



Edexcel AS Further Mathematics

8FM0/28 - Decision Mathematics 2

Tuesday 24 May 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Four workers A, B, C and D are to be assigned one each to tasks P, Q, R and S. Worker C cannot do Q and worker D cannot do S. Their completion times, in minutes, are

	P	Q	R	S
A	54	48	51	52
B	55	51	53	58
C	52	—	53	54
D	67	63	68	—

- (a) Modify the table so that the Hungarian algorithm may be used.
 (b) Reducing rows first, use the Hungarian algorithm to find an allocation with minimum total time. Explain the row and column reductions and how optimality is checked at each stage.

Model solution

(a) The Hungarian algorithm needs a numerical entry in every cell. An impossible assignment must therefore be given a time too large to be selected. Since every permitted time is at most 68, using 100 safely prevents either forbidden assignment from appearing in a minimum allocation.

	P	Q	R	S
A	54	48	51	52
B	55	51	53	58
C	52	100	53	54
D	67	63	68	100

Replace CQ and DS by 100.

(1)

(b) Row reduction is appropriate first because subtracting the same amount from every entry in a row does not alter which allocation is cheapest. Subtract the row minima 48, 51, 52, 63 from rows A, B, C and D respectively:

	P	Q	R	S
A	6	0	3	4
B	4	0	2	7
C	0	48	1	2
D	4	0	5	37

The column minima are 0, 0, 1, 2. Subtracting these from columns P, Q, R, S gives

	P	Q	R	S
A	6	0	2	2
B	4	0	1	5
C	0	48	0	0
D	4	0	4	35

All the zeros can be covered by just two lines: row C and column Q. As fewer than four lines cover the zeros, the table is not optimal. The smallest uncovered entry is 1. Subtract

it from every uncovered entry and add it to the entry covered twice; entries covered once are unchanged.

	P	Q	R	S
A	5	0	1	1
B	3	0	0	4
C	0	49	0	0
D	3	0	3	34

Now three lines cover all zeros, for example row C and columns Q and R. This is still fewer than four, so repeat the adjustment. The smallest uncovered entry is again 1:

	P	Q	R	S
A	4	0	1	0
B	2	0	0	3
C	0	50	1	0
D	2	0	3	33

Four lines are now required to cover all the zeros, so the table is optimal. Choose independent zeros, one in each row and column: $A \rightarrow S$, $B \rightarrow R$, $C \rightarrow P$, $D \rightarrow Q$. These correspond to permitted entries of the original table.

$$\text{Minimum total time} = 52 + 53 + 52 + 63 = 220 \text{ minutes.}$$

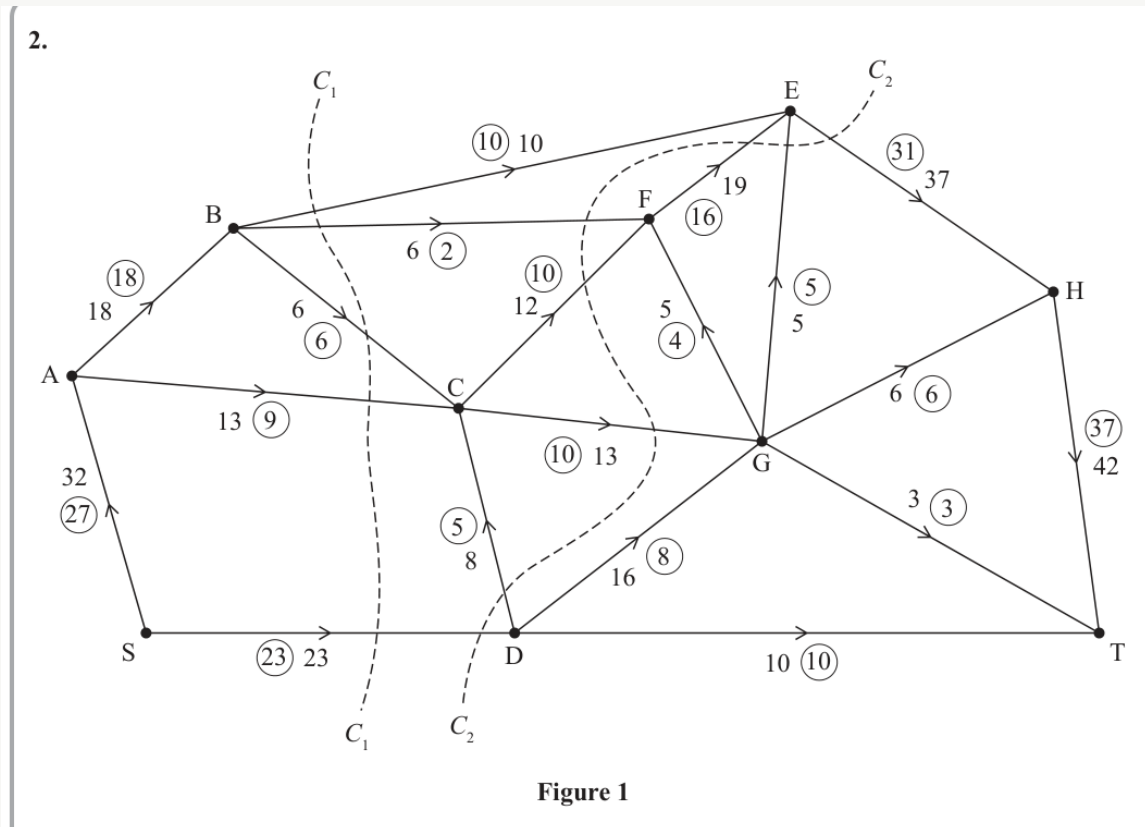
$A \rightarrow S$, $B \rightarrow R$, $C \rightarrow P$, $D \rightarrow Q$, with minimum total time 220 minutes.

