



Edexcel AS Further Mathematics

8FM0/28 - Decision Mathematics 2

November 2021

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the Pearson Edexcel Summer 2021 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Five workers A, B, C, D and E are to be assigned, at most one each, to four tasks P, Q, R and S. Their earnings (in pounds) are

	P	Q	R	S
A	55	56	58	57
B	60	61	—	64
C	59	60	62	63
D	64	66	71	69
E	65	68	72	66

Worker B cannot do R. Use the Hungarian algorithm to find the maximum total earnings. Explain the modification needed, then reduce rows first and explain the optimality tests.

Model solution

(a) The Hungarian algorithm is a minimisation algorithm on a square array. We therefore turn each earning into a cost by subtracting it from a common constant, choosing 72 (the largest earning). This reverses the preference: a larger earning gives a smaller cost. There are five workers but only four real tasks, so a dummy task X is added; assignment to X means that worker is unused. Give every X-entry cost 72, so it corresponds to earnings of £0. The forbidden assignment BR is given a very large cost, say 100, so it cannot be selected.

	P	Q	R	S	X
A	17	16	14	15	72
B	12	11	100	8	72
C	13	12	10	9	72
D	8	6	1	3	72
E	7	4	0	6	72

(2)

(b) First subtract the smallest entry in each row. The row reductions are respectively 14, 8, 9, 1, 0. Then subtract the smallest entry in each resulting column: 3, 2, 0, 0, 58 from columns P, Q, R, S, X. This preserves which complete assignment has minimum cost.

	P	Q	R	S	X
A	0	0	0	1	0
B	1	1	92	0	6
C	1	1	1	0	5
D	4	3	0	2	13
E	4	2	0	6	14

The zeros can be covered by only three lines, for example row A and columns R and S. Since fewer than five lines are needed, an assignment of five independent zeros is not yet possible. The least uncovered entry is 1. Subtract 1 from every uncovered entry and add 1

at intersections of two covering lines, leaving singly covered entries unchanged.

	P	Q	R	S	X
A	0	0	1	2	0
B	0	0	92	0	5
C	0	0	1	0	4
D	3	2	0	2	12
E	3	1	0	6	13

Now four lines are needed to cover all zeros, for example rows A, B and C together with column R, so we repeat the adjustment. The least uncovered entry is again 1.

	P	Q	R	S	X
A	0	0	2	2	0
B	0	0	93	0	5
C	0	0	2	0	4
D	2	1	0	1	11
E	2	0	0	5	12

This time five lines are required to cover all zeros, so the array is optimal. One set of independent zeros is $A \rightarrow X$, $B \rightarrow S$, $C \rightarrow P$, $D \rightarrow R$, $E \rightarrow Q$. The dummy assignment leaves A unassigned, so the real earnings are

$$64 + 59 + 71 + 68 = 262.$$

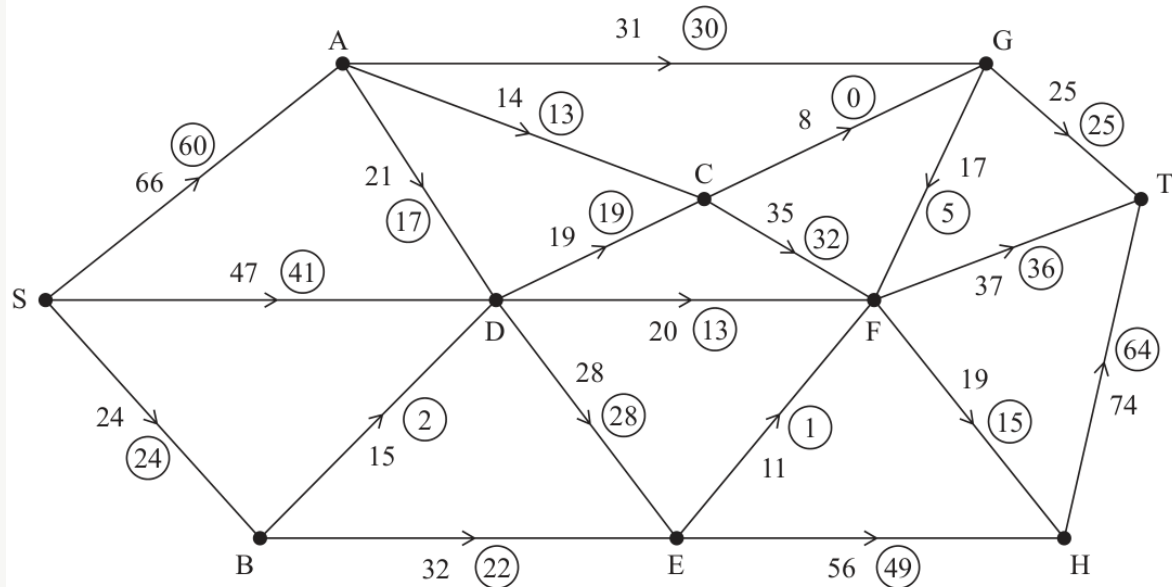
Maximum total earnings = \$262, with $B \rightarrow S$, $C \rightarrow P$, $D \rightarrow R$, $E \rightarrow Q$ (and A unused).

