Maths with Matt

Physics and maths resources

OCR A Level Mathematics B (MEI)

H640/01 - Pure Mathematics and Mechanics

Tuesday 7 June 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Tuesday 7 June 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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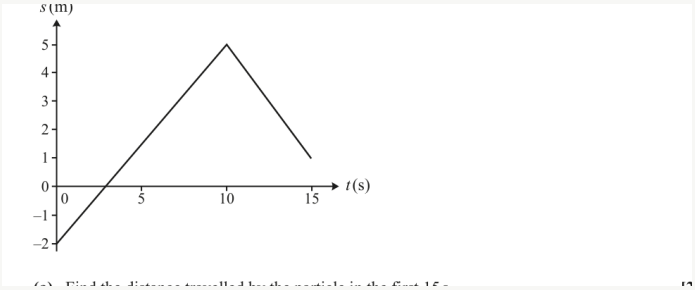
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Question 1

Question 1

A particle moves along a straight line. The displacement s m at time t s is shown in the displacement–time graph below. The graph consists of straight line segments joining the points $(0, -2)$, $(10, 5)$ and $(15, 1)$.



(a) Find the distance travelled by the particle in the first 15 s. [2]
(b) Calculate the velocity of the particle between $t = 10$ and $t = 15$. [2]

Model solution

(a) Distance counts the size of each change in displacement, whether the particle travels forwards or backwards.

$$\textit{distance} = |5 - (-2)| + |1 - 5| = 7 + 4 = 11 \textit{ m}.$$

11 *m* *(2)*

(b) Velocity is the gradient of a displacement--time graph.

$$v = \frac{1 - 5}{15 - 10} = -\frac{4}{5} = -0.8 \textit{ m s}^{-1}.$$

-0.8 *m* $\textit{s}^{-1}$ *(2)*

Question 2

Question 2

Express $\dfrac{13-x}{(x-3)(x+2)}$ in partial fractions. [3]

Model solution

The denominator has two distinct linear factors, so use one constant numerator over each factor.

$$\frac{13-x}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}.$$

Multiplying by $(x-3)(x+2)$ makes an identity that can be solved by choosing convenient values of x.

$$13 - x = A(x+2) + B(x-3).$$

$$x = 3: \quad 10 = 5A \quad \Rightarrow \quad A = 2, \qquad x = -2: \quad 15 = -5B \quad \Rightarrow \quad B = -3.$$

$$\frac{13-x}{(x-3)(x+2)} = \frac{2}{x-3} - \frac{3}{x+2}$$ *(3)*

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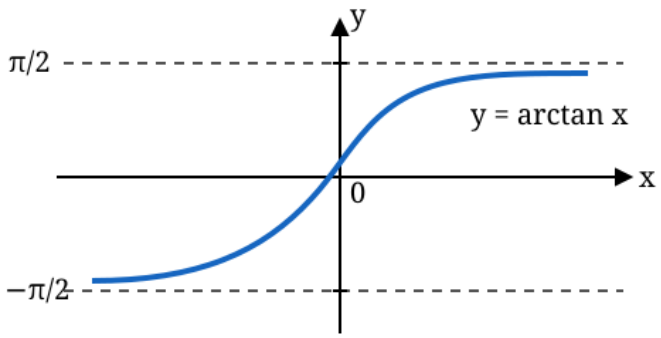
Question 3

Question 3

(a) Sketch the graph of $y = \arctan x$, where x is in radians. [2]
(b) In this question you must show detailed reasoning.
Find all points of intersection of the curves $y = 3\sin x \cos x$ and $y = \cos^2 x$ for $-\pi \leq x \leq \pi$. [6]

Model solution

(a) The inverse tangent is defined for every real x, is increasing, passes through the origin, and approaches but never reaches $y = \pm\pi/2$.



