



Edexcel AS Further Mathematics

8FM0/28 - Decision Mathematics 2

Thursday 17 May 2018

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2018 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Four workers, A, B, C and D, are to be assigned to tasks P, Q, R and S, one task to each worker and one worker to each task. The completion times, in hours, are

	P	Q	R	S
A	7.5	3.5	8	9.5
B	5	2	7	7.5
C	4	3.5	3.5	8
D	6	5	3.5	4

Reducing rows first, use the Hungarian algorithm to obtain an allocation which minimises the total time. Explain the method and show the table after each stage.

Model solution

Subtracting the smallest entry in each row gives

	P	Q	R	S
A	4	0	4.5	6
B	3	0	5	5.5
C	0.5	0	0	4.5
D	2.5	1.5	0	0.5

The column minima are 0.5, 0, 0, 0.5, so column reduction gives

	P	Q	R	S
A	3.5	0	4.5	5.5
B	2.5	0	5	5
C	0	0	0	4
D	2	1.5	0	0

The zeros cannot yet be covered by four independent selections. Cover them with three lines: rows C and D, and column Q. The least uncovered number is 2.5. Subtract 2.5 from each uncovered entry, add 2.5 at each intersection of two covering lines, and leave singly covered entries unchanged. This gives

	P	Q	R	S
A	1	0	2	3
B	0	0	2.5	2.5
C	0	2.5	0	4
D	2	4	0	0

There are now four independent zeros: $A \rightarrow Q$, $B \rightarrow P$, $C \rightarrow R$, and $D \rightarrow S$. Thus this is an optimal allocation.

The minimum total time is $3.5 + 5 + 3.5 + 4 = 16$ hours.

The Hungarian algorithm preserves the optimal assignment while creating a set of independent zeros.

A--Q, B--P, C--R and D--S; minimum total time 16 hours.

(5)

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Question 2

Question 2

(a) Explain what is meant by a zero-sum game.

In each quiz round 10 points are available. The entries below are the expected points gained by Team A.

	Paul	Qaasim	Rashid
Mischa	5	6	3
Noel	4	1	7
Olive	4	5	8

(b) State Team B's expected score if Team A chooses Noel and Team B chooses Rashid.

(c) Explain why subtracting 5 from every entry models the quiz as a zero-sum game.

(d) For the resulting zero-sum game, (i) find the play-safe strategies and (ii) explain why the game is not stable.

Olive then becomes unavailable. Team A chooses Mischa with probability p and Noel with probability $1 - p$.

(e) Use a graphical method to find the optimal value of p , and hence Team A's best strategy.

(f) For this value of p , find the expected points per round awarded to (i) Team A and (ii) Team B.

Model solution

(a) In a zero-sum game, one player's gain is exactly balanced by the other player's loss; the two pay-offs sum to zero. (1)

(b) Team A receives 7 points, so Team B receives $10 - 7 = 3$ points. (1)

(c) If A receives x points, B receives $10 - x$ points. After subtracting 5, the pay-offs are $x - 5$ and $(10 - x) - 5 = 5 - x$, whose sum is zero. (1)