(7)

Question 2

Question 2

The capacitated directed network below has the circled initial flows. Answer the questions about its initial flow, cuts, labelling procedure and maximum flow.



Model solution

(a) The value of a flow is the total leaving the source S. The initial outgoing flows are 60, 41 and 24.

$$60 + 41 + 24 = 125.$$

Initial flow = 125.

(1)

(b) The capacity of a cut is found by adding the capacities, not the current flows, of every arc crossed from the source side to the sink side. Although the cut crosses GF, that arc points in the reverse direction and so does not contribute.

$$31 + 8 + 37 + 19 + 56 = 151.$$

Capacity of the cut = 151.

(1)

(c) In the labelling procedure, each arc has a forward residual capacity $c - f$ and a backward

residual capacity f . Thus the required labels are

arc	forward residual	backward residual
SD	$47 - 41 = 6$	41
BD	$15 - 2 = 13$	2
BE	$32 - 22 = 10$	22
GF	$17 - 5 = 12$	5

The forward arrow follows the original arc; the backward arrow is available because previously sent flow can be cancelled. (2)

(d) We use augmenting routes in the residual network. For each route, the amount added is the smallest residual capacity on that route, so no capacity is exceeded. A valid sequence is

flow-augmenting route	additional flow
$S \rightarrow A \rightarrow D \rightarrow F \rightarrow H \rightarrow T$	4
$S \rightarrow D \rightarrow B \rightarrow E \rightarrow H \rightarrow T$	2
$S \rightarrow A \rightarrow C \rightarrow F \rightarrow T$	1
$S \rightarrow A \rightarrow G \rightarrow F \rightarrow E \rightarrow H \rightarrow T$	1