Increasing $y = \arctan x$ curve through $(0, 0)$, with horizontal asymptotes $y = \pm\pi/2$. (2)

(b) At an intersection the two expressions for y are equal. Factoring before dividing is important, since $\cos x = 0$ may itself give solutions.

$$3\sin x \cos x = \cos^2 x$$

$$\cos x(3\sin x - \cos x) = 0.$$

The first factor gives

$$\cos x = 0 \quad \Rightarrow \quad x = -\frac{\pi}{2}, \ \frac{\pi}{2}, \qquad y = 0.$$

For the other factor, $3\sin x = \cos x$, so $\tan x = 1/3$. Within the stated interval this gives

$$x = \arctan\left(\frac{1}{3}\right) = 0.322\ldots \quad \textit{or} \quad x = \arctan\left(\frac{1}{3}\right) - \pi = -2.82\ldots.$$

At either of these values, $\tan x = 1/3$, hence $\cos^2 x = 9/10 = 0.9$.

$\left(-\frac{\pi}{2}, 0\right)$, $\left(\frac{\pi}{2}, 0\right)$, $(0.322, 0.9)$, $(-2.82, 0.9)$ *(6)*

Question 4

Question 4

Using an appropriate expansion show that, for sufficiently small values of x,

$$\frac{1-x}{(2+x)^2} \approx \frac{1}{4} - \frac{1}{2}x + \frac{7}{16}x^2.$$

[4]

Model solution

Write the denominator in the form $1+u$, since the binomial expansion for a negative power then applies when x is small.

$$\frac{1-x}{(2+x)^2} = \frac{1-x}{4\left(1+\frac{x}{2}\right)^2} = \frac{1}{4}(1-x)\left(1+\frac{x}{2}\right)^{-2}.$$

Using $(1+u)^{-2} \approx 1-2u+3u^2$, with $u = x/2$, gives

$$\left(1+\frac{x}{2}\right)^{-2} \approx 1 - x + \frac{3}{4}x^2.$$

Multiplying and retaining terms through x^2,

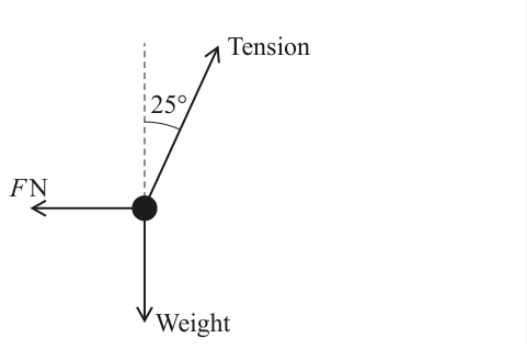
$$\frac{1}{4}(1-x)\left(1-x+\frac{3}{4}x^2\right) \approx \frac{1}{4}\left(1-2x+\frac{7}{4}x^2\right) = \frac{1}{4} - \frac{1}{2}x + \frac{7}{16}x^2.$$

$$\frac{1-x}{(2+x)^2} \approx \frac{1}{4} - \frac{1}{2}x + \frac{7}{16}x^2 \quad \textit{(4)}$$

Question 5

Question 5

A sphere of mass 3 kg hangs on a string. A horizontal force of magnitude F N acts on the sphere so that it hangs in equilibrium with the string making an angle of $25°$ to the vertical. The force diagram for the sphere is shown below.



(a) Sketch the triangle of forces for these forces.

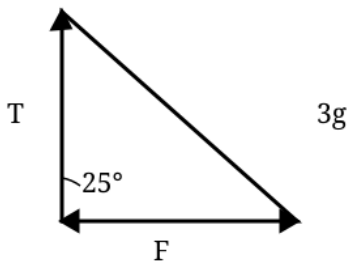
(a) Sketch the triangle of forces for these forces. [2]
(b) Hence or otherwise determine each of the following:

- the tension in the string
- the value of F.

[3]

Model solution

(a) Equilibrium means the three force vectors form a closed, head-to-tail triangle. The tension is $25°$ from the vertical weight.



