



Edexcel AS Further Mathematics

8FM0/22 - Further Pure Mathematics 2

Thursday 17 May 2018

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2018 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

- (i) Using a suitable algorithm, and without performing any division, determine whether 23 738 is divisible by 11. (2)
- (ii) Use the Euclidean algorithm to find the highest common factor of 2322 and 654. (3)

Model solution

(i) For the divisibility test for 11, subtract the sum of the alternate digits from the sum of the remaining digits:

$$(2 + 7 + 8) - (3 + 3) = 11.$$

This is a multiple of 11, so 23 738 is divisible by 11.

23 738 is divisible by 11.

(ii) Apply the Euclidean algorithm:

$$2322 = 3(654) + 360,$$

$$654 = 1(360) + 294,$$

$$360 = 1(294) + 66,$$

$$294 = 4(66) + 30,$$

$$66 = 2(30) + 6,$$

$$30 = 5(6) + 0.$$

The last non-zero remainder is 6. Hence

$$\text{HCF}(2322, 654) = 6.$$

The alternating-digit test is the required no-division algorithm in part (i); the Euclidean algorithm is shown through to remainder zero in part (ii).

Question 2

Question 2

An equilateral triangle is acted on by $T = \{I, X, Y, Z, R_1, R_2\}$, where I is the identity, X, Y, Z are reflections in its three medians, and R_1, R_2 are anticlockwise rotations about its centre through 120° and 240° , respectively. The operation $*$ means “followed by”; thus the row operation is performed first and the column operation second.

- (a)(i) Complete the Cayley table. (ii) Given associativity, show that T is a group under $*$. (6)
 (b) Show that R_2 has order 3. (2)
 (c) Explain why T is not cyclic. (1)
 (d) Write down the elements of a subgroup of T of order 3. (1)

Model solution

(a)(i) The completed table is

$*$	I	X	Y	Z	R_1	R_2
I	I	X	Y	Z	R_1	R_2
X	X	I	R_2	R_1	Z	Y
Y	Y	R_1	I	R_2	X	Z
Z	Z	R_2	R_1	I	Y	X
R_1	R_1	Y	Z	X	R_2	I
R_2	R_2	Z	X	Y	I	R_1

(a)(ii) The table shows closure: every product of two elements of T is again an element of T . It also shows that I is an identity, since $I * A = A * I = A$ for every $A \in T$.

The inverses are

$$I^{-1} = I, \quad X^{-1} = X, \quad Y^{-1} = Y, \quad Z^{-1} = Z, \quad R_1^{-1} = R_2, \quad R_2^{-1} = R_1.$$

Associativity is given. Therefore T is closed and associative, has an identity, and every element has an inverse; hence T is a group under $*$.

(b) From the table,

$$R_2 * R_2 = R_1, \quad R_2 * R_2 * R_2 = R_1 * R_2 = I.$$

Neither R_2 nor $R_2 * R_2 = R_1$ is I , so the least positive power giving the identity is 3.

R_2 has order 3.

(c) A cyclic group of order 6 must have an element of order 6. Here R_1 and R_2 have order 3, while X, Y, Z have order 2, and I has order 1. Thus no element generates all of T .

T is not cyclic.

(d)

$\{I, R_1, R_2\}$.

The table records the stated order of composition: an entry in row A , column B is $A * B$, so A is applied before B .

Question 3

Question 3

A tree grows by 15 cm each year. On the first day of each year it is measured and immediately trimmed by 3% of that height. Its height before trimming at the start of year 1 is 6 m. Let H_n be the height in metres immediately before trimming at the start of year n .

- (a) Explain why $H_{n+1} = 0.97H_n + 0.15$, $H_1 = 6$. (3)
- (b) Prove by induction that $H_n = 0.97^{n-1} + 5$, for $n \geq 1$. (4)
- (c) Explain what happens to the pre-trimming height in the long term. (1)
- (d) Find the fixed percentage by which the tree should instead be trimmed annually so that its pre-trimming height tends to 4 m. (2)

Model solution

(a) Trimming removes 3% of H_n , leaving 97%, or $0.97H_n$, metres. During the following year the tree grows by 15 cm = 0.15 m. Hence

$$H_{n+1} = 0.97H_n + 0.15.$$

Before the first trimming the stated height is 6 m, so $H_1 = 6$.

