



Edexcel AS Further Mathematics

8FM0/21 - Further Pure Mathematics 1

Thursday 16 May 2019

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2019 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) Write down the t -formula for $\sin x$. (1)

(b) Use the answer to part (a)

(i) to find the exact value of $\sin x$ when $\tan \frac{x}{2} = \sqrt{2}$;

(ii) to show that $\cos x = \frac{1-t^2}{1+t^2}$. (4)

(c) Use the t -formulae to solve, for $0 < \theta \leq 360^\circ$,

$$7 \sin \theta + 9 \cos \theta + 3 = 0,$$

giving your answers to one decimal place. (4)

Model solution

(a) With $t = \tan \frac{x}{2}$, the required formula is

$$\sin x = \frac{2t}{1+t^2}.$$

$$\sin x = \frac{2t}{1+t^2}.$$

(b)(i) Substitute $t = \sqrt{2}$ into the formula from part (a).

$$\sin x = \frac{2\sqrt{2}}{1+(\sqrt{2})^2} = \frac{2\sqrt{2}}{3}.$$

$$\sin x = \frac{2\sqrt{2}}{3}.$$

(b)(ii) Since $\sin^2 x + \cos^2 x = 1$,

$$\cos^2 x = 1 - \left(\frac{2t}{1+t^2} \right)^2 = \frac{(1+t^2)^2 - 4t^2}{(1+t^2)^2} = \frac{(1-t^2)^2}{(1+t^2)^2}.$$

Choosing the sign consistent with $t = \tan(x/2)$ gives

$$\cos x = \frac{1-t^2}{1+t^2},$$

as required.

(c) Put $t = \tan(\theta/2)$, so that $\sin \theta = \frac{2t}{1+t^2}$ and $\cos \theta = \frac{1-t^2}{1+t^2}$.

$$7 \left(\frac{2t}{1+t^2} \right) + 9 \left(\frac{1-t^2}{1+t^2} \right) + 3 = 0.$$

$$14t + 9 - 9t^2 + 3 + 3t^2 = 0 \implies 3t^2 - 7t - 6 = 0.$$

$$(t - 3)(3t + 2) = 0,$$

so $t = 3$ or $t = -\frac{2}{3}$. Therefore

$$\theta = 2 \tan^{-1}(3) = 143.130 \dots^\circ$$

or, using the angle in the interval $(90^\circ, 180^\circ)$ for $\theta/2$,

$$\theta = 2 \left(180^\circ + \tan^{-1} \left(-\frac{2}{3} \right) \right) = 292.620 \dots^\circ.$$

$$\theta = 143.1^\circ, 292.6^\circ \text{ (to 1 d.p.)}.$$

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Question 2

Question 2

A student attempts to solve

$$\frac{x}{x-24} > \frac{1}{x+11}.$$

Their attempt is

$$\begin{aligned} (x)(x-24)(x+11)^2 &> (x+11)(x-24)^2, \\ x(x-24)(x+11)^2 - (x+11)(x-24)^2 &> 0, \\ (x-24)(x+11)[x(x+11) - x - 24] &> 0 \\ (x-24)(x+11)(x^2 + 10x - 24) &> 0, \\ (x-24)(x+11)(x+12)(x-2) &> 0, \\ x = 24, \quad x = -11, \quad x = -12, \quad x = 2, \end{aligned}$$

and concludes

$$x \in \{x : -12 < x < -11\} \cup \{x : 2 < x < 24\}.$$

(a) Identify the error in (i) line 3 and (ii) line 7. (2)

(b) Find a correct solution of the inequality. (4)

Model solution

(a)(i) The final two terms in the bracket must be kept together: the bracket should be $x(x+11) - (x-24)$, not $x(x+11) - x - 24$. (a)(ii) The stated intervals are the regions where the resulting expression is negative; the inequality requires the regions where it is positive.

(b) Multiply by $(x-24)^2(x+11)^2$, which is positive wherever the original expression is defined:

$$\frac{x}{x-24} > \frac{1}{x+11} \implies x(x-24)(x+11)^2 > (x+11)(x-24)^2.$$

$$\implies (x-24)(x+11)[x(x+11) - (x-24)] > 0.$$

$$(x-24)(x+11)(x+6)(x+4) > 0.$$

The critical values are $-11, -6, -4, 24$. A sign chart shows the product is positive on the two outside intervals and the middle interval.

$$x \in \{x : x < -11\} \cup \{x : -6 < x < -4\} \cup \{x : x > 24\}.$$

Question 3

Question 3

Julie initially has 20 rabbits. After t years, the number R is modelled by

$$\frac{dR}{dt} = 2R + 4 \sin t, \quad t > 0.$$

She needs at least 40 rabbits before selling them. Use two iterations of

$$y_{n+1} \approx y_n + h \left(\frac{dy}{dx} \right)_n$$

to decide whether she can start selling after 4 months.