(6)

Question 2

Question 2

Figure 1 is a capacitated directed network from S to T. Capacities are written on the arcs and circled values give a feasible flow.



- State the value of the flow.
 - List the eight saturated arcs.
 - Explain why EH can never be full to capacity.
 - Find the capacities of cuts C_1 and C_2 .
 - Write down a flow-augmenting route that increases the flow by three units.
- Given that the flow is increased by three units, (f) prove that the new flow is maximal.

Model solution

(a) The value of a flow is the total amount leaving the source (equivalently, entering the sink). Adding the circled flows out of S gives 50.

The flow has value 50.

(1)

(b) An arc is saturated when its flow equals its capacity. Comparing each circled flow with the capacity on that arc gives

AB, BC, SD, BE, DT, GE, GH and GT.

(1)

(c) To fill EH, enough flow must first reach E. The total possible inflow to E is bounded by the capacities of its three incoming arcs:

$$\text{cap}(\text{BE}) + \text{cap}(\text{FE}) + \text{cap}(\text{GE}) = 10 + 19 + 5 = 34.$$

But EH has capacity 37, which is greater than this maximum possible inflow.

EH can carry at most 34 units, so it can never reach its capacity 37.

(1)

(d) The capacity of a cut is the sum of capacities of the arcs crossing it from the source side to the sink side. Reading the arcs crossed by each dashed cut from the diagram,

$$\text{cap}(C_1) = 10 + 6 + 6 + 13 + 23 = 58,$$

$$\text{cap}(C_2) = 37 + 6 + 12 + 13 + 23 = 91.$$

(i) 58 (ii) 91.

(2)

(e) A flow-augmenting route must use residual capacity throughout. The route below has residual capacity 3, so it permits an increase of three units.

$S \rightarrow A \rightarrow C \rightarrow B \rightarrow F \rightarrow E \rightarrow H \rightarrow T.$

(1)

(f) Increasing the original flow by three gives a flow of value 53. To prove it is maximal, use the max-flow min-cut theorem: if a cut has capacity 53, no flow can exceed 53. After augmentation, the arcs BE, FE, GE, GH, GT, DT form such a cut and are all saturated.

$$\text{cap}(\text{BE}, \text{FE}, \text{GE}, \text{GH}, \text{GT}, \text{DT}) = 10 + 19 + 5 + 6 + 3 + 10 = 53.$$

The flow has value 53, equal to the capacity of this cut. Therefore the maximum flow equals the minimum-cut capacity.

The new flow is maximal, with value 53.

(3)

Question 3

Question 3

Terry and June play a zero-sum game. The entries are Terry's scores.

	June X	June Y
Terry A	1	4
Terry B	-2	6
Terry C	-1	5
Terry D	8	-4

- (a) Explain "zero-sum".
 (b) Verify that there is no stable solution.
 (c) Write down June's pay-off matrix.
 (d)(i) Find June's best strategy, defining variables used. (ii) State the value of the game to Terry. Let Terry play A with probability t .
 (e) Find Terry's best strategy by writing down a linear equation in t .

Model solution

(a) In a zero-sum game, the two players' scores always add to zero. Thus any number of points gained by Terry is lost by June, and vice versa.

One player's gain is exactly the other player's loss.

(1)

(b) A stable solution would require the maximin and minimax values to be equal. Terry first considers the worst score in each row:

row minima: 1, -2, -1, -4, maximin = 1.

June considers the largest score Terry could obtain in each column:

column maxima: 8, 6, minimax = 6.

Since $1 \neq 6$, there is no stable solution.

(2)

(c) June's pay-off is the negative of Terry's score in every outcome.

