



Edexcel AS Further Mathematics

8FM0/22 - Further Pure Mathematics 2

Thursday 16 May 2019

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2019 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Given $A = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$.

(a) Find the characteristic equation for A , simplifying your answer. (2)

(b) Hence find A^{-1} in the form $\lambda A + \mu I$, where λ and μ are constants. (3)

Model solution

(a) The characteristic equation is obtained from

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 2 \\ 2 & 2 - \lambda \end{vmatrix} = 0.$$

Therefore

$$(3 - \lambda)(2 - \lambda) - 4 = 0 \implies \lambda^2 - 5\lambda + 2 = 0.$$

$$\lambda^2 - 5\lambda + 2 = 0$$

(2)

(b) By the Cayley--Hamilton theorem, A satisfies its characteristic equation:

$$A^2 - 5A + 2I = 0.$$

Multiplying by A^{-1} gives

$$A - 5I + 2A^{-1} = 0,$$

so

$$A^{-1} = -\frac{1}{2}A + \frac{5}{2}I.$$

$$A^{-1} = -\frac{1}{2}A + \frac{5}{2}I$$

(3)

The determinant gives the characteristic equation. Cayley--Hamilton then turns this scalar polynomial into a matrix equation, which can be rearranged for the inverse.

Question 2

Question 2

- (i) Determine all possible integers a , where $a > 3$, such that $15 \equiv 3 \pmod{a}$. (2)
- (ii) Show that if p is prime, x is an integer and $x^2 \equiv 1 \pmod{p}$, then either $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$. (3)
- (iii) A company has \$13 940 220 to share between 11 charities. Without performing any division, and showing all working, decide whether it can be shared equally. (2)

Model solution

(i) $15 \equiv 3 \pmod{a}$ means that $a \mid (15 - 3)$, so $a \mid 12$. The positive divisors of 12 that are greater than 3 are 4, 6, 12.

$$a = 4, 6, 12$$

(2)

(ii) From $x^2 \equiv 1 \pmod{p}$,

$$x^2 - 1 \equiv 0 \pmod{p} \implies (x - 1)(x + 1) \equiv 0 \pmod{p}.$$

Thus $p \mid (x - 1)(x + 1)$. Since p is prime, Euclid's lemma implies that $p \mid (x - 1)$ or $p \mid (x + 1)$. Hence

$$x \equiv 1 \pmod{p} \text{ or } x \equiv -1 \pmod{p}.$$

(3)

(iii) Apply the divisibility test for 11, using alternating digit sums:

$$1 - 3 + 9 - 4 + 0 - 2 + 2 - 0 = 3.$$

As 3 is not divisible by 11, 13 940 220 is not divisible by 11. Therefore the money cannot be shared equally between the 11 charities.

It is not possible to share the money equally.

(2)

For a congruence, subtracting the two congruent numbers produces a multiple of the modulus. The proof uses the fact that a prime divisor of a product divides at least one factor.

Question 3

Question 3

A curve C in the complex plane is described by

$$|z - 1 - 8i| = 3|z - 1|.$$

- (a) Show that C is a circle, and find its centre and radius. (4)
 (b) Using part (a), determine whether $z = 3 - 3i$ satisfies $|z - 1 - 8i| \geq 3|z - 1|$. (2)
 (c) Shade, on an Argand diagram, the set of points satisfying both

$$|z - 1 - 8i| \geq 3|z - 1| \quad \text{and} \quad 0 \leq \arg(z + i) \leq \frac{\pi}{4}.$$

(4)

Model solution

(a) Let $z = x + iy$. Squaring both sides gives

$$(x - 1)^2 + (y - 8)^2 = 9((x - 1)^2 + y^2).$$

Expanding and collecting terms,

$$8x^2 - 16x + 8y^2 + 16y - 56 = 0,$$

so, on dividing by 8 and completing the square,

$$x^2 - 2x + y^2 + 2y - 7 = 0 \implies (x - 1)^2 + (y + 1)^2 = 9.$$

This is a circle.

Centre $(1, -1)$ (that is, $1 - i$); radius 3.

(4)

(b) The point $3 - 3i$ has coordinates $(3, -3)$. Its distance from the centre $(1, -1)$ is

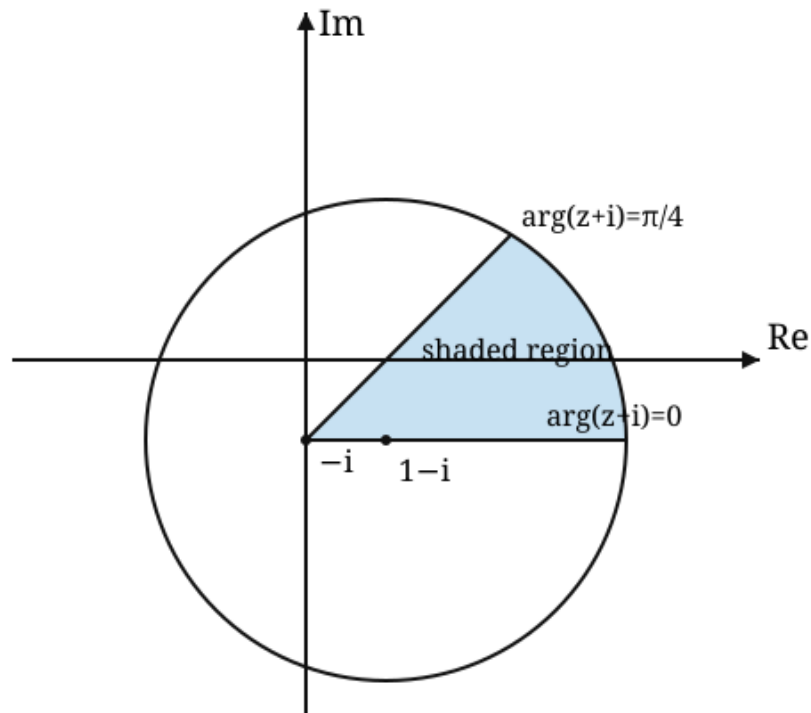
$$\sqrt{(3 - 1)^2 + (-3 - (-1))^2} = \sqrt{8} < 3.$$