(7)

Model solution

Four months is $\frac{1}{3}$ year. Two equal iterations therefore have step length

$$h = \frac{1/3}{2} = \frac{1}{6}.$$

Initially, $t_0 = 0$ and $R_0 = 20$, so

$$\left. \frac{dR}{dt} \right|_{(0,20)} = 2(20) + 4 \sin 0 = 40.$$

$$R_1 = 20 + \frac{1}{6}(40) = \frac{80}{3}.$$

For the second iteration, $t_1 = \frac{1}{6}$, hence

$$\left. \frac{dR}{dt} \right|_{(1/6, 80/3)} = 2 \left(\frac{80}{3} \right) + 4 \sin \left(\frac{1}{6} \right) = 53.9969 \dots$$

$$R_2 = \frac{80}{3} + \frac{1}{6}(53.9969 \dots) = 35.666 \dots$$

This is the model's estimate at $t = \frac{1}{3}$, i.e. after four months. It is less than 40.

No. After 4 months the estimate is 35.7 rabbits (about 36), which is fewer than 40.

Question 4

Question 4

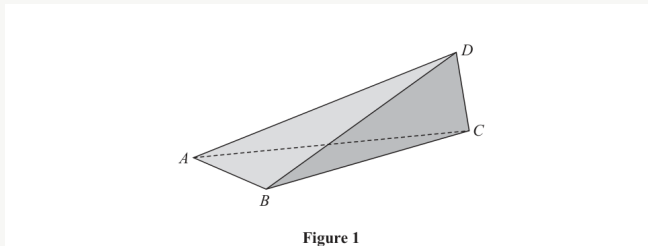


Figure 1

A wooden doorstep is modelled by a tetrahedron with vertices

$$A(2, 1, 4), \quad B(6, 1, 2), \quad C(4, 10, 3), \quad D(5, 8, d),$$

where $d > 0$ and distances are in centimetres.

(a) Find the area of triangle ABC.

(4)

Given that the volume is 21 cm^3 , (b) find d .

(4)

Model solution

(a) Use two sides of triangle ABC:

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix}.$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 18 \\ 0 \\ 36 \end{pmatrix}.$$

$$\text{Area of ABC} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \sqrt{18^2 + 0^2 + 36^2} = 9\sqrt{5}.$$

$$\text{Area of ABC} = 9\sqrt{5} \text{ cm}^2 (= 20.1 \text{ cm}^2, 3 \text{ s.f.})$$

(b) Take A as the common vertex. Then

$$\overrightarrow{AD} = \begin{pmatrix} 3 \\ 7 \\ d-4 \end{pmatrix}.$$

The tetrahedron volume is one sixth of the magnitude of the scalar triple product:

$$21 = \frac{1}{6} |\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})| = \frac{1}{6} \left| \begin{pmatrix} 3 \\ 7 \\ d-4 \end{pmatrix} \cdot \begin{pmatrix} 18 \\ 0 \\ 36 \end{pmatrix} \right|.$$

$$21 = \frac{1}{6} |54 + 36(d-4)| = \frac{1}{6} |36d - 90|.$$

Since $d > 0$, the valid solution is obtained from $36d - 90 = 126$:

$$d = 6.$$

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Question 5

Question 5

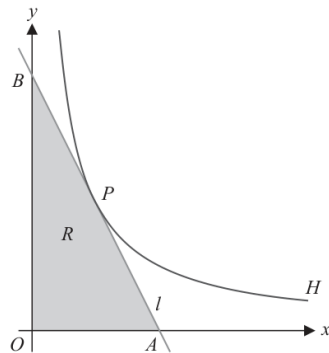


Figure 2

Figure 2 shows a sketch of part of the rectangular hyperbola H with equation

The rectangular hyperbola H has equation $xy = c^2$, $x > 0$, where $c > 0$. The point

$$P\left(ct, \frac{c}{t}\right)$$

lies on H . Its tangent l meets the axes at A and B , enclosing triangular region $R = OAB$.

Given that $OB = 2OA$ and that the area of R is 32, find the exact coordinates of P .

(10)

Model solution

Differentiate $xy = c^2$ implicitly:

$$x \frac{dy}{dx} + y = 0, \quad \frac{dy}{dx} = -\frac{y}{x} = -\frac{c^2}{x^2}.$$

At $P\left(ct, \frac{c}{t}\right)$, the gradient of the tangent is

$$m = -\frac{c/t}{ct} = -\frac{1}{t^2}.$$

Thus the tangent has equation

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct),$$

or

$$y = -\frac{x}{t^2} + \frac{2c}{t}.$$

At the x -intercept A , $y = 0$, and at the y -intercept B , $x = 0$. Hence

$$OA = 2ct, \quad OB = \frac{2c}{t}.$$

The condition $OB = 2OA$ gives

$$\frac{2c}{t} = 2(2ct) \implies t^2 = \frac{1}{2}.$$

As P lies in the first quadrant, $t > 0$, so $t = \frac{1}{\sqrt{2}}$. Also

$$32 = \text{Area}(\text{OAB}) = \frac{1}{2}(\text{OA})(\text{OB}) = \frac{1}{2}(2ct) \left(\frac{2c}{t} \right) = 2c^2.$$

$$c^2 = 16, \quad c = 4$$

because $c > 0$. Therefore

$$P = \left(ct, \frac{c}{t} \right) = \left(4 \cdot \frac{1}{\sqrt{2}}, \frac{4}{1/\sqrt{2}} \right) = (2\sqrt{2}, 4\sqrt{2}).$$

$$P(2\sqrt{2}, 4\sqrt{2}).$$

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