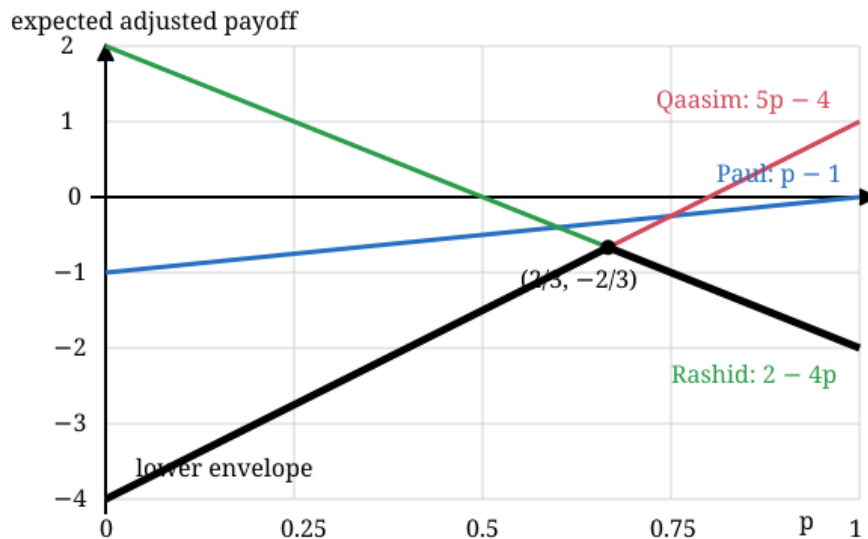
(d) The adjusted payoff matrix for A is

	Paul	Qaasim	Rashid
Mischa	0	1	-2
Noel	-1	-4	2
Olive	-1	0	3

The row minima are $-2, -4, -1$, so the maximin is -1 , attained by Olive. The column maxima are $0, 1, 3$, so the minimax is 0 , attained by Paul. Hence the play-safe strategies are Olive for Team A and Paul for Team B. Since $-1 \neq 0$, the maximin and minimax differ, so there is no saddle point and the game is not stable. (4)

(e) With Olive unavailable, use the adjusted pay-offs. For B's three pure strategies, A's expected pay-off is respectively

B's choice	A's expected adjusted pay-off
Paul	$0p + (-1)(1 - p) = p - 1$
Qaasim	$1p + (-4)(1 - p) = 5p - 4$
Rashid	$-2p + 2(1 - p) = 2 - 4p$



On a graph of these three straight lines against p , Team B chooses the lowest line and Team A chooses p to maximise that lower envelope. Its maximum occurs where the two limiting lines intersect:

$$5p - 4 = 2 - 4p \Rightarrow 9p = 6 \Rightarrow p = \frac{2}{3}.$$

At $p = \frac{2}{3}$, the Paul line has value $-\frac{1}{3}$, while the other two both have value $-\frac{2}{3}$, confirming that the lower envelope is $-\frac{2}{3}$. Therefore Team A should choose Mischa with probability $\frac{2}{3}$ and Noel with probability $\frac{1}{3}$. (6)

(f) The adjusted value to A is $-\frac{2}{3}$. Adding back 5 points gives A's expected score

$$5 - \frac{2}{3} = \frac{13}{3}.$$

As the round contains 10 points, B's expected score is

$$10 - \frac{13}{3} = \frac{17}{3}.$$

(2)

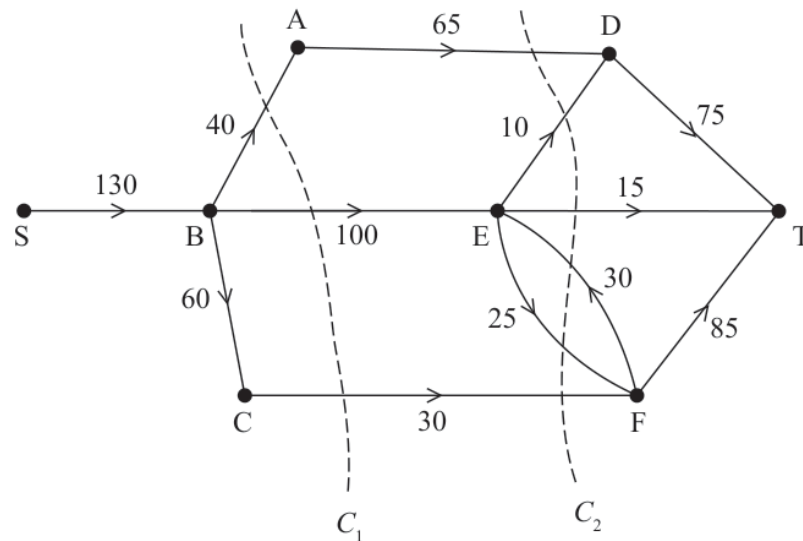
Subtracting 5 merely re-centres the pay-offs: it does not change either team's optimal choices.

(a) Gains and losses balance. (b) 3. (d) A: Olive; B: Paul; not stable. (e) $p = \frac{2}{3}$, so Mischa $\frac{2}{3}$, Noel $\frac{1}{3}$. (f) A: $\frac{13}{3}$, B: $\frac{17}{3}$ points per round.

