



Edexcel AS Further Mathematics

8FM0/21 - Further Pure Mathematics 1

Friday 19 May 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) Use algebra to determine the values of x for which

$$\frac{5x}{x-2} \geq 12.$$

(b) Hence, given that x is an integer, deduce the value of x .

Model solution

(a) A rational inequality can change sign only where its numerator is zero or its denominator is zero. We first put everything over one denominator; $x = 2$ must be excluded.

$$\frac{5x}{x-2} - 12 \geq 0 \implies \frac{5x - 12(x-2)}{x-2} \geq 0$$

$$\frac{24 - 7x}{x-2} \geq 0.$$

The critical values are $x = 2$ (where the expression is undefined) and $x = \frac{24}{7}$ (where the numerator is zero). Testing the three resulting intervals gives:

x	$x < 2$	$2 < x < \frac{24}{7}$	$x > \frac{24}{7}$
$\frac{24-7x}{x-2}$	-	+	-

We need the non-negative interval. The zero at $x = \frac{24}{7}$ is included, but $x = 2$ is not.

$$2 < x \leq \frac{24}{7}.$$

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(4)

(b) The only integer strictly greater than 2 and no greater than $\frac{24}{7} \approx 3.43$ is 3.

$$x = 3.$$

(1)

Question 2

Question 2

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$3 \cos x - 2 \sin x = 1$$

can be written as $2t^2 + 2t - 1 = 0$.

(b) Hence solve, for $-180^\circ < x < 180^\circ$, the equation $3 \cos x - 2 \sin x = 1$, giving answers to one decimal place.

Model solution

(a) The half-angle substitution is signalled in the question. It turns both sine and cosine into rational expressions in the one variable t :

$$\cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}.$$

Substitute these into the equation, then multiply by $1+t^2$, which is always positive.

$$3 \left(\frac{1-t^2}{1+t^2} \right) - 2 \left(\frac{2t}{1+t^2} \right) = 1$$

$$3 - 3t^2 - 4t = 1 + t^2$$

$$4t^2 + 4t - 2 = 0$$

$$\boxed{2t^2 + 2t - 1 = 0}.$$

$$2t^2 + 2t - 1 = 0.$$

(3)

(b) We now solve the quadratic from part (a), then reverse the substitution $t = \tan(x/2)$.

$$t = \frac{-2 \pm \sqrt{2^2 - 4(2)(-1)}}{2(2)} = \frac{-1 \pm \sqrt{3}}{2}.$$

$$t \approx 0.366025 \quad \text{or} \quad t \approx -1.366025.$$

Since $-180^\circ < x < 180^\circ$, we have $-90^\circ < x/2 < 90^\circ$, where $\tan$ is one-to-one. Thus each value gives one value of x .

$$x = 2 \tan^{-1}(0.366025) \approx 40.2078^\circ,$$

$$x = 2 \tan^{-1}(-1.366025) \approx -107.5877^\circ.$$

$$x = -107.6^\circ \quad \text{or} \quad x = 40.2^\circ.$$

(4)

Question 3

Question 3

The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant. The line l has equation $x - 2y = c$. The points P and Q are the points of intersection of H and l .

(a) Determine, in terms of c , the coordinates of P and Q .

The point R is the midpoint of PQ .

(b) Show that, as c varies, the coordinates of R satisfy the equation

$$xy = \frac{c^2}{a},$$

where a is a constant to be determined.

Model solution

(a) At an intersection both equations hold. The line gives x in terms of y , so substituting it into $xy = c^2$ produces a quadratic whose two roots give the two intersection points.

$$x = 2y + c$$

$$(2y + c)y = c^2$$

$$2y^2 + cy - c^2 = 0$$

$$(2y - c)(y + c) = 0.$$

Hence $y = \frac{c}{2}$ or $y = -c$. Substitution into $x = 2y + c$ gives the matching x -coordinates:

$$(x, y) = \left(2c, \frac{c}{2}\right) \quad \text{or} \quad (x, y) = (-c, -c).$$

P and Q are $\left(2c, \frac{c}{2}\right)$ and $(-c, -c)$, in either order.

(3)

(b) Use the midpoint formula coordinate by coordinate. This gives the moving point R , from which its locus equation can be found.

$$R = \left(\frac{2c + (-c)}{2}, \frac{\frac{c}{2} + (-c)}{2}\right) = \left(\frac{c}{2}, -\frac{c}{4}\right).$$

Therefore, if (x, y) are the coordinates of R ,

$$xy = \left(\frac{c}{2}\right)\left(-\frac{c}{4}\right) = -\frac{c^2}{8} = \frac{c^2}{-8}.$$

$$xy = \frac{c^2}{-8}, \text{ so } a = -8.$$

(2)

Question 4

Question 4

A teacher made a cup of coffee. Its temperature $\theta^\circ\text{C}$, t minutes after it was made, is modelled by

$$\frac{d\theta}{dt} + 0.05(\theta - 20) = 0.$$

Initially the coffee is 95°C . It is safe to drink only below 70°C . The coffee was made at 1.15 pm and the teacher needs to start drinking it by 1.20 pm.

