Maths with Matt

Physics and maths resources

# OCR A Level Mathematics B (MEI)

## H640/01 - Pure Mathematics and Mechanics

Tuesday 4 June 2024

## *Questions and Fully Worked Model Solutions*

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Tuesday 4 June 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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# Question 1

## Question 1

A student states that $1 + x^2 < (1 + x)^2$ for all values of $x$.
Using a counter example, show that the student is wrong. [2]

## *Model solution*

*A statement asserted for every value can be disproved by one value for which it fails. Choose the simple value* $x = 0$.

$$1 + 0^2 = 1, \qquad (1 + 0)^2 = 1.$$

*Thus the two sides are equal, rather than the left-hand side being less than the right-hand side.*

*The statement is false.* *(2)*

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# Question 2

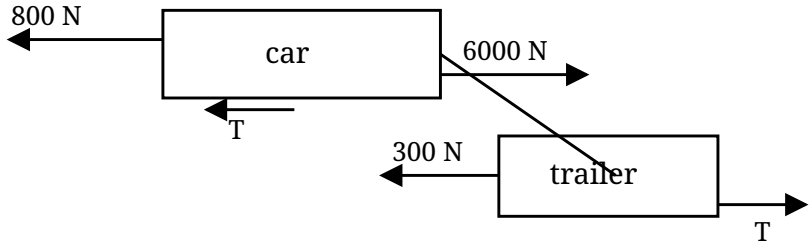
## Question 2

A car of mass 1400 kg pulls a trailer of mass 400 kg along a straight horizontal road. The engine of the car produces a driving force of 6000 N. A resistance of 800 N acts on the car. A resistance of 300 N acts on the trailer. The tow-bar between the car and the trailer is light and horizontal.
(a) Draw a force diagram showing all the horizontal forces on the car and the trailer. [2]
(b) Calculate the acceleration of the car and trailer. [3]

## *Model solution*

*(a) The tow-bar has a common tension $T$: it pulls the car backwards and the trailer forwards. Resistances oppose the motion.*

| | |
|---|---|
| *car* | $\leftarrow 800$ **N**, $\leftarrow T$, $6000$ **N** $\rightarrow$ |
| *trailer* | $\leftarrow 300$ **N**, $T \rightarrow$ |




*The force diagram is as labelled above.* *(2)*

*(b) Treat the car and trailer as one system. The tow-bar tension is internal to this system, so it cancels; Newton's second law then gives the common acceleration.*

$$6000 - 800 - 300 = (1400 + 400)a,$$

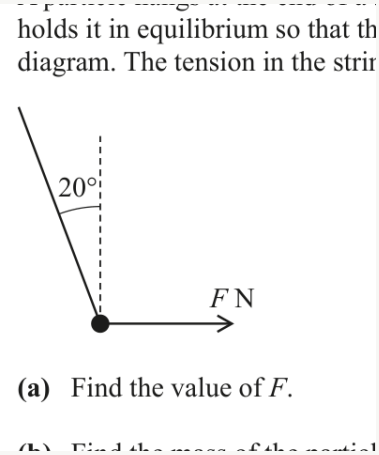
$$4900 = 1800a, \qquad a = 2.722\ldots$$

$a = 2.72$ ***m s***$^{-2}$ *(3 s.f.).* *(3)*

# Question 3

## Question 3

A particle hangs at the end of a string. A horizontal force of magnitude $F$ N acting on the particle holds it in equilibrium so that the string makes an angle of $20°$ with the vertical, as shown in the diagram. The tension in the string is 12 N.




(a) Find the value of $F$. [2]
(b) Find the mass of the particle. [3]

## Model solution

*The particle is in equilibrium, so resolve the tension into horizontal and vertical components and balance each direction separately.*
*(a) Horizontally, the applied force balances the horizontal component of the tension.*

$$F = 12\sin 20° = 4.104\ldots$$

$F = 4.10$ **N.** **(2)**

*(b) Vertically, the upward component of the tension balances the weight.*

$$mg = 12\cos 20°, \qquad m = \frac{12\cos 20°}{9.8} = 1.150\ldots$$

$m = 1.15$ **kg.** **(3)**

# Question 4

## Question 4

The vectors $\mathbf{v}_1 = 2a\mathbf{i} + b\mathbf{j}$ and $\mathbf{v}_2 = b\mathbf{i} - 3\mathbf{j}$ are defined, where $a$ and $b$ are constants.
Given that $3\mathbf{v}_1 + \mathbf{v}_2 = 22\mathbf{i} - 9\mathbf{j}$, find the values of $a$ and $b$. [4]

## *Model solution*

*Expand the vector equation and equate the coefficients of the independent unit vectors* $\mathbf{i}$ *and* $\mathbf{j}$.

$$3(2a\mathbf{i} + b\mathbf{j}) + (b\mathbf{i} - 3\mathbf{j}) = 22\mathbf{i} - 9\mathbf{j},$$

$$(6a + b)\mathbf{i} + (3b - 3)\mathbf{j} = 22\mathbf{i} - 9\mathbf{j}.$$

*Hence*

$$6a + b = 22, \qquad 3b - 3 = -9.$$

*The second equation gives* $b = -2$, *and then* $6a - 2 = 22$, *so* $a = 4$.

