrahedron.

Question 5

Question 5

The rectangular hyperbola H has equation $xy = c^2$, where $c \neq 0$. The point

$$P\left(cp, \frac{c}{p}\right), \quad p \neq 0,$$

lies on H .

(a) Use calculus to show that an equation of the normal to H at P is

$$p^3x - py + c(1 - p^4) = 0.$$

The normal at P meets H again at Q .

(b) Find the coordinates of the midpoint of PQ in terms of c and p , simplifying where possible.

Model solution

(a) From $xy = c^2$,

$$y = \frac{c^2}{x}, \quad \frac{dy}{dx} = -\frac{c^2}{x^2}.$$

At P , where $x = cp$, the gradient of the tangent is

$$m_T = -\frac{c^2}{(cp)^2} = -\frac{1}{p^2}.$$

Thus the gradient of the normal is $m_N = p^2$. Its equation through P is

$$y - \frac{c}{p} = p^2(x - cp).$$

Multiplying by p and rearranging,

$$py - c = p^3x - cp^4$$

$$p^3x - py + c(1 - p^4) = 0,$$

as required. (4)

(b) On H , $y = \frac{c^2}{x}$. Substitution into the normal equation gives

$$p^3x - p\left(\frac{c^2}{x}\right) + c(1 - p^4) = 0.$$

Multiplying by x ,

$$p^3x^2 + c(1 - p^4)x - c^2p = 0.$$

One root is the x -coordinate of P , namely $x = cp$. Factoring,

$$(x - cp)(p^3x + c) = 0.$$

The other root is therefore $x = -\frac{c}{p^3}$, and the corresponding y -coordinate is

$$y = \frac{c^2}{-c/p^3} = -cp^3.$$

Thus

$$Q\left(-\frac{c}{p^3}, -cp^3\right).$$

The midpoint of P and Q is

$$\left(\frac{cp - c/p^3}{2}, \frac{c/p - cp^3}{2}\right) = \left(\frac{c}{2}\left(p - \frac{1}{p^3}\right), \frac{c}{2}\left(\frac{1}{p} - p^3\right)\right).$$

$$\left(\frac{c}{2}\left(p - \frac{1}{p^3}\right), \frac{c}{2}\left(\frac{1}{p} - p^3\right)\right).$$

(6)

The known intersection P provides one factor of the quadratic formed after substituting the hyperbola into the normal.

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