Closed triangle of forces, labelled T, $3g$, and F, with the $25°$ angle shown. **(2)**

(b) Resolving vertically is convenient because the horizontal force has no vertical component.

$$T\cos 25° = 3g \quad \Rightarrow \quad T = \frac{3(9.8)}{\cos 25°} = 32.4 \text{ N}.$$

The horizontal component of the tension balances F.

$$F = T\sin 25° = 13.7 \text{ N}.$$

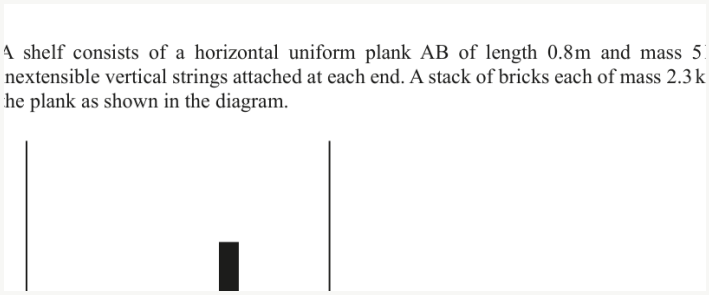
Tension $= 32.4$ ***N;*** $F = 13.7$ ***N.*** ***(3)***

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Question 6

Question 6

A shelf consists of a horizontal uniform plank AB of length 0.8 m and mass 5 kg with light inextensible vertical strings attached at each end. A stack of bricks each of mass 2.3 kg is placed on the plank as shown in the diagram.



(a) Explain the meaning of each of the following modelling assumptions.

- The stack of bricks is modelled as a particle.
- The plank is modelled as uniform.

[2]

Either of the strings will break if the tension exceeds 75 N.

(b) Find the greatest number of bricks that can be placed at the centre of the plank without breaking the strings. [2]

(c) Find an expression for the moment about A of the weight of a stack of n bricks when the stack is at a distance of x m from A. State the units for your answer. [2]

(d) Calculate the greatest distance from A that the largest stack of bricks can be placed without a string breaking. [3]

Model solution

(a) A particle has negligible size, so the stack's contact force and weight can be treated as acting at one point. A uniform plank has its mass evenly distributed, so its weight acts at its centre.

Bricks: negligible size, with their force acting at a point. Plank: mass evenly distributed, so its weight acts at the midpoint. **(2)**

(b) At the centre the arrangement is symmetric, so the two string tensions are equal. At the limiting case each is 75 N.

$$2(75) = (5 + 2.3n)g \quad \Rightarrow \quad n = \frac{150/g - 5}{2.3} = 4.48\ldots.$$

Only a whole number of bricks is possible.

4 *bricks* **(2)**

(c) Moment equals force times perpendicular distance. The stack has weight $2.3ng$ and is x m from A.

moment about $A = 2.3gnx$ N m.

$2.3gnx$ *N m, clockwise about A.* **(2)**

(d) The largest stack is the four-brick stack. As it is moved towards B, the tension at B is the first to reach 75 N. Take moments about A.

$$5g(0.4) + 4(2.3g)x = 75(0.8).$$

$$2g + 9.2gx = 60 \quad \Rightarrow \quad x = \frac{60 - 2g}{9.2g} = 0.448 \text{ m}.$$

0.448 *m from* A **(3)**

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Question 7

Question 7

In this question the x- and y-directions are horizontal and vertically upwards respectively and the origin is on horizontal ground.

A ball is thrown from a point 5 m above the origin with an initial velocity $\begin{pmatrix} 14 \\ 7 \end{pmatrix}$ m s^{-1}.