Use two iterations of $y_{n+1} = y_n + h \left(\frac{dy}{dx} \right)_n$ to estimate whether the teacher will be able to start drinking the coffee at 1.20 pm.

Model solution

Two iterations must cover the five-minute interval from 1.15 pm to 1.20 pm, so use a step size of $h = \frac{5}{2} = 2.5$ minutes. The initial values are $t_0 = 0$ and $\theta_0 = 95$.

$$\frac{d\theta}{dt} = -0.05(\theta - 20).$$

First find the initial cooling rate, then use Euler's approximation formula for the first step.

$$\left. \frac{d\theta}{dt} \right|_{t=0} = -0.05(95 - 20) = -3.75,$$

$$\theta_1 = 95 + 2.5(-3.75) = 85.625.$$

The temperature is now estimated at $t = 2.5$. Recalculate the rate there before taking the second step.

$$\left. \frac{d\theta}{dt} \right|_{\theta=85.625} = -0.05(85.625 - 20) = -3.28125,$$

$$\theta_2 = 85.625 + 2.5(-3.28125) = 77.421875.$$

This is the estimate at $t = 5$, namely 1.20 pm. Since it remains above the safe temperature, the coffee is still too hot.

$\theta(5) \approx 77.4^\circ\text{C} > 70^\circ\text{C}$, so the teacher will not be able to start drinking it at 1.20 pm.

(6)

Question 5

Question 5

The points A, B and C are vertices of a triangle. Given that

$$\overrightarrow{AB} = \begin{pmatrix} p \\ 4 \\ 6 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} q \\ 4 \\ 5 \end{pmatrix},$$

where p and q are constants, and that $\overrightarrow{AB} \times \overrightarrow{AC}$ is parallel to $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$:

- (a) determine p and q;
- (b) hence determine the exact area of triangle ABC.

Model solution

(a) The cross product is given to be parallel to a known vector. Calculate the cross product, then equate it to a scalar multiple of $(2, 3, 4)$; this supplies equations for p and q.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p & 4 & 6 \\ q & 4 & 5 \end{vmatrix} = -4\mathbf{i} + (6q - 5p)\mathbf{j} + (4p - 4q)\mathbf{k}.$$

Its i-component is -4 , while the corresponding component of the parallel vector is 2. Therefore the scalar multiple is -2 .

$$\overrightarrow{AB} \times \overrightarrow{AC} = -2(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = -4\mathbf{i} - 6\mathbf{j} - 8\mathbf{k}.$$

Equating the j- and k-components gives two simultaneous equations.

$$6q - 5p = -6, \quad 4p - 4q = -8.$$

$$p - q = -2 \implies q = p + 2.$$

$$6(p + 2) - 5p = -6 \implies p = -18,$$

$$q = -18 + 2 = -16.$$

$$p = -18, \quad q = -16.$$

(7)

(b) The magnitude of $\overrightarrow{AB} \times \overrightarrow{AC}$ is the area of the parallelogram formed by the two side vectors. A triangle has half this area.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -4 \\ -6 \\ -8 \end{pmatrix},$$

$$\text{Area of } \triangle ABC = \frac{1}{2} \sqrt{(-4)^2 + (-6)^2 + (-8)^2} = \frac{1}{2} \sqrt{116} = \sqrt{29}.$$

$\sqrt{29}$ square units.

(2)

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Question 6

Question 6

The parabola C has equation $y^2 = 4ax$, where a is a positive constant. The point $P(at^2, 2at)$, $t \neq 0$, lies on C . The normal to C at P is parallel to $y = 2x$.

(a) For the point P , show that $t = -2$.

The normal to C at P intersects C again when $x = 9$.

(b) Determine a , giving a reason for your answer.

Model solution

(a) Differentiate the parabola to obtain the tangent gradient. Since the normal is parallel to $y = 2x$, its gradient is 2, so the tangent gradient must be the negative reciprocal, $-\frac{1}{2}$.

$$y^2 = 4ax \implies 2y \frac{dy}{dx} = 4a \implies \frac{dy}{dx} = \frac{2a}{y}.$$

At P , $y = 2at$, so

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}.$$

$$\frac{1}{t} = -\frac{1}{2} \implies t = -2.$$

$$t = -2.$$

(3)

(b) With $t = -2$, first find P and its normal. Then use the given other intersection $x = 9$ in both the normal and the parabola to determine a .

$$P = (at^2, 2at) = (4a, -4a).$$

The normal has gradient 2, so its equation through P is

$$y + 4a = 2(x - 4a),$$

$$y = 2x - 12a.$$

At the stated second intersection $x = 9$, the normal gives $y = 18 - 12a$. This point lies on $y^2 = 4ax$, hence

$$(18 - 12a)^2 = 4a(9),$$

$$324 - 432a + 144a^2 = 36a,$$

$$4a^2 - 13a + 9 = 0,$$

$$(4a - 9)(a - 1) = 0.$$

$$a = \frac{9}{4} \quad \text{or} \quad a = 1.$$

If $a = \frac{9}{4}$, then P itself has $x = 4a = 9$. That contradicts the statement that $x = 9$ is the other intersection, so this root must be rejected.

$$a = 1.$$

(5)

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