



Edexcel AS Further Mathematics

8FM0/21 - Further Pure Mathematics 1

Tuesday 24 May 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Use algebra to find the set of values of x for which

$$x \geq \frac{2x + 15}{2x + 3}.$$

Model solution

The denominator can be positive or negative, so multiplying the inequality by it directly could reverse the inequality. Instead, move everything to one side and make a sign table.

$$x - \frac{2x + 15}{2x + 3} \geq 0$$

$$\frac{x(2x + 3) - (2x + 15)}{2x + 3} \geq 0$$

$$\frac{2x^2 + x - 15}{2x + 3} \geq 0$$

$$\frac{(2x - 5)(x + 3)}{2x + 3} \geq 0.$$

The critical values are the two zeros of the numerator, $x = -3$ and $x = \frac{5}{2}$, together with $x = -\frac{3}{2}$, where the expression is undefined. Testing the sign in the four intervals gives

x	$(-\infty, -3)$	$(-3, -\frac{3}{2})$	$(-\frac{3}{2}, \frac{5}{2})$	$(\frac{5}{2}, \infty)$
$\frac{(2x - 5)(x + 3)}{2x + 3}$	-	+	-	+

Equality is allowed at zeros of the numerator, but $x = -\frac{3}{2}$ must be excluded because it makes the original fraction undefined.

$$x \in \left[-3, -\frac{3}{2}\right) \cup \left[\frac{5}{2}, \infty\right).$$

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Question 2

Question 2

A population of deer is modelled by

$$\frac{dP}{dt} = \frac{P}{5000} \left(1000 - \frac{P(t+1)}{6t+5} \right), \quad t > 0.$$

There were estimated to be 540 deer six months after introduction. Use two applications of

$$\left. \frac{dy}{dx} \right|_n \approx \frac{y_{n+1} - y_n}{h}$$

to estimate the number of deer ten months after introduction.

Model solution

The data are given at $t = \frac{1}{2}$ year and the target is $t = \frac{5}{6}$ year. Two equal Euler steps therefore have size

$$h = \frac{\frac{5}{6} - \frac{1}{2}}{2} = \frac{1}{6}, \quad t_0 = \frac{1}{2}, \quad t_1 = \frac{2}{3}, \quad t_2 = \frac{5}{6}, \quad P_0 = 540.$$

First evaluate the differential equation at the known population, then use $P_{n+1} \approx P_n + h \left. \frac{dP}{dt} \right|_n$.

$$\left. \frac{dP}{dt} \right|_0 = \frac{540}{5000} \left(1000 - \frac{540(1/2 + 1)}{6(1/2) + 5} \right) = 97.065 \dots$$

$$P_1 \approx 540 + \frac{1}{6}(97.065 \dots) = 556.1775.$$

This is the estimated population after eight months, so it is now used as the starting value for the second step.

$$\left. \frac{dP}{dt} \right|_1 = \frac{556.1775}{5000} \left(1000 - \frac{556.1775(2/3 + 1)}{6(2/3) + 5} \right) = 99.778706 \dots$$

$$P_2 \approx 556.1775 + \frac{1}{6}(99.778706 \dots) = 572.807 \dots$$

The estimated number of deer after 10 months is 573 deer.

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Question 3

Question 3

(a) Use $t = \tan \frac{\theta}{2}$ to show, where both sides are defined, that

$$\frac{29 - 21 \sec \theta}{20 - 21 \tan \theta} \equiv \frac{5t + 2}{2t + 5}.$$

(b) Hence, again using $t = \tan \frac{\theta}{2}$, prove, where both sides are defined, that

$$\frac{20 + 21 \tan \theta}{29 + 21 \sec \theta} \equiv \frac{29 - 21 \sec \theta}{20 - 21 \tan \theta}.$$