(b) For $n = 1$,

$$0.97^{1-1} + 5 = 1 + 5 = 6 = H_1,$$

so the formula is true initially. Assume it is true for $n = k$, so that

$$H_k = 0.97^{k-1} + 5.$$

Then, using the recurrence relation,

$$\begin{aligned} H_{k+1} &= 0.97H_k + 0.15 \\ &= 0.97(0.97^{k-1} + 5) + 0.15 \\ &= 0.97^k + 4.85 + 0.15 \\ &= 0.97^k + 5. \end{aligned}$$

This is $0.97^{(k+1)-1} + 5$, so the result is true for $k + 1$. Since it is true for $n = 1$, it follows by induction that

$$H_n = 0.97^{n-1} + 5 \text{ for all positive integers } n.$$

(c) Since $0.97^{n-1} \rightarrow 0$ as $n \rightarrow \infty$,

the height immediately before trimming approaches 5 m.

(d) Let the proportion left after each trim be x . At the limiting height,

$$4 = 4x + 0.15, \quad x = \frac{3.85}{4} = 0.9625.$$

Thus the proportion removed is $1 - 0.9625 = 0.0375$.

The tree should be trimmed by 3.75% each year.

All heights in the recurrence are in metres; converting the annual growth is essential.

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Question 4

Question 4

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$

Find a matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

(7)

Model solution

The characteristic equation is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & 1 \\ -2 & 4 - \lambda \end{vmatrix} = 0 \\ (1 - \lambda)(4 - \lambda) + 2 &= 0 \\ \lambda^2 - 5\lambda + 6 &= 0, \end{aligned}$$

so the eigenvalues are $\lambda = 2$ and $\lambda = 3$.

For $\lambda = 2$,

$$(A - 2I) \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad -x + y = 0,$$

so an eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda = 3$,

$$(A - 3I) \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad -2x + y = 0,$$

so an eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Putting these eigenvectors, in the same order as their eigenvalues, into the columns of P ,

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Indeed $AP = PD$, and hence $D = P^{-1}AP$.

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

The columns of P are eigenvectors corresponding, in order, to the diagonal entries of D .

Question 5

Question 5

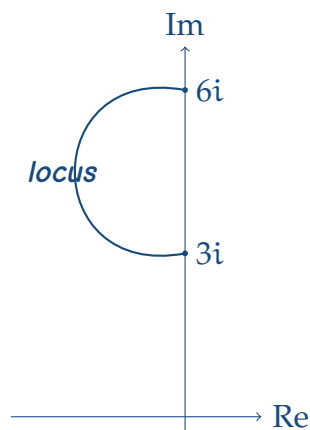
A complex number z is represented by P on an Argand diagram. Given that

$$\arg\left(\frac{z-6i}{z-3i}\right) = \frac{\pi}{3},$$

- (a) sketch the locus of P as z varies; (3)
 (b) find the exact maximum possible value of $|z|$. (5)

Model solution

(a) The argument is the directed angle between the lines from P to $3i$ and $6i$. Thus the locus is the major arc through $3i$ and $6i$ which subtends an angle $\pi/3$. It lies wholly to the left of the imaginary axis and above the real axis. The endpoints $3i$ and $6i$ are excluded because the given quotient is then undefined:



(b) The chord joining $3i$ and $6i$ has midpoint with imaginary coordinate 4.5 . Its length is 3 , so its half-length is 1.5 . The centre C is to the left of the imaginary axis and

$$\operatorname{Re}(C) = -\frac{1.5}{\tan(\pi/3)} = -\frac{\sqrt{3}}{2}, \quad C = -\frac{\sqrt{3}}{2} + 4.5i.$$

The radius of the circle is

$$r = \frac{1.5}{\sin(\pi/3)} = \sqrt{3}.$$

The greatest distance from the origin to a point on this major arc occurs on the line through the origin and C , on the far side of C . Now

$$|C| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + (4.5)^2} = \sqrt{\frac{3}{4} + \frac{81}{4}} = \sqrt{21}.$$

Therefore the maximum possible modulus is $|C| + r$, namely

$$\sqrt{21} + \sqrt{3}.$$

The major arc, rather than the other arc through $3i$ and $6i$, is selected by the positive directed

argument $\pi/3$.

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