



Edexcel AS Further Mathematics

8FM0/22 - Further Pure Mathematics 2

Tuesday 24 May 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Sketch on an Argand diagram the region

$$\left\{z \in \mathbb{C} : -\frac{\pi}{4} < \arg(z+2) < \frac{\pi}{4}\right\} \cap \{z \in \mathbb{C} : -1 < \operatorname{Re}(z) \leq 1\}.$$

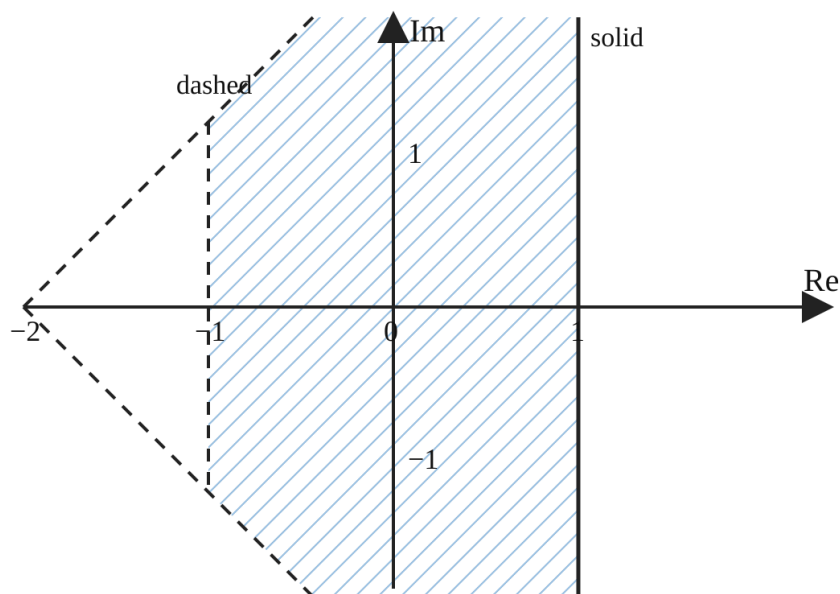
Shade the included region, using solid and dashed boundary lines as appropriate.

Model solution

Write $z = x + iy$. The clue $\arg(z+2)$ means that the vertex of the angular region is not the origin: it is the point $(-2, 0)$. Arguments between $-\pi/4$ and $\pi/4$ give the open sector between the two rays of gradients -1 and 1 .

$$-\frac{\pi}{4} < \arg(z+2) < \frac{\pi}{4} \iff -(x+2) < y < x+2.$$

The real-part condition is the vertical strip $-1 < x \leq 1$. Its left-hand boundary is excluded, while its right-hand boundary is included. Intersect this strip with the sector; the angular boundaries are excluded because both argument inequalities are strict.



The shaded region is $-1 < x \leq 1$ between the dashed lines $y = \pm(x+2)$; $x = 1$ is solid and $x = -1$ is dashed.

(4)

Question 2

Question 2

$$M = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}.$$

Find a matrix P and a diagonal matrix D such that $P^{-1}MP = D$.

Model solution

Diagonalisation uses a basis of eigenvectors. The eigenvalues will become the diagonal entries of D , in the same order as their eigenvectors are placed as columns of P .

$$\det(M - \lambda I) = \begin{vmatrix} 4 - \lambda & 2 \\ 3 & -1 - \lambda \end{vmatrix} = (4 - \lambda)(-1 - \lambda) - 6 = \lambda^2 - 3\lambda - 10.$$

$$\lambda^2 - 3\lambda - 10 = 0 \implies (\lambda - 5)(\lambda + 2) = 0,$$

$$\lambda = 5 \quad \text{or} \quad \lambda = -2.$$

For each eigenvalue, solve $(M - \lambda I)\mathbf{v} = \mathbf{0}$. For $\lambda = -2$,

$$\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies 3x + y = 0.$$

$$\mathbf{v}_{-2} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

For $\lambda = 5$,

$$\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x = 2y, \quad \mathbf{v}_5 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Put these eigenvectors in columns, retaining the order $-2, 5$. This makes the corresponding diagonal matrix immediate.

$$P = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}.$$

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(7)

Question 3

Question 3

- (i) Let G be a group of order 5291848. Without performing any division, use proof by contradiction to show that G cannot have a subgroup of order 11.
- (ii)(a) Complete the Cayley table for $X = \{2, 4, 8, 14, 16, 22, 26, 28\}$ under multiplication modulo 30.
- (ii)(b) Hence determine whether $(X, \times_{30})$ forms a group. You may assume multiplication modulo n is associative.

Model solution

(i) A subgroup order must divide the group order by Lagrange's theorem, so assume the proposed subgroup exists and test divisibility by 11.

$$\text{Assume that } H \leq G \text{ and } |H| = 11.$$

$$\text{Then } 11 \mid |G| = 5291848.$$

The alternating sum of the digits tests divisibility by 11:

$$5 - 2 + 9 - 1 + 8 - 4 + 8 = 23.$$

Since 23 is not divisible by 11, 5291848 is not divisible by 11. This contradicts the necessary consequence of Lagrange's theorem.