Question 3

Question 3

Figure 1 is a directed flow network from source S to sink T , with capacities in litres per minute:



arc	SB	BA	BC	BE	AD	CF	ED	EF	FE	DT	ET	FT
capacity	130	40	60	100	65	30	10	25	30	75	15	85

The diagram shows cuts C_1 , crossing arcs BA , BE , CF , and C_2 , crossing arcs AD , ED , ET , EF , FE as drawn.

- Find the capacities of (i) C_1 and (ii) C_2 .
 - Using only these two cuts, state what can be deduced about the maximum possible flow.
 - Show how a flow of 120 litres per minute can be achieved.
 - Prove that 120 litres per minute is the maximum possible flow.
- A new pipe SA , of capacity x litres per minute, is added.
- Find the maximum possible flow in the new system, in terms of x where necessary.

Model solution

(a) Cut C_1 has capacity

$$40 + 100 + 30 = 170 \text{ litres per minute.}$$

For C_2 , the directed arcs crossing from the source side to the sink side have total capacity

$$65 + 10 + 15 + 25 + 30 = 145 \text{ litres per minute.}$$

(2)

(b) A flow cannot exceed the capacity of any cut. Therefore the maximum possible flow is at most 145 litres per minute.

(1)

(c) One valid flow is

arc	SB	BA	BC	BE	AD	CF	ED	EF	DT	ET	FT
flow	120	40	30	50	40	30	10	25	50	15	55

with zero flow on FE. At each intermediate vertex, inflow equals outflow; for example, the 50 entering E leaves as $10 + 25 + 15$, and the 55 entering F leaves to T. The flow into T is $50 + 15 + 55 = 120$ litres per minute. (2)

(d) Consider the cut with source side $\{S, B, C, E\}$ and sink side $\{A, D, F, T\}$. The forward crossing arcs are BA, CF, ED, ET and EF, so its capacity is

$$40 + 30 + 10 + 25 + 15 = 120 \text{ litres per minute.}$$

The flow in part (c) has value 120, equal to this cut capacity. By the maximum-flow/minimum-cut theorem, the flow is maximal. (2)

(e) The new arc permits a further x litres per minute to reach A and then D, until the existing cut of capacity 145 becomes limiting. Thus

$$\text{maximum flow} = \begin{cases} 120 + x, & 0 < x \leq 25, \\ 145, & x > 25. \end{cases}$$

(3)

For a directed cut, only arcs crossing from the source side to the sink side contribute to its capacity.

(a) 170, 145 L min⁻¹. (b) At most 145 L min⁻¹. (e) $120 + x$ for $0 < x \leq 25$, otherwise 145 L min⁻¹.

Question 4

Question 4

A village's expected population is modelled by

$$u_{n+1} = 1.02u_n + 50,$$

where u_n is the expected population n years from now. Its natural growth rate is $r\%$ per year and N people are expected to move in each year.

(a) State (i) r , and (ii) N .

Given $u_1 = 560$:

(b) solve the recurrence relation for u_n ;

(c) hence determine algebraically the first number of years from now for which the model predicts a population greater than 3000.

Model solution

(a) Since the multiplier is 1.02, the natural growth rate is 2% per year. The constant addition is 50 people per year. Hence $r = 2$ and $N = 50$. (2)

(b) The complementary solution of $u_{n+1} = 1.02u_n$ is $A(1.02)^n$. Seek a constant particular solution $u_n = \lambda$. Substitution gives

$$\lambda = 1.02\lambda + 50, \quad -0.02\lambda = 50, \quad \lambda = -2500.$$

Therefore

$$u_n = A(1.02)^n - 2500.$$

Using $u_1 = 560$,

$$560 = 1.02A - 2500 \Rightarrow A = 3000.$$

Thus

$$u_n = 3000(1.02)^n - 2500.$$

(5)

(c)

$$3000(1.02)^n - 2500 > 3000 \Rightarrow (1.02)^n > \frac{11}{6}.$$

Taking logarithms,

$$n > \frac{\log(11/6)}{\log(1.02)} = 30.6088 \dots$$

Since n is an integer number of years, the population first exceeds 3000 after 31 years. (3)

The particular solution is negative because a population below the equilibrium offset would be needed to balance the constant inward migration in the recurrence equation.

(a) $r = 2$, $N = 50$. (b) $u_n = 3000(1.02)^n - 2500$. (c) 31 years.