(a) Find the position vector of the ball at time t s after it is thrown. [3]

(b) Find the distance between the origin and the point at which the ball lands on the ground. [3]

Model solution

(a) Horizontal acceleration is zero and vertical acceleration is $-g$. Apply the constant-acceleration displacement formula separately to the two components.

$$x = 14t, \qquad y = 5 + 7t - \frac{1}{2}gt^2 = 5 + 7t - 4.9t^2.$$

$$\mathbf{r} = \begin{pmatrix} 14t \\ 5 + 7t - 4.9t^2 \end{pmatrix} \text{ m} \qquad \textit{(3)}$$

(b) The ball lands when its vertical coordinate is zero. Solve for the positive time, then use the horizontal coordinate.

$$5 + 7t - 4.9t^2 = 0 \quad \Rightarrow \quad t = 1.95 \text{ s}.$$

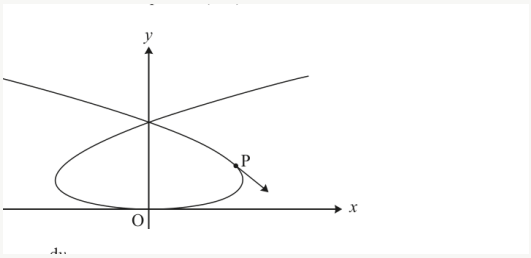
$$x = 14(1.95) = 27.3 \text{ m}.$$

$$27.3 \text{ m} \qquad \textit{(3)}$$

Question 8

Question 8

A particle moves in the x-y plane so that its position at time t s is given by $x = t^3 - 8t$, $y = t^2$ for $-3.5 \leq t \leq 3.5$. The units of distance are metres. The graph shows the path of the particle and the direction of travel at the point $P(8, 4)$.



(a) Find $\dfrac{dy}{dx}$ in terms of t. [3]

(b) Hence show that the value of $\dfrac{dy}{dx}$ at P is -1. [2]

(c) Find the time at which the particle is travelling in the direction opposite to that at P. [2]

(d) Find the cartesian equation of the path, giving x^2 as a function of y. [3]

Model solution

(a) For parametric equations, differentiate both coordinates with respect to the parameter and divide.

$$\frac{dx}{dt} = 3t^2 - 8, \qquad \frac{dy}{dt} = 2t, \qquad \frac{dy}{dx} = \frac{2t}{3t^2 - 8}.$$

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 8}$$ *(3)*

(b) Point P has $y = 4$, so $t^2 = 4$. The stated direction selects $t = -2$, since this gives $x = (-2)^3 - 8(-2) = 8$.

$$\left.\frac{dy}{dx}\right|_P = \frac{2(-2)}{3(-2)^2 - 8} = \frac{-4}{4} = -1.$$

$\dfrac{dy}{dx} = -1$ *at P.* *(2)*

(c) The opposite direction on the same path has the same tangent gradient but the other parameter value. Set the derivative equal to -1.

$$\frac{2t}{3t^2 - 8} = -1 \quad \Rightarrow \quad 3t^2 + 2t - 8 = 0 \quad \Rightarrow \quad (3t - 4)(t + 2) = 0.$$

The root $t = -2$ is P, so the other root is required.

$t = \dfrac{4}{3}$ *s* *(2)*

(d) Eliminate t using $y = t^2$. Squaring the equation for x avoids a $\pm\sqrt{y}$ term.

$$x = t(t^2 - 8) = t(y - 8),$$

$$x^2 = t^2(y - 8)^2 = y(y - 8)^2 = y^3 - 16y^2 + 64y.$$

$x^2 = y^3 - 16y^2 + 64y$ **(3)**

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Question 9

Question 9

In this question, the vectors **i** and **j** are directed east and north respectively.
The velocity **v** m s^{-1} of a particle at time t s is given by $\mathbf{v} = kt^2\mathbf{i} + 6t\mathbf{j}$, where k is a positive constant.
The magnitude of the acceleration when $t = 2$ is 10 m s^{-2}.
(a) Calculate the value of k. [4]
The particle is at the origin when $t = 0$.
(b) Determine an expression for the position vector of the particle at time t. [2]
(c) Determine the time when the particle is directly north-east of the origin. [2]

Model solution

(a) Acceleration is the derivative of velocity. Use its given magnitude at $t = 2$.

$$\mathbf{a} = 2kt\mathbf{i} + 6\mathbf{j}, \qquad \mathbf{a}\big|_{t=2} = 4k\mathbf{i} + 6\mathbf{j}.$$

$$\sqrt{(4k)^2 + 6^2} = 10 \quad \Rightarrow \quad 16k^2 + 36 = 100 \quad \Rightarrow \quad k^2 = 4.$$

Since k is positive, choose the positive root.