The inequality $|z - 1 - 8i| \geq 3|z - 1|$ holds on the centre side of the circle (for example, it holds at $z = 1 - i$); hence it holds inside and on the circle. Therefore $3 - 3i$ satisfies the inequality.

Yes, $z = 3 - 3i$ satisfies the inequality.

(2)

(c) The first inequality gives the closed disc with centre $1 - i$ and radius 3. The argument condition is the closed sector from $-i$, between the horizontal ray to the right and the ray of gradient 1. Their intersection, including both boundary rays and the circular boundary, is shaded below.



(4)

Squaring moduli converts the locus into Cartesian coordinates. A test point, the circle's centre, identifies which side of the circle satisfies the inequality.

Question 4

Question 4

The set $\{e, p, q, r, s\}$ forms a group A under $*$. Given that e is the identity element and

$$p * p = s, \quad s * s = r, \quad p * p * p = q,$$

(a) Show that (i) $p * q = r$, and (ii) $s * p = q$. (2)

(b) Hence complete the Cayley table. (2)

(c) Use the table to find $p * q * r * s$. (1)

A student states that there is a subgroup of A of order 3.

(d) Comment on the validity of this statement, giving a reason. (2)

Model solution

(a) By associativity,

$$p * q = p * (p * p * p) = p * p * p * p = (p * p) * (p * p) = s * s = r.$$

Also

$$s * p = (p * p) * p = p * p * p = q.$$

$$p * q = r, \quad s * p = q$$

(2)

(b) The powers $p^0 = e, p^1 = p, p^2 = s, p^3 = q, p^4 = r$ are the five distinct elements of A . Thus p has order 5, so $p^5 = e$. Multiplication therefore corresponds to adding powers of p modulo 5:

$*$	e	p	q	r	s
e	e	p	q	r	s
p	p	s	r	e	q
q	q	r	p	s	e
r	r	e	s	q	p
s	s	q	e	p	r

(2)

(c) Reading successively from the table,

$$p * q * r * s = r * r * s = q * s = e.$$

$$p * q * r * s = e$$

(1)

(d) By Lagrange's theorem, the order of a subgroup must divide the order of its group. Here $|A| = 5$, and $3 \nmid 5$. Therefore there cannot be a subgroup of order 3.

The student's statement is invalid.

(2)

The given products show that the elements are successive powers of p , giving a cyclic group of order 5.

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Question 5

Question 5

On Jim's 11th birthday, \$1000 is invested in a savings account. The account earns 2% interest each year and, on each subsequent birthday, another \$500 is added. Let U_n be the amount n years after his 11th birthday, after that year's interest has been paid and the \$500 added.

- (a) Explain why the model is $U_n = 1.02U_{n-1} + 500$, $U_0 = 1000$, $n \in \mathbb{Z}^+$. (3)
 - (b) State an assumption needed for this model. (1)
 - (c) Solve the recurrence relation. (5)
- Jim hopes to buy a \$4500 car on his 18th birthday.
- (d) Use part (c) to decide whether he will have enough money. (2)

Model solution

(a) U_{n-1} is the balance $n - 1$ years after Jim's 11th birthday. It receives 2% interest, so it is multiplied by 1.02, giving $1.02U_{n-1}$. On the subsequent birthday \$500 is added, giving

$$U_n = 1.02U_{n-1} + 500.$$

At $n = 0$, on his 11th birthday, the initial investment is \$1000, so $U_0 = 1000$. (3)

(b) For example, assume that the interest rate remains 2% in every year. (1)

(c) For the complementary function, $U_n = C(1.02)^n$. Seek a constant particular solution $U_n = L$. Then

$$L = 1.02L + 500,$$

so $-0.02L = 500$, giving $L = -25\,000$. Thus

$$U_n = C(1.02)^n - 25\,000.$$

Using $U_0 = 1000$,

$$1000 = C - 25\,000 \implies C = 26\,000.$$

$$U_n = 26\,000(1.02)^n - 25\,000, \quad n \geq 0$$

(5)

(d) Jim is 18 seven years after his 11th birthday, so

$$U_7 = 26\,000(1.02)^7 - 25\,000 = 4865.827 \dots$$

Thus he will have \$4865.83 (to the nearest penny), which exceeds \$4500.

Yes; Jim has enough money to buy the car.

(2)

The recurrence describes annual compound interest followed by the annual deposit. A constant particular solution accounts for the repeated \$500 addition.