



Edexcel AS Further Mathematics

8FM0/22 - Further Pure Mathematics 2

October 2020

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2020 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

The set

$$G = \{1, 3, 7, 9, 11, 13, 17, 19\}$$

under multiplication modulo 20 forms a group.

- Find the inverse of each element of G .
- Find the order of each element of G .
- Find a subgroup of G of order 4.
- Explain how the subgroup found in part (c) satisfies Lagrange's theorem.

Model solution

(a) An inverse of g is an element $h \in G$ for which $gh \equiv 1 \pmod{20}$. Checking products modulo 20, four elements are self-inverse:

$$1^2 \equiv 9^2 \equiv 11^2 \equiv 19^2 \equiv 1 \pmod{20}.$$

The remaining elements pair up as follows:

$$3 \cdot 7 \equiv 1 \pmod{20}, \quad 13 \cdot 17 \equiv 1 \pmod{20}.$$

$$1^{-1} = 1, \quad 3^{-1} = 7, \quad 7^{-1} = 3, \quad 9^{-1} = 9, \quad 11^{-1} = 11, \quad 13^{-1} = 17, \quad 17^{-1} = 13, \quad 19^{-1} = 19.$$

(3)

(b) The order is the least positive power that gives the identity 1. The self-inverse elements other than 1 have order 2; the inverse pairs need one further squaring.

$$3^2 \equiv 9, \quad 3^4 \equiv 9^2 \equiv 1 \pmod{20}, \quad 7^2 \equiv 9, \quad 7^4 \equiv 1 \pmod{20},$$

$$13^2 \equiv 9, \quad 13^4 \equiv 1 \pmod{20}, \quad 17^2 \equiv 9, \quad 17^4 \equiv 1 \pmod{20}.$$

The orders of 1, 3, 7, 9, 11, 13, 17, 19 are respectively 1, 4, 4, 2, 2, 4, 4, 2.

(3)

(c) A subgroup must contain the identity and be closed under the group operation. One suitable set is the elements generated by 3.

$$\langle 3 \rangle = \{1, 3, 3^2, 3^3\} = \{1, 3, 9, 7\}.$$

$\{1, 3, 7, 9\}$ is a subgroup of order 4.

(1)

(d) Lagrange's theorem says that the order of a subgroup divides the order of its finite group. Here $|G| = 8$ and the subgroup has order 4.

Since $4 \mid 8$, Lagrange's theorem is satisfied.

(1)

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Question 2

Question 2

The highest common factor of 963 and 657 is c .

- (a) Use the Euclidean algorithm to find c .
 (b) Hence find integers a and b such that $963a + 657b = c$.

Model solution

(a) The Euclidean algorithm repeatedly divides, replacing a pair by the divisor and remainder. The last non-zero remainder is the highest common factor.

$$963 = 1(657) + 306,$$

$$657 = 2(306) + 45,$$

$$306 = 6(45) + 36,$$

$$45 = 1(36) + 9,$$

$$36 = 4(9) + 0.$$

$$c = \text{HCF}(963, 657) = 9.$$

(3)

(b) "Hence" signals that the Euclidean algorithm should now be worked backwards, expressing the final non-zero remainder as a linear combination of the original numbers.

$$\begin{aligned} 9 &= 45 - 36 \\ &= 45 - (306 - 6 \cdot 45) \\ &= 7 \cdot 45 - 306 \\ &= 7(657 - 2 \cdot 306) - 306 \\ &= 7 \cdot 657 - 15 \cdot 306 \\ &= 7 \cdot 657 - 15(963 - 657) \\ &= -15 \cdot 963 + 22 \cdot 657. \end{aligned}$$

$$a = -15 \text{ and } b = 22.$$

(3)

Question 3

Question 3

(i) $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

(a) Show that the characteristic equation for A is $\lambda^2 - 5\lambda + 6 = 0$.

(b) Use the Cayley–Hamilton theorem to find integers p and q such that $A^3 = pA + qI$.

(ii) The 2×2 matrix M has eigenvalues $-1 + i$ and $-1 - i$, with eigenvectors $\begin{pmatrix} 1 \\ 2 - i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 + i \end{pmatrix}$ respectively. Find M .

Model solution

(i)(a) *Eigenvalues satisfy $\det(A - \lambda I) = 0$, so this determinant produces the required characteristic equation.*

$$\begin{aligned} 0 &= \det \begin{pmatrix} 1 - \lambda & -2 \\ 1 & 4 - \lambda \end{pmatrix} \\ &= (1 - \lambda)(4 - \lambda) + 2 \\ &= \lambda^2 - 5\lambda + 6. \end{aligned}$$

$$\lambda^2 - 5\lambda + 6 = 0, \text{ as required.}$$

(2)

(i)(b) *Cayley–Hamilton allows the characteristic polynomial to be applied to the matrix itself. It gives a relation for A^2 , which can then be multiplied by A to obtain A^3 .*

$$A^2 - 5A + 6I = 0, \quad \text{so} \quad A^2 = 5A - 6I.$$

$$\begin{aligned} A^3 &= A(A^2) = A(5A - 6I) \\ &= 5A^2 - 6A \\ &= 5(5A - 6I) - 6A \\ &= 19A - 30I. \end{aligned}$$

$$p = 19 \text{ and } q = -30.$$

(3)