$k = 2$ **(4)**

(b) Position is the integral of velocity. The zero initial position makes the integration constant the zero vector.

$$\mathbf{r} = \int (2t^2\mathbf{i} + 6t\mathbf{j})\, dt = \frac{2}{3}t^3\mathbf{i} + 3t^2\mathbf{j}.$$

$\mathbf{r} = \frac{2}{3}t^3\mathbf{i} + 3t^2\mathbf{j}$ *m* **(2)**

(c) North-east means equal positive east and north components.

$$\frac{2}{3}t^3 = 3t^2 \quad \Rightarrow \quad t^2(2t - 9) = 0.$$

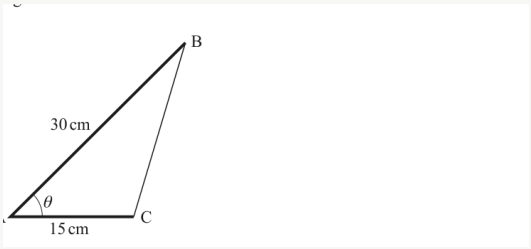
The root $t = 0$ is the origin, not a north-east position.

$t = 4.5$ *s* **(2)**

Question 10

Question 10

A triangle ABC is made from two thin rods hinged together at A and a piece of elastic which joins B and C. AB is a 30 cm rod and AC is a 15 cm rod. The angle BAC is θ radians as shown in the diagram.



The angle θ increases at a rate of 0.1 radians per second.
Determine the rate of change of the length BC when $\theta = \pi/3$. [8]

Model solution

The side whose rate is required is opposite the changing included angle, so begin with the cosine rule. Let $l = BC$, measured in cm.

$$l^2 = 30^2 + 15^2 - 2(30)(15)\cos\theta = 1125 - 900\cos\theta.$$

Differentiate implicitly with respect to θ, then use the chain rule to introduce the given angular rate.

$$2l\frac{dl}{d\theta} = 900\sin\theta \quad \Rightarrow \quad \frac{dl}{d\theta} = \frac{450\sin\theta}{l}, \qquad \frac{d\theta}{dt} = 0.1.$$

$$\frac{dl}{dt} = \frac{450\sin\theta}{l}(0.1).$$

At $\theta = \pi/3$, the cosine rule gives $l^2 = 675$, so $l = 15\sqrt{3}$. Therefore

$$\frac{dl}{dt} = \frac{450(\sqrt{3}/2)}{15\sqrt{3}}(0.1) = 1.5.$$

1.5 *cm s*$^{-1}$ *(8)*

Question 11

Question 11

Given that k is a positive constant, show that

$$\int_k^{2k} \frac{2}{(2x+k)^2}\,dx$$

is inversely proportional to k. [6]

Model solution

The denominator is a power of the linear expression $2x+k$, whose derivative is 2, so substitute $u = 2x+k$.

$$u = 2x+k, \qquad du = 2\,dx.$$

The limits change to $u = 3k$ when $x = k$, and $u = 5k$ when $x = 2k$.

$$\int_k^{2k} \frac{2}{(2x+k)^2}\,dx = \int_{3k}^{5k} u^{-2}\,du = \left[-u^{-1}\right]_{3k}^{5k}.$$

$$= -\frac{1}{5k} + \frac{1}{3k} = \frac{2}{15k}.$$

The bracket is a constant independent of k, so the result has the required form C/k.

