



Edexcel AS Further Mathematics

8FM0/22 - Further Pure Mathematics 2

Friday 19 May 2023

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2023 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

The operation $*$ on $G = \{0, 1, 2, 3\}$ is defined by

$$x * y \equiv x + y - 2xy \pmod{4}.$$

- (a) Complete the Cayley table. (b) Show that G is a group under $*$, assuming associativity. (c) State the order of each element. (d) State whether G is cyclic, giving a reason.

Model solution

(a) A Cayley table is found by applying the defining rule to every row--column pair, always reducing the result modulo 4. For example, $1 * 2 \equiv 1 + 2 - 2(1)(2) \equiv -1 \equiv 3 \pmod{4}$.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

The completed Cayley table is shown above.

(2)

(b) The table contains only elements of G , so G is closed under $*$. The first row and column show that $0 * x = x * 0 = x$ for every $x \in G$; thus 0 is the identity. The diagonal entries are all 0, so

$$0^{-1} = 0, \quad 1^{-1} = 1, \quad 2^{-1} = 2, \quad 3^{-1} = 3.$$

Each element therefore has an inverse in G . Together with the given associativity, closure and an identity, these are the group axioms. Hence G is a group under $*$.

G is a group under $*$.

(3)

(c) The order is the least positive number of times an element must be combined with itself to obtain the identity. Thus 0 already is the identity, while each non-zero element squares to 0.

$$\text{ord}(0) = 1, \quad \text{ord}(1) = \text{ord}(2) = \text{ord}(3) = 2.$$

0 has order 1; 1, 2 and 3 each have order 2.

(2)

(d) A cyclic group of order 4 must have a generator of order 4. No element here has order 4, so no generator exists.

G is not cyclic.

(1)

Question 2

Question 2

The linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has matrix

$$M = \begin{pmatrix} 5 & 1 \\ k & -3 \end{pmatrix},$$

where k is a constant. Given that M has a repeated eigenvalue, determine (a)(i) k , (a)(ii) the eigenvalue, and (b) a Cartesian equation of the invariant line under T .

Model solution

(a)(i) A repeated eigenvalue means that the characteristic equation has one repeated root. We first form $\det(M - \lambda I) = 0$, then make its discriminant zero.

$$\begin{aligned} 0 &= \det \begin{pmatrix} 5 - \lambda & 1 \\ k & -3 - \lambda \end{pmatrix} \\ &= (5 - \lambda)(-3 - \lambda) - k \\ &= \lambda^2 - 2\lambda - 15 - k. \end{aligned}$$

$$(-2)^2 - 4(1)(-15 - k) = 0 \implies 4 + 60 + 4k = 0 \implies k = -16.$$

$$k = -16.$$

(4)

(a)(ii) Substituting this value makes the characteristic equation a perfect square, so its root is the repeated eigenvalue.

$$\lambda^2 - 2\lambda - 15 - (-16) = 0 \implies \lambda^2 - 2\lambda + 1 = 0 \implies (\lambda - 1)^2 = 0.$$

The repeated eigenvalue is $\lambda = 1$.

(2)

(b) An invariant line through the origin is spanned by an eigenvector. Use the repeated eigenvalue in $(M - I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to find its direction.

$$\begin{pmatrix} 5 & 1 \\ -16 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{pmatrix} 4 & 1 \\ -16 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$4x + y = 0 \implies y = -4x.$$

$$y = -4x.$$

(2)

Question 3

Question 3

A complex number z is represented by P on an Argand diagram. Given that

$$\arg\left(\frac{z-4-i}{z-2-7i}\right) = \frac{\pi}{2},$$

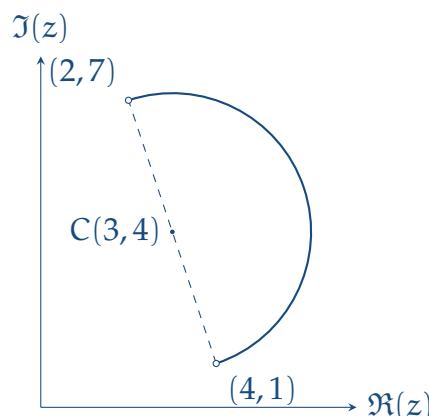
(a) sketch the locus of P as z varies; (b) determine the exact maximum possible value of $|z|$.

Model solution

(a) The numerator and denominator are the vectors from $4+i$ and $2+7i$ respectively to P . Their arguments differ by $\pi/2$, so these two vectors are perpendicular. Therefore the angle subtended by the fixed points $(4,1)$ and $(2,7)$ at P is a right angle. By the circle theorem, P lies on the circle with these points as endpoints of a diameter. The sign of the directed angle selects the right-hand semicircle.