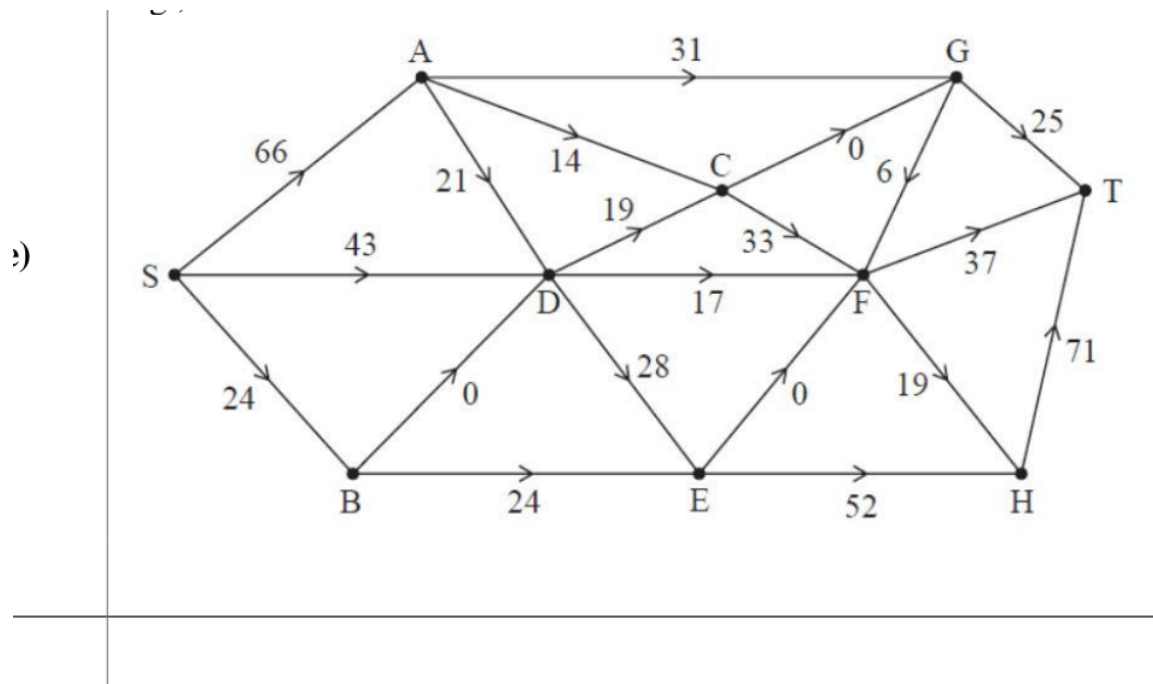
On the second route, $D \rightarrow B$ is a backward residual arc, so it reduces the existing flow on $B \rightarrow D$ by 2. The total increase is $4 + 2 + 1 + 1 = 8$, giving a flow value $125 + 8 = 133$.

A flow of value 133 is obtained.

(3)

(e) Updating the initial flows along the four routes gives the following maximum-flow pattern. Listing the flow on every arc makes the diagram unambiguous.

arc	SA	SD	SB	AG	AC	AD	DC	BD	BE	DE	CG	CF	DF	EF	GF	GT	FT	FH	EH
flow	66	43	24	31	14	21	19	0	24	28	0	33	17	0	6	25	37	19	52



Maximum-flow pattern as listed; its value is 133.

(1)

(f) To prove optimality, use the max-flow min-cut theorem. Take the cut through GT, FT, FH, EF, DE, BD, SB. Arcs EF and BD point from the sink side back to the source side, so they are crossed but do not contribute to the cut capacity. The forward arcs give

$$25 + 37 + 19 + 28 + 24 = 133.$$

Every flow is at most the capacity of any cut, but the constructed flow already has value 133. Therefore the maximum flow equals this cut capacity.

The flow of value 133 is maximal.

(3)

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Question 3

Question 3

For the zero-sum game with pay-offs to A

	X	Y
Q	4	-3
R	2	-1
S	-3	5
T	-1	3

verify that there is no stable solution. If B plays X with probability p , use a graphical method to find B's best strategy, the value to A, and A's best strategy.

Model solution

(a) A stable solution would occur when A's maximin equals B's minimax. The row minima are $-3, -1, -3, -1$, so A's maximin is -1 . The column maxima are $4, 5$, so B's minimax is 4 .