$\frac{2}{15k}$, hence inversely proportional to k. ***(6)***

Question 12

Question 12

Prove by contradiction that 3 is the only prime number which is 1 less than a square number. [4]

Model solution

Assume, for a contradiction, that a prime p is one less than a square. Then for some integer $n \geq 2$,

$$p = n^2 - 1 = (n-1)(n+1).$$

If $n = 2$, this gives $p = (1)(3) = 3$, which is indeed prime. If $n > 2$, then $n - 1 > 1$ and $n + 1 > 1$, so p is expressed as a product of two integers greater than 1.
That contradicts the assumption that p is prime. Hence no value other than 3 can be a prime that is one less than a square.

3 is the only such prime. ***(4)***

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Question 13

Question 13

A toy train consists of an engine of mass 0.5 kg pulling a coach of mass 0.4 kg. The coupling between the engine and the coach is light and inextensible. The train is pulled along with a string attached to the front of the engine.
At first, the train is pulled from rest along a horizontal carpet where there is a resistance to motion of 0.8 N on each part of the train. The string is horizontal, and the tension in the string is 5 N.
(a) Determine the velocity of the train after 1.5 s. [4]
The train is then pulled up a track inclined at $20°$ to the horizontal. The string is parallel to the track and the tension in the string is P N. The resistance on each part of the train along the track is R N.
(b) Draw a diagram showing all the forces acting on the train modelled as two connected particles. [3]
(c) Find the equation of motion for the train modelled as a single particle. [2]
(d) The acceleration of the train when $P = 5.5$ is double the acceleration when $P = 5$.
Calculate the value of R. [3]

Model solution

(a) Treat both connected parts as one system: the coupling force is internal and cancels. Take the direction of motion as positive.

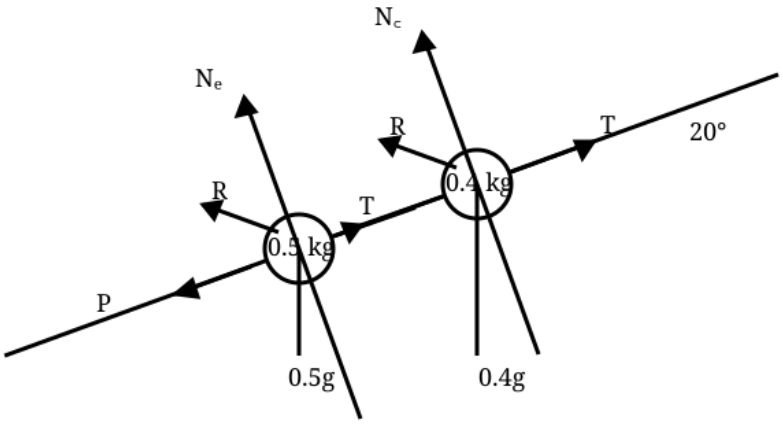
$$5 - 2(0.8) = (0.5 + 0.4)a \quad \Rightarrow \quad a = \frac{3.4}{0.9} = \frac{34}{9} \text{ m s}^{-2}.$$

The train starts from rest, so $v = u + at$ *gives*

$$v = 0 + \frac{34}{9}(1.5) = \frac{17}{3} = 5.67 \text{ m s}^{-1}.$$

5.67 *m s*$^{-1}$ *(4)*

(b) Each particle has its weight vertically down and a normal reaction perpendicular to the track. The string and coupling tensions act along the track; both resistances act down the track.