Model solution

(a) The half-angle substitution is appropriate because it expresses both $\sec \theta$ and $\tan \theta$ as rational functions of t :

$$\sec \theta = \frac{1 + t^2}{1 - t^2}, \quad \tan \theta = \frac{2t}{1 - t^2}.$$

$$\frac{29 - 21 \sec \theta}{20 - 21 \tan \theta} = \frac{29 - 21 \frac{1 + t^2}{1 - t^2}}{20 - 21 \frac{2t}{1 - t^2}}.$$

Multiply the numerator and denominator by $1 - t^2$ to clear the smaller fractions.

$$\begin{aligned} &= \frac{29(1 - t^2) - 21(1 + t^2)}{20(1 - t^2) - 42t} = \frac{8 - 50t^2}{20 - 42t - 20t^2} \\ &= \frac{-2(5t - 2)(5t + 2)}{-2(5t - 2)(2t + 5)} = \frac{5t + 2}{2t + 5}. \end{aligned}$$

$$\frac{29 - 21 \sec \theta}{20 - 21 \tan \theta} \equiv \frac{5t + 2}{2t + 5}.$$

(4)

(b) Use the same identities on the left-hand side. If it simplifies to the expression found in part (a), both sides of the required identity are equal.

$$\begin{aligned} \frac{20 + 21 \tan \theta}{29 + 21 \sec \theta} &= \frac{20 + 21 \frac{2t}{1 - t^2}}{29 + 21 \frac{1 + t^2}{1 - t^2}} \\ &= \frac{20(1 - t^2) + 42t}{29(1 - t^2) + 21(1 + t^2)} = \frac{20 + 42t - 20t^2}{50 - 8t^2} \\ &= \frac{-2(5t + 2)(2t - 5)}{-2(2t - 5)(2t + 5)} = \frac{5t + 2}{2t + 5}. \end{aligned}$$

This is exactly the expression found in part (a). From part (a),

$$\frac{29 - 21 \sec \theta}{20 - 21 \tan \theta} = \frac{5t + 2}{2t + 5}.$$

Hence the two sides are identical.

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Question 4

Question 4

The parabola C has equation $y^2 = 10x$, and F is its focus. Point P on C has y -coordinate q , where $q > 0$.

- Write down the coordinates of F .
- Show that the tangent to C at P has equation $10x - 2qy + q^2 = 0$.
- The tangent at P meets the directrix at A . Point B lies on the directrix such that PB is parallel to the x -axis. Show that the intersection of the diagonals of quadrilateral $PBAF$ always lies on the y -axis.

Model solution

(a) Compare $y^2 = 10x$ with $y^2 = 4ax$. Here $4a = 10$, so $a = \frac{5}{2}$.

$$F = \left(\frac{5}{2}, 0\right).$$

(1)

(b) Implicit differentiation is the natural method because the curve is given in terms of y^2 .

$$2y \frac{dy}{dx} = 10 \implies \frac{dy}{dx} = \frac{5}{y}.$$

At P , $y = q$ and $q^2 = 10x$, so $P = \left(\frac{q^2}{10}, q\right)$ and the gradient is $\frac{5}{q}$. Thus

$$y - q = \frac{5}{q} \left(x - \frac{q^2}{10}\right).$$

$$10qy - 10q^2 = 50x - 5q^2 \implies 10x - 2qy + q^2 = 0.$$

$$10x - 2qy + q^2 = 0.$$

(3)

(c) The directrix is $x = -\frac{5}{2}$. Since PB is horizontal, it has the same y -coordinate as P , giving

$$B = \left(-\frac{5}{2}, q\right).$$

The diagonal BF joins $\left(-\frac{5}{2}, q\right)$ to $\left(\frac{5}{2}, 0\right)$, so its equation is

$$y = -\frac{q}{5} \left(x - \frac{5}{2}\right) = -\frac{q}{5}x + \frac{q}{2}.$$

The other diagonal is AP , which is the tangent found in part (b). Substitute the equation of BF into that tangent to locate their intersection.

$$10x - 2q \left(-\frac{q}{5}x + \frac{q}{2}\right) + q^2 = 0$$

$$10x + \frac{2q^2}{5}x = 0$$

$$x \left(10 + \frac{2q^2}{5} \right) = 0.$$

Since $q > 0$, the bracket is positive and cannot be zero. Hence $x = 0$, so the intersection lies on the y -axis.

