



Edexcel AS Further Mathematics

8FM0/21 - Further Pure Mathematics 1

November 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

Contents

Question 1	2
Question 2	3
Question 3	4
Question 4	6
Question 5	8

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Question 1

Question 1

Use algebra to determine the values of x for which

$$x(x-1) > \frac{x-1}{x},$$

giving your answer in set notation.

Model solution

Move everything to one side and write it as one fraction. This is useful because the sign of a rational expression can then be found from the signs of its factors.

$$x(x-1) - \frac{x-1}{x} > 0$$

$$\frac{x^2(x-1) - (x-1)}{x} > 0$$

$$\frac{(x-1)(x^2-1)}{x} > 0$$

$$\frac{(x-1)^2(x+1)}{x} > 0.$$

The critical values are the zeros of the numerator and denominator: $x = -1$, $x = 0$ and $x = 1$. The value $x = 0$ is excluded because the original expression is undefined. Since $(x-1)^2$ is positive except at $x = 1$, the expression is positive on $x < -1$, $0 < x < 1$ and $x > 1$. The endpoints -1 and 1 are not included because the inequality is strict.

$$\{x \in \mathbb{R} : x < -1\} \cup \{x \in \mathbb{R} : 0 < x < 1\} \cup \{x \in \mathbb{R} : x > 1\}.$$

(6)

Question 2

Question 2

The variables x and y satisfy

$$\frac{d^2y}{dx^2} + 15\frac{dy}{dx} - 3y^2 = 2x,$$

where $y = 1$ at $x = 0$ and $y = 2$ at $x = 0.1$. Use

$$\left(\frac{d^2y}{dx^2}\right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}, \quad \left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h},$$

with $h = 0.1$ to estimate y when $x = 0.3$.

Model solution

The two known adjacent values allow the central-difference formulae to be applied first at $x = 0.1$. Let y_n denote the value at $x = 0.1n$, so $y_0 = 1$ and $y_1 = 2$. The differential equation then becomes an equation for the next value, y_2 .

$$\frac{y_2 - 2y_1 + y_0}{0.1^2} + 15\left(\frac{y_2 - y_0}{2(0.1)}\right) - 3y_1^2 = 2(0.1)$$

$$100(y_2 - 4 + 1) + 75(y_2 - 1) - 3(2)^2 = 0.2$$

$$175y_2 = 387.2, \text{ so } y_2 = \frac{1936}{875} = 2.212571\dots$$

This estimates y at $x = 0.2$. Repeat the same central-difference process at $x = 0.2$ to obtain y_3 , the required estimate at $x = 0.3$.

$$\frac{y_3 - 2y_2 + y_1}{0.1^2} + 15\left(\frac{y_3 - y_1}{2(0.1)}\right) - 3y_2^2 = 2(0.2)$$

$$100(y_3 - 2y_2 + 2) + 75(y_3 - 2) - 3y_2^2 = 0.4$$

$$175y_3 = 0.4 + 200y_2 - 50 + 3y_2^2$$

$$y_3 = 2.32914\dots$$

$$y(0.3) \approx 2.329.$$

(6)

Question 3

Question 3

On a particular day, the depth of water in a river estuary is modelled by

$$D = 2 \sin\left(\frac{x}{3}\right) + 3 \cos\left(\frac{x}{3}\right) + 6, \quad 0 \leq x \leq 7\pi,$$

where D is the depth in metres at time x hours after midnight.

(a) Write down the depth at midnight.

Using $t = \tan\left(\frac{x}{6}\right)$,

(b) show that $D = \frac{3t^2 + 4t + 9}{t^2 + 1}$.

(c) Hence determine the first time after midnight when the depth is 5 metres, to the nearest minute.

Model solution

(a) At midnight $x = 0$, so substitute 0 directly into the model.

$$D = 2 \sin 0 + 3 \cos 0 + 6 = 9.$$

$$9 \text{ m.}$$

(1)

(b) Put $\theta = \frac{x}{6}$, so that $t = \tan \theta$. The double-angle clue is that $\frac{x}{3} = 2\theta$. Expressing $\sin \theta$ and $\cos 2\theta$ in terms of $\tan \theta$ gives a rational expression in t .

$$\sin \theta = \frac{2t}{1+t^2}, \quad \cos 2\theta = \frac{1-t^2}{1+t^2}.$$

$$\begin{aligned} D &= 2 \left(\frac{2t}{1+t^2} \right) + 3 \left(\frac{1-t^2}{1+t^2} \right) + 6 \\ &= \frac{4t + 3 - 3t^2 + 6 + 6t^2}{1+t^2} = \frac{3t^2 + 4t + 9}{t^2 + 1}. \end{aligned}$$

$$D = \frac{3t^2 + 4t + 9}{t^2 + 1}.$$

(3)

(c) Set the model equal to the specified depth. The substitution has reduced the trigonometric equation to a quadratic, which can be solved exactly.

$$\frac{3t^2 + 4t + 9}{t^2 + 1} = 5$$

$$3t^2 + 4t + 9 = 5t^2 + 5$$

$$t^2 - 2t - 2 = 0.$$

$$t = 1 \pm \sqrt{3}.$$

For the first time after midnight, $0 < x < 3\pi$, hence $0 < \frac{x}{6} < \frac{\pi}{2}$ and $t > 0$. Therefore choose $t = 1 + \sqrt{3}$.

$$\tan\left(\frac{x}{6}\right) = 1 + \sqrt{3}$$

$$x = 6 \tan^{-1}(1 + \sqrt{3}) = 7.32 \dots \text{ hours.}$$

$$0.32 \dots \times 60 = 19 \dots \text{ minutes.}$$

07:19 (the first time after midnight).