Force diagram shown, including weights, normal reactions, string and coupling tensions, and both resistances. *(3)*

(c) For the combined train, take up the slope as positive. There are two resistances, and the component of the total weight down the slope is $0.9g \sin 20°$.

$$P - 2R - 0.9g \sin 20° = 0.9a.$$

$P - 2R - 0.9g\sin 20° = 0.9a$ **(2)**

(d) Let the acceleration when $P = 5$ be a; then when $P = 5.5$ it is $2a$. Substitute each condition into the equation of motion.

$$5 - 2R - 0.9g\sin 20° = 0.9a,$$

$$5.5 - 2R - 0.9g\sin 20° = 1.8a.$$

Subtracting eliminates R and the slope term: $0.5 = 0.9a$, so $a = 5/9$. Substitution into the first equation gives

$$2R = 5 - 0.9(9.8)\sin 20° - 0.9\left(\frac{5}{9}\right), \qquad R = 0.742 \text{ N}.$$

$R = 0.742$ **N** **(3)**

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Question 14

Question 14

Alex places a hot object into iced water and records the temperature $\theta\,^{\circ}\text{C}$ of the object every minute. The temperature of an object t minutes after being placed in iced water is modelled by

$$\theta = \theta_0 e^{-kt}$$

where θ_0 and k are constants whose values depend on the characteristics of the object.
The temperature of Alex's object is $82°\text{C}$ when it is placed into the water. After 5 minutes the temperature is $27°\text{C}$.
(a) Find the values of θ_0 and k that best model the data. [3]
(b) Explain why the model may not be suitable in the long term if Alex does not top up the ice in the water. [1]
(c) Show that the model with the values found in part (a) can be written as $\ln\theta = a - bt$ where a and b are constants to be determined. [2]
Ben places a different object into iced water at the same time as Alex. The model for Ben's object is $\ln\theta = 3.4 - 0.08t$.
(d) Determine each of the following:

- the initial temperature of Ben's object
- the rate at which Ben's object is cooling initially.

[4]
(e) According to the models, there is a time at which both objects have the same temperature. Find this time and the corresponding temperature. [3]

Model solution

(a) At $t = 0$, the exponential factor is 1, so the initial temperature is θ_0. Use the second reading to find the decay constant.

$$\theta_0 = 82, \qquad 27 = 82e^{-5k}.$$

$$k = -\frac{1}{5}\ln\left(\frac{27}{82}\right) = 0.222 \text{ min}^{-1} \quad \textit{(3 s.f.)}.$$

$\theta_0 = 82$, $k = 0.222$ *min*$^{-1}$ *(3)*

(b) The model tends to $0°\text{C}$. Without more ice, however, the water warms up, so it will not remain at $0°\text{C}$ and the object will not continue cooling towards zero.

The water warms as the ice melts, whereas the model incorrectly continues to predict a limiting temperature of $0°\text{C}$. *(1)*

(c) Taking natural logarithms linearises the exponential model.

$$\ln\theta = \ln\theta_0 - kt = \ln 82 - 0.222t.$$

$\ln\theta = 4.41 - 0.222t$, *so* $a = \ln 82$ *and* $b = 0.222$. *(2)*

(d) At $t = 0$, Ben's model gives $\ln\theta = 3.4$, so $\theta = e^{3.4}$. Rewrite it exponentially before differentiating.

$$\theta = e^{3.4-0.08t} = e^{3.4}e^{-0.08t}.$$

$$\frac{d\theta}{dt} = -0.08e^{3.4}e^{-0.08t}.$$

$$\theta(0) = e^{3.4} = 29.96\ldots \approx 30.0°\text{C}, \qquad \left.\frac{d\theta}{dt}\right|_{t=0} = -2.3968\ldots \approx -2.4°\text{C } \textit{min}^{-1}.$$

Initial temperature $= 30.0°\text{C}$*; initially cooling at* $2.4°\text{C } \textit{min}^{-1}$*.* **(4)**

(e) Equal temperatures mean equal logarithms, so equate the two linear log-temperature expressions.

$$\ln 82 - 0.222t = 3.4 - 0.08t \quad \Rightarrow \quad t = 7.09\ldots \textit{ min}.$$

$$\ln\theta = 3.4 - 0.08(7.089\ldots) = 2.8328\ldots \quad \Rightarrow \quad \theta = 17.0\ldots{}^\circ\text{C}.$$

$t = 7.1$ *min (about* 7 *min* 5 *s), and* $\theta = 17°\text{C}$*.* **(3)**

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