The diagonals always intersect on $x = 0$, the y -axis.

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Question 5

Question 5

The vertices of a tetrahedron are $A(3, 2, -4)$, $B(9, -4, 2)$, $C(-6, -10, 8)$ and $D(-4, -5, 10)$. The plane $z = 0$ meets AB , AC and AD at M , N and P respectively.

Determine (a) the coordinates of M , N and P , (b) the area of triangle MNP , and (c) the exact volume of solid $BCDPNM$.

Model solution

(a) *Parametrise each edge from A. Setting the z-coordinate equal to zero finds where that edge meets the plane.*

$$\mathbf{r}_{AB} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -6 \\ 6 \end{pmatrix}.$$

$$-4 + 6\lambda = 0 \implies \lambda = \frac{2}{3}, \quad M = (7, -2, 0).$$

$$\mathbf{r}_{AC} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -9 \\ -12 \\ 12 \end{pmatrix}, \quad -4 + 12\mu = 0 \implies \mu = \frac{1}{3},$$

$$N = (0, -2, 0).$$

$$\mathbf{r}_{AD} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \nu \begin{pmatrix} -7 \\ -7 \\ 14 \end{pmatrix}, \quad -4 + 14\nu = 0 \implies \nu = \frac{2}{7},$$

$$P = (1, 0, 0).$$

$$M = (7, -2, 0), \quad N = (0, -2, 0) \text{ and } P = (1, 0, 0).$$

(3)

(b) *All three points lie in $z = 0$. Segment MN is horizontal in this plane, with length 7, and P is 2 units from the line $y = -2$. Use $\frac{1}{2} \times \text{base} \times \text{perpendicular height}$.*

$$\text{Area}(MNP) = \frac{1}{2} \times 7 \times 2 = 7.$$

$$\text{Area}(MNP) = 7 \text{ square units.}$$

(2)

(c) *The required solid is the whole tetrahedron $ABCD$ with the smaller tetrahedron $AMNP$ removed. We calculate both volumes using $V = \frac{1}{6} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.*

$$\overrightarrow{AB} = \begin{pmatrix} 6 \\ -6 \\ 6 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} -9 \\ -12 \\ 12 \end{pmatrix}, \quad \overrightarrow{AD} = \begin{pmatrix} -7 \\ -7 \\ 14 \end{pmatrix}.$$

$$\overrightarrow{AC} \times \overrightarrow{AD} = \begin{pmatrix} -84 \\ 42 \\ -21 \end{pmatrix}, \quad V_{ABCD} = \frac{1}{6} \left| \begin{pmatrix} 6 \\ -6 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -84 \\ 42 \\ -21 \end{pmatrix} \right| = \frac{882}{6} = 147.$$

For the smaller tetrahedron, use the three vectors from A to its other vertices.

$$\overrightarrow{AM} = \begin{pmatrix} 4 \\ -4 \\ 4 \end{pmatrix}, \quad \overrightarrow{AN} = \begin{pmatrix} -3 \\ -4 \\ 4 \end{pmatrix}, \quad \overrightarrow{AP} = \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}.$$

$$\overrightarrow{AN} \times \overrightarrow{AP} = \begin{pmatrix} -8 \\ 4 \\ -2 \end{pmatrix}, \quad V_{AMNP} = \frac{1}{6} \left| \begin{pmatrix} 4 \\ -4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 4 \\ -2 \end{pmatrix} \right| = \frac{56}{6} = \frac{28}{3}.$$

Subtracting removes precisely the portion below the plane $z = 0$.

$$V_{BCDPNM} = 147 - \frac{28}{3} = \frac{413}{3}.$$

$$\text{Vol}(BCDPNM) = \frac{413}{3} \text{ cubic units.}$$

(6)

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