(5)

Question 4

Question 4

With respect to a fixed origin O,

$$\vec{OA} = 18\mathbf{i} - 14\mathbf{j} - 2\mathbf{k}, \quad \vec{OB} = -7\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}, \quad \vec{OC} = -2\mathbf{i} - 9\mathbf{j} - 6\mathbf{k}.$$

The points O, A, B, C are vertices of a tetrahedron.

(a) Show that the area of face ABC is 108 to 3 significant figures.

(b) Find the volume of the tetrahedron.

The tetrahedron is an oak block, with centimetres as the units. The density of oak is 0.85 g cm^{-3} .

(c) Determine its mass in kg.

Model solution

(a) Two side vectors in triangle ABC give its area by a cross product: $\frac{1}{2} \|\vec{AB} \times \vec{AC}\|$.

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} -25 \\ 9 \\ 5 \end{pmatrix}, \quad \vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix} -20 \\ 5 \\ -4 \end{pmatrix}.$$

$$\vec{AB} \times \vec{AC} = \begin{pmatrix} -61 \\ -200 \\ 55 \end{pmatrix}.$$

$$\text{Area of ABC} = \frac{1}{2} \sqrt{(-61)^2 + (-200)^2 + 55^2} = \frac{1}{2} \sqrt{46746} = 108.101 \dots$$

$$\text{Area} = 108 \text{ (to 3 s.f.)}$$

(3)

(b) The three edges from O form the tetrahedron. One sixth of the magnitude of their scalar triple product is its volume.

$$V = \frac{1}{6} \left| \vec{OA} \cdot (\vec{OB} \times \vec{OC}) \right| = \frac{1}{6} \left| \begin{vmatrix} 18 & -14 & -2 \\ -7 & -5 & 3 \\ -2 & -9 & -6 \end{vmatrix} \right|.$$

$$\begin{vmatrix} 18 & -14 & -2 \\ -7 & -5 & 3 \\ -2 & -9 & -6 \end{vmatrix} = 18(57) - (-14)(48) + (-2)(53) = 1592.$$

$$V = \frac{1592}{6} = \frac{796}{3} \text{ cm}^3.$$

(4)

(c) Density is mass per unit volume, so multiply the volume in cm^3 by 0.85 g cm^{-3} , then convert grams to kilograms.

$$m = \frac{796}{3} \times 0.85 = 225.533 \dots \text{ g} = 0.225533 \dots \text{ kg}.$$

0.226 kg.

(2)

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Question 5

Question 5

The point $P(ap^2, 2ap)$, where a is positive, lies on $y^2 = 4ax$. The normal at P meets the parabola again at $Q(aq^2, 2aq)$.

(a) Show that $q = -p - \frac{2}{p}$.

(b) Hence show that

$$PQ^2 = \frac{ka^2}{p^4}(p^2 + 1)^n,$$

where k and n are integers to be determined.

Model solution

(a) Differentiate the parabola to find the tangent gradient. The normal has the negative reciprocal gradient, so its equation can be evaluated at the second point Q .

$$y^2 = 4ax \implies 2y \frac{dy}{dx} = 4a, \quad \frac{dy}{dx} = \frac{2a}{y}.$$

$$\text{At } P, \quad \frac{dy}{dx} = \frac{2a}{2ap} = \frac{1}{p}, \quad \text{so normal gradient} = -p.$$

$$y - 2ap = -p(x - ap^2).$$

Both P and Q are on this normal. Substitute the coordinates of Q and factor out the known root $q = p$, which corresponds to $Q = P$; the other factor gives the second intersection.

$$2aq - 2ap = -p(aq^2 - ap^2)$$

$$2(q - p) = -p(q - p)(q + p)$$

$$(q - p)(2 + p(q + p)) = 0.$$

$$q \neq p \implies 2 + p(q + p) = 0$$

$$q = -p - \frac{2}{p}.$$

(5)

(b) Use Pythagoras on the horizontal and vertical changes from P to Q , then substitute the result from part (a) so that the distance is in terms of p only.

$$PQ^2 = (aq^2 - ap^2)^2 + (2aq - 2ap)^2$$

$$= a^2(p - q)^2((p + q)^2 + 4).$$

$$p - q = p - \left(-p - \frac{2}{p}\right) = \frac{2(p^2 + 1)}{p}, \quad p + q = -\frac{2}{p}.$$

$$PQ^2 = a^2 \left(\frac{2(p^2 + 1)}{p}\right)^2 \left(\frac{4}{p^2} + 4\right) = \frac{16a^2(p^2 + 1)^3}{p^4}.$$

$$PQ^2 = \frac{16a^2}{p^4}(p^2 + 1)^3, \quad k = 16, \quad n = 3.$$

(5)

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