(ii) *Write the unknown matrix in general form. Applying $Mv = \lambda v$ to each eigenpair gives enough simultaneous equations to determine its four entries.*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$a + (2 - i)b = -1 + i, \quad c + (2 - i)d = -1 + 3i,$$

$$a + (2 + i)b = -1 - i, \quad c + (2 + i)d = -1 - 3i.$$

Subtracting the first equation in each column from the second eliminates a or c :

$$2ib = -2i \Rightarrow b = -1, \quad 2id = -6i \Rightarrow d = -3.$$

$$b = -1 \Rightarrow a = -1 + i - (2 - i)(-1) = 1,$$

$$d = -3 \Rightarrow c = -1 + 3i - (2 - i)(-3) = 5.$$

$$M = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}.$$

(5)

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Question 4

Question 4

Sam borrows \$10 000. The bank charges 5% annual interest on the amount not yet repaid. Immediately after interest is added at the end of each year, Sam pays a fixed amount \$F. If \$A_n, where A_n ≥ 0, is the amount owed at the start of year n:

- (a) write an expression for A_{n+1} in terms of A_n and F;
 (b) prove by induction that, for n ≥ 1,

$$A_n = (10\,000 - 20F)1.05^{n-1} + 20F;$$

- (c) find the smallest F for which the loan is repaid by the start of year 16.

Model solution

(a) From the start of one year to the next, the debt is first multiplied by 1.05 for interest, then the repayment is subtracted.

$$A_{n+1} = 1.05A_n - F.$$

(1)

(b) Induction is appropriate because the formula is claimed for every integer n ≥ 1. First check the starting year. At the start of year 1, no repayments have yet been made, so A₁ = 10 000.

$$(10\,000 - 20F)1.05^{1-1} + 20F = 10\,000 - 20F + 20F = 10\,000 = A_1.$$

The statement is therefore true for n = 1. Assume it is true for n = k, so that

$$A_k = (10\,000 - 20F)1.05^{k-1} + 20F.$$

Use the recurrence from part (a) to move from year k to year k + 1:

$$\begin{aligned} A_{k+1} &= 1.05A_k - F \\ &= 1.05((10\,000 - 20F)1.05^{k-1} + 20F) - F \\ &= (10\,000 - 20F)1.05^k + 21F - F \\ &= (10\,000 - 20F)1.05^k + 20F. \end{aligned}$$

This is exactly the stated formula with n = k + 1. Since the base case and inductive step both hold, the formula is true for all n ≥ 1.

$$A_n = (10\,000 - 20F)1.05^{n-1} + 20F \text{ for all } n \geq 1.$$

(5)

(c) Being repaid by the start of year 16 means A₁₆ ≤ 0. Substitute n = 16 into the proven formula and solve the inequality for the repayment.

$$\begin{aligned} (10\,000 - 20F)1.05^{15} + 20F &\leq 0 \\ 10\,000(1.05^{15}) &\leq 20F(1.05^{15} - 1) \\ F &\geq \frac{10\,000(1.05^{15})}{20(1.05^{15} - 1)} \\ &= 963.422876\dots \end{aligned}$$

A payment is in whole pennies, so it must be rounded up rather than to the nearest penny; \$963.42 would be slightly too small.

The smallest fixed annual payment is \$963.43.

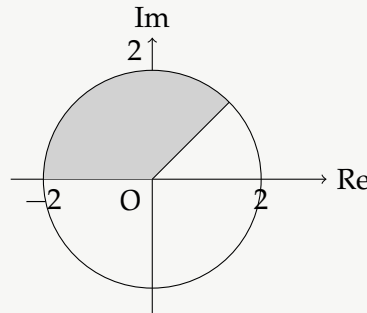
(4)

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Question 5

Question 5

Figure 1 is an Argand diagram. The shaded region is the part of the circle $|z| \leq 2$ between the rays $\arg z = \pi/4$ and $\arg z = \pi$, including its boundaries.



The set of points in the shaded region is

$$A = \{z : p \leq \arg(z) \leq q\} \cap \{z : |z| \leq r\},$$

where p, q, r are positive constants.

(a) Write down p, q, r .

Given $w = -2\sqrt{3} + 2i$ and $z \in A$,

(b) find the maximum value of $|w - z|^2$, in exact simplified form.

Model solution

(a) The two radial boundaries have arguments $\pi/4$ and π , and the curved boundary is a circle of radius 2.

$$p = \pi/4, \quad q = \pi, \quad r = 2.$$

(2)

(b) The point $w = -2\sqrt{3} + 2i$ is outside the circle, in quadrant II. To maximise the distance to w , choose the boundary point of the allowed sector furthest from it. This is where the ray $y = x$ meets $|z| = 2$ in quadrant I.

$$z = \sqrt{2} + \sqrt{2}i.$$

The modulus of w is 4. Since $\arg(w) = 5\pi/6$ and $\arg(z) = \pi/4$, the angle between Ow and Oz is $5\pi/6 - \pi/4 = 7\pi/12$. Applying the cosine rule to triangle Owz ,

$$\begin{aligned} |w - z|^2 &= 4^2 + 2^2 - 2(4)(2) \cos\left(\frac{7\pi}{12}\right) \\ &= 20 - 16 \cos\left(\frac{7\pi}{12}\right) \\ &= 20 - 16 \left(\frac{\sqrt{2} - \sqrt{6}}{4}\right) \\ &= 20 - 4\sqrt{2} + 4\sqrt{6}. \end{aligned}$$

$$\max |w - z|^2 = 20 - 4\sqrt{2} + 4\sqrt{6}.$$

(4)

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