Locus: the right-hand semicircle with diameter joining $(4,1)$ to $(2,7)$,

excluding the endpoints $(4,1)$ and $(2,7)$, where the argument is undefined.



The required sketch is this right-hand semicircle.

(2)

(b) The greatest modulus is the greatest distance from the origin. The circle has centre equal to the midpoint of its diameter, so we can use the centre--radius geometry to find the farthest point.

$$C = \left(\frac{4+2}{2}, \frac{1+7}{2}\right) = (3,4),$$

$$r = \sqrt{(4-3)^2 + (1-4)^2} = \sqrt{10}, \quad OC = \sqrt{3^2 + 4^2} = 5.$$

The point on this right-hand arc farthest from the origin is on the ray from the origin through C , so its distance is $OC + r$.

$$|z|_{\max} = 5 + \sqrt{10}.$$

$$5 + \sqrt{10}.$$

(5)

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Question 4

Question 4

A student borrows £1000. Each month, 0.5% interest is added and then £50 is repaid. The model is

$$U_n = 1.005U_{n-1} - A, \quad U_0 = 1000.$$

(a)(i) State A . (ii) Explain 1.005 in context. (b) Solve the recurrence. (c) Hence determine the number of months required to repay the loan completely.

Model solution

(a)(i) The fixed repayment made after each interest calculation is £50, so it is the quantity subtracted in the recurrence.

$$A = 50.$$

(1)

(a)(ii) Monthly interest of 0.5% changes the outstanding balance to $100\% + 0.5\% = 100.5\%$ of its previous value.

1.005 is the monthly interest multiplier applied to the amount owed before the £50 repayment.

(1)

(b) This is a first-order linear recurrence. Its homogeneous part has multiplier 1.005, so write the general form as a geometric complementary solution plus a constant particular solution:

$$U_n = c(1.005)^n + \ell.$$

For a constant balance ℓ , substituting into the recurrence gives

$$\ell = 1.005\ell - 50 \implies 0.005\ell = 50 \implies \ell = 10000.$$

Now use the initial debt to determine c .

$$1000 = c(1.005)^0 + 10000 \implies c = -9000.$$

$$U_n = 10000 - 9000(1.005)^n.$$

$$U_n = 10000 - 9000(1.005)^n.$$

(5)

(c) The model reaches repayment when the amount owed becomes zero. Solve the expression from part (b) for this threshold; because repayments occur monthly, the answer must be the next whole month.

$$10000 - 9000(1.005)^n = 0 \implies (1.005)^n = \frac{10}{9},$$

$$n = \frac{\log(10/9)}{\log(1.005)} \approx 21.1.$$

After 21 full months the balance is still positive, so the loan is cleared during the 22nd payment.

22 months.

(2)

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Question 5

Question 5

- (i) Using modulo arithmetic, show that 214^6 is divisible by 8.
 (ii) The seven-digit number $a5b8ab0$ is divisible by 11. (a) Determine a . Given that it is also divisible by 3, (b) determine the possible values of b .

Model solution

(i) To prove divisibility by 8, it is enough to show that the number is congruent to 0 modulo 8. Since 214 has a simple remainder on division by 8, reduce the base first.

$$214 \equiv 6 \pmod{8},$$

$$214^6 \equiv 6^6 = (6^2)^3 \equiv 4^3 = 64 \equiv 0 \pmod{8}.$$

Thus the remainder on division by 8 is zero.

214^6 is divisible by 8.

(3)

(ii)(a) The divisibility test for 11 says that the alternating sum of the digits must be a multiple of 11. Starting from the left gives

$$a - 5 + b - 8 + a - b + 0 = 2a - 13.$$

As a is a digit, $2a - 13$ lies from -13 to 5 ; the only possible multiple of 11 in this range that gives an integer digit is -11 .

$$2a - 13 = -11 \implies a = 1.$$

$a = 1$.

(2)

(ii)(b) A number is divisible by 3 exactly when its digit sum is divisible by 3. Substitute $a = 1$ into the digit sum.

$$1 + 5 + b + 8 + 1 + b + 0 = 15 + 2b.$$

$$15 + 2b \equiv 0 \pmod{3} \implies 2b \equiv 0 \pmod{3} \implies b \equiv 0 \pmod{3}.$$

The digit values satisfying this congruence are

$b = 0, 3, 6$ or 9 .

(3)