	Terry A	Terry B	Terry C	Terry D
June X	-1	2	1	-8
June Y	-4	-6	-5	4

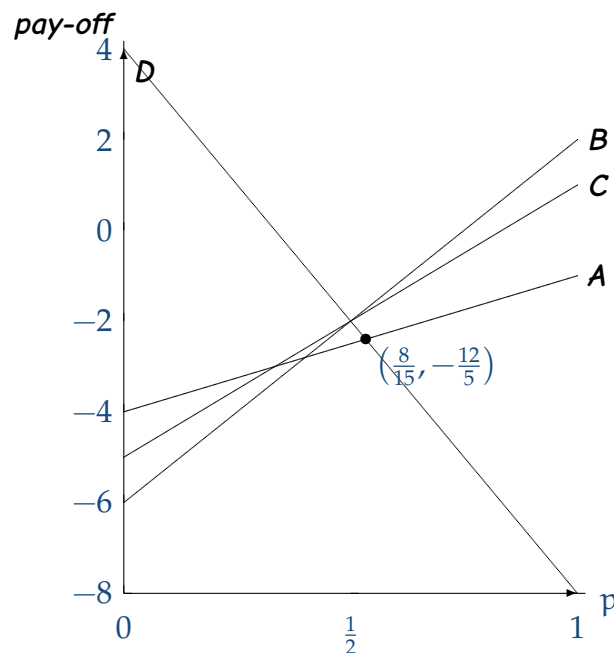
June's pay-off matrix is $\begin{pmatrix} -1 & 2 & 1 & -8 \\ -4 & -6 & -5 & 4 \end{pmatrix}$, with rows X, Y and columns A, B, C, D.

(1)

(d)(i) Let June choose X with probability p , and hence Y with probability $1 - p$. For each possible pure strategy of Terry, calculate June's expected pay-off. These straight-line expressions allow June to choose p so that her worst possible pay-off is as large as possible.

Terry's choice	June's expected pay-off
A	$(-1)p + (-4)(1 - p) = 3p - 4$
B	$2p + (-6)(1 - p) = 8p - 6$
C	$p + (-5)(1 - p) = 6p - 5$
D	$(-8)p + 4(1 - p) = 4 - 12p$

Plotting the four expected pay-offs for $0 \leq p \leq 1$ gives the following graph.



The graph shows that the lower envelope is maximised where the lines for A and D meet, at $p = \frac{8}{15}$. Using their equations to obtain this exact intersection makes Terry indifferent between those two choices:

$$3p - 4 = 4 - 12p \implies 15p = 8 \implies p = \frac{8}{15}.$$

At this value, the B and C pay-offs are above -2.4 , so A and D are indeed the limiting choices.

June plays X with probability $\frac{8}{15}$ and Y with probability $\frac{7}{15}$.

(6)

(d)(ii) Substitute $p = \frac{8}{15}$ into either limiting expected pay-off:

$$3\left(\frac{8}{15}\right) - 4 = -\frac{12}{5} = -2.4.$$

This is June's value, so the value to Terry is its negative.

The value of the game to Terry is $\frac{12}{5} = 2.4$ points.

(1)

(e) The work above shows that only A and D are used by Terry. If Terry uses A with probability t , he uses D with probability $1 - t$. June must be indifferent between X and Y at the optimal mix, so equate Terry's expected score for these two columns:

$$1t + 8(1 - t) = 4t + (-4)(1 - t).$$

$$8 - 7t = 4t - 4 \implies 12 = 11t \implies t = \frac{12}{11}.$$

Terry plays A with probability 0.8, D with probability 0.2, and never plays B or C.

(3)

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Question 4

Question 4

For $n \geq 0$, a sequence satisfies

$$u_{n+1} + 3u_n = n + k,$$

where $k \neq 0$, and $u_0 = 1$.

(a) Solve the recurrence relation, giving u_n in terms of k and n .

Given that u_n is a linear function of n , (b) find u_{100} .

Model solution

(a) First solve the associated homogeneous recurrence. Its auxiliary equation is $m + 3 = 0$, so $m = -3$, giving complementary solution $A(-3)^n$. The right-hand side is linear in n , so try a linear particular solution $u_n = an + b$.

$$a(n+1) + b + 3(an+b) = n + k.$$

Expand and compare the coefficients of n and the constants:

$$4an + a + 4b = n + k,$$

$$4a = 1, \quad a + 4b = k.$$

$$a = \frac{1}{4}, \quad b = \frac{k}{4} - \frac{1}{16} = \frac{4k-1}{16}.$$

Combining complementary and particular solutions gives

$$u_n = A(-3)^n + \frac{1}{4}n + \frac{4k-1}{16}.$$

Now use $u_0 = 1$ to determine A :

$$1 = A + \frac{4k-1}{16} \implies A = \frac{17-4k}{16}.$$

$$u_n = \frac{17-4k}{16}(-3)^n + \frac{1}{4}n + \frac{4k-1}{16}.$$

(7)

(b) A linear function of n cannot contain a non-zero $(-3)^n$ term. Therefore its coefficient in the answer to part (a) must be zero:

$$\frac{17-4k}{16} = 0 \implies k = \frac{17}{4}.$$

Substitution removes the exponential term and gives

$$u_n = \frac{1}{4}n + \frac{4(17/4)-1}{16} = \frac{1}{4}n + 1.$$

$$u_{100} = \frac{100}{4} + 1 = 26.$$

$$u_{100} = 26.$$

(3)

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