Therefore G cannot have a subgroup of order 11.

(3)

(ii)(a) Each entry is found by multiplying the row and column headings and reducing modulo 30. For example, $14 \times 22 = 308 \equiv 8 \pmod{30}$.

$\times_{30}$	2	4	8	14	16	22	26	28
2	4	8	16	28	2	14	22	26
4	8	16	2	26	4	28	14	22
8	16	2	4	22	8	26	28	14
14	28	26	22	16	14	8	4	2
16	2	4	8	14	16	22	26	28
22	14	28	26	8	22	4	2	16
26	22	14	28	4	26	2	16	8
28	26	22	14	2	28	16	8	4

The completed Cayley table is shown above.

(3)

(ii)(b) To establish a group, check closure, an identity and inverses; associativity is supplied in the question. The row and column headed 16 reproduce the headings, so 16 is the identity.

x	2	4	8	14	16	22	26	28
x^{-1}	8	4	2	14	16	28	26	22

Every product in the completed table remains in X , so the operation is closed. Every element has an inverse shown above, and multiplication modulo 30 is associative.

$(X, \times_{30})$ is a group, with identity 16.

(3)

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Question 4

Question 4

- (i)(a) Use the Euclidean algorithm to find the highest common factor h of 416 and 72.
 (i)(b) Hence determine integers a and b such that $416a + 72b = h$.
 (i)(c) Determine $c \in \{0, 1, 2, \dots, 415\}$ such that $23 \times 72 \equiv c \pmod{416}$.
 (ii) Evaluate $5^{10} \pmod{13}$, giving the smallest positive integer solution.

Model solution

(i)(a) The Euclidean algorithm repeatedly replaces the larger number by a remainder. The last non-zero remainder is the highest common factor.

$$416 = 5(72) + 56, \quad 72 = 1(56) + 16, \quad 56 = 3(16) + 8, \quad 16 = 2(8) + 0.$$

$$h = 8.$$

(3)

(i)(b) Work backwards from the last non-zero remainder, so that 8 is expressed as a linear combination of 416 and 72.

$$8 = 56 - 3(16) = 56 - 3(72 - 56) = 4(56) - 3(72)$$

$$= 4(416 - 5 \cdot 72) - 3(72) = 4(416) - 23(72).$$

$$a = 4 \text{ and } b = -23.$$

(3)

(i)(c) Part (b) already gives a multiple of 72, so reduce it modulo 416:

$$23 \times 72 = 4 \times 416 - 8 \equiv -8 \pmod{416}.$$

The required representative must lie between 0 and 415, so replace -8 by the congruent value $416 - 8$.

$$c = 408.$$

(2)

(ii) Repeated powers are useful for a large exponent. Since $5^2 = 25 \equiv -1 \pmod{13}$,

$$5^{10} = (5^2)^5 \equiv (-1)^5 = -1 \equiv 12 \pmod{13}.$$

$$12.$$

(3)

Question 5

Question 5

A person takes a course of a vitamin. Initially two tablets are taken, then one tablet at the same time on each following day. Each tablet contains 10 mg. Between doses, the amount decreases by 60%.

Let u_n mg be the amount immediately after a tablet is taken, n days after the initial two tablets.

(a) Explain why $u_0 = 20$ and $u_{n+1} = 0.4u_n + 10$.

The general solution has form $u_n = a(0.4)^n + b$.

(b) Determine a and b .

(c) The course is effective only if the amount remains above 6 mg at all times. Determine whether it will be effective, giving a reason.

Model solution

(a) The sequence is measured immediately after each dose. Initially two 10 mg tablets have been taken, giving 20 mg. Over the following day, a 60% reduction leaves 40% = 0.4 of the amount; the next tablet then adds 10 mg.

$$u_0 = 20, \quad u_{n+1} = 0.4u_n + 10.$$

$$u_0 = 20, \quad u_{n+1} = 0.4u_n + 10.$$

(2)

(b) Use two consecutive values of the recurrence to determine the two constants in the given general form. First,

$$u_1 = 0.4(20) + 10 = 18.$$

$$u_0 = a + b = 20, \quad u_1 = 0.4a + b = 18.$$

Subtracting the second equation from the first eliminates b .

$$0.6a = 2 \implies a = \frac{10}{3}, \quad b = 20 - \frac{10}{3} = \frac{50}{3}.$$

$$a = \frac{10}{3}, \quad b = \frac{50}{3}, \text{ so } u_n = \frac{10}{3}(0.4)^n + \frac{50}{3}.$$

(4)

(c) The smallest amount occurs immediately before a dose, not immediately after it. The model approaches a limiting post-dose amount because $(0.4)^n \rightarrow 0$:

$$u_n \longrightarrow \frac{50}{3} \text{ mg.}$$

Just before the next tablet, 60% has been lost, so the limiting lowest amount is

$$0.4 \left(\frac{50}{3} \right) = \frac{20}{3} \text{ mg} = 6\frac{2}{3} \text{ mg.}$$

This limiting lower bound is greater than 6 mg. Earlier pre-dose amounts are larger because the sequence decreases towards, but never reaches, its limit.

The course will be effective : the amount is always above 6mg.

(3)

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