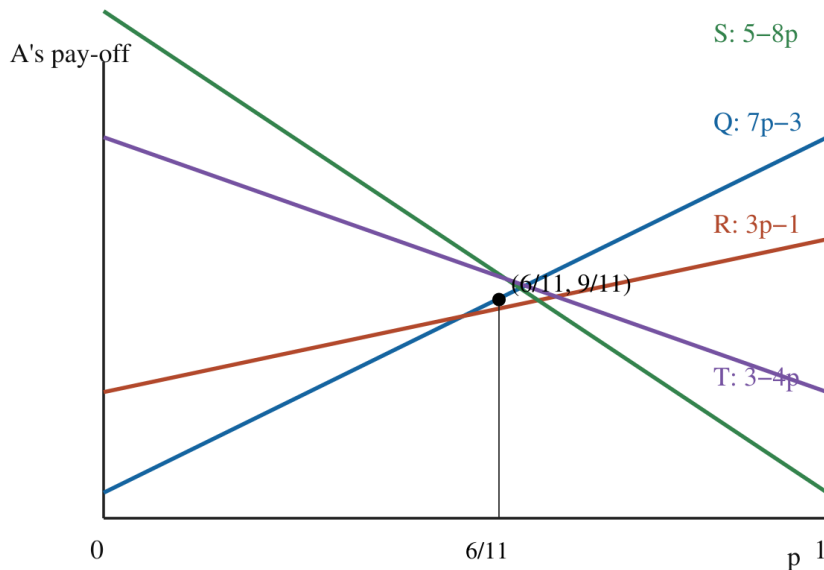
$$-1 \neq 4.$$

There is no stable solution.

(2)

(b) Because B has only two options, a graph in p is appropriate. For each pure option of A, find A's expected pay-off. A will choose the greatest of these values, so B minimises the upper envelope of the four lines.

A's option	A's expected pay-off
Q	$4p - 3(1 - p) = 7p - 3$
R	$2p - 1(1 - p) = 3p - 1$
S	$-3p + 5(1 - p) = 5 - 8p$
T	$-p + 3(1 - p) = 3 - 4p$



Plotting these four straight lines for $0 \leq p \leq 1$, the minimum point of their upper envelope is where the active lines for Q and T meet. Hence

$$7p - 3 = 3 - 4p \implies 11p = 6 \implies p = \frac{6}{11}.$$

B should play X with probability $\frac{6}{11}$ and Y with probability $\frac{5}{11}$.

(6)

(c) Substitute the optimal probability into either active line to obtain the value to A.

$$V(A) = 7\left(\frac{6}{11}\right) - 3 = \frac{9}{11}.$$

Value of the game to A = $\frac{9}{11}$.

(1)

(d) Only Q and T are active at the optimum, so let A play Q with probability q and T with probability $1 - q$. A chooses q to make B indifferent between X and Y. The expected pay-off to A when B chooses X is $4q - (1 - q)$, and when B chooses Y it is $-3q + 3(1 - q)$.

$$4q - (1 - q) = -3q + 3(1 - q) \implies 5q - 1 = 3 - 6q \implies q = \frac{4}{11}.$$

A should play Q with probability $\frac{4}{11}$, T with probability $\frac{7}{11}$, and never play R or S.

(2)

Question 4

Question 4

Sarah's mortgage balance satisfies

$$u_{n+1} = 1.0025u_n - x,$$

where x is her fixed monthly repayment. Find the monthly interest rate, solve the recurrence, and determine the least repayment that clears the mortgage in exactly 30 years.

Model solution

(a) The multiplier 1.0025 is $1 + r/100$, because monthly interest is added before the repayment.

$$1 + \frac{r}{100} = 1.0025 \implies r = 0.25.$$

$$r = 0.25\% \text{ per month.}$$

(1)

(b) This is a first-order linear recurrence with a constant repayment term. The complementary solution comes from the homogeneous recurrence $u_{n+1} = 1.0025u_n$, giving $A(1.0025)^n$. For a particular solution, try a constant $u_n = \lambda$, since the non-homogeneous term is constant.

$$\lambda = 1.0025\lambda - x \implies -0.0025\lambda = -x \implies \lambda = 400x.$$

Thus

$$u_n = A(1.0025)^n + 400x.$$

With the initial £155 000 balance represented by u_0 , substitute $n = 0$ to fix the otherwise arbitrary constant:

$$155000 = A + 400x \implies A = 155000 - 400x.$$

$$u_n = (155000 - 400x)(1.0025)^n + 400x.$$

(5)

(c) Thirty years is $30 \times 12 = 360$ monthly repayments. To be paid off exactly then, set $u_{360} = 0$ and solve for x .

$$0 = (155000 - 400x)(1.0025)^{360} + 400x,$$

$$x = \frac{155000(1.0025)^{360}}{400((1.0025)^{360} - 1)} = 653.49 \dots$$

A smaller repayment leaves a positive balance after 360 months, so this is the least possible monthly repayment.

$$x = \$653.49 \text{ per month.}$$

(3)