$a = 4, \quad b = -2.$ ***(4)***

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# Question 5

## Question 5

(a) Make $y$ the subject of the formula $\log_{10}(y-k) = x\log_{10} 2$, where $k$ is a positive constant. [2]
(b) Sketch the graph of $y$ against $x$. [3]

## *Model solution*

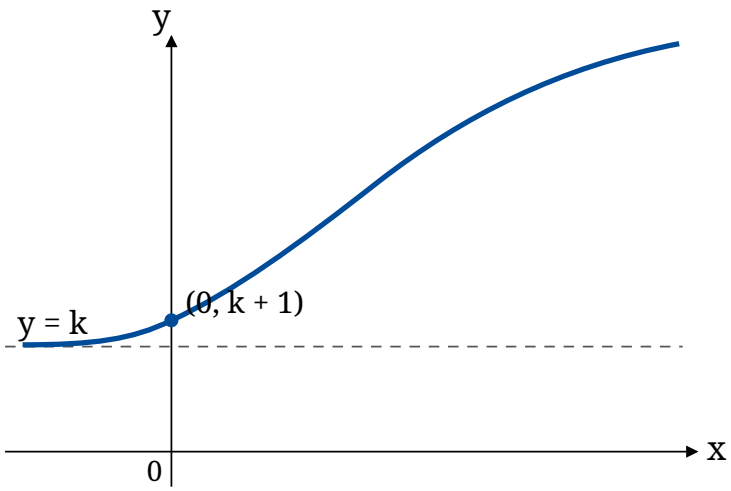
*(a) Rewrite the right-hand side as one logarithm, then use the inverse relationship between logarithms and powers of* $10$.

$$\log_{10}(y-k) = \log_{10}(2^x).$$

*Therefore* $y-k=2^x$, *so*

$y = k + 2^x$. *(2)*

*(b) This is the increasing exponential* $2^x$, *translated upwards by* $k$. *Thus it has horizontal asymptote* $y=k$ *and crosses the* $y$*-axis at* $k+1$.




*Increasing exponential curve with asymptote* $y=k$ *and intercept* $(0, k+1)$. *(3)*

# Question 6

## Question 6

Given that $f(x) = 2x^2 + 3$, show from first principles that $f'(x) = 4x$. [4]

## Model solution

*Use the limit definition of a derivative, because the question specifically asks for first principles.*

$$\frac{f(x+h) - f(x)}{h} = \frac{2(x+h)^2 + 3 - (2x^2+3)}{h} = \frac{2x^2 + 4xh + 2h^2 + 3 - 2x^2 - 3}{h}.$$

$$\frac{f(x+h) - f(x)}{h} = \frac{4xh + 2h^2}{h} = 4x + 2h.$$

*Taking the limit as* $h \to 0$,

$$f'(x) = \lim_{h \to 0}(4x + 2h) = 4x.$$

$f'(x) = 4x.$ **(4)**

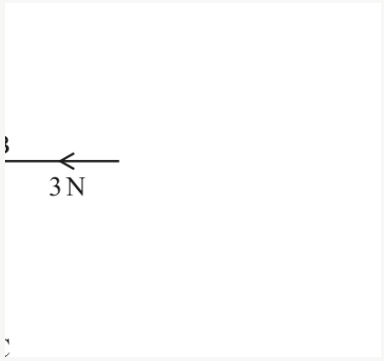
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# Question 7

## Question 7

A rectangular book $ABCD$ rests on a smooth horizontal table. The length of $AB$ is 28 cm and the length of $AD$ is 18 cm. The following five forces act on the book, as shown in the diagram.

- 4 N at $A$ in the direction $AD$
- 5 N at $B$ in the direction $BC$
- 3 N at $B$ in the direction $BA$
- 9 N at $D$ in the direction $DA$
- 3 N at $D$ in the direction $DC$




(a) Show that the resultant of the forces acting on the book has zero magnitude. [2]
(b) Find the total moment of the forces about the centre of the book. Give your answer in N m. [3]
(c) Describe how the book will move under the action of these forces. [2]

## *Model solution*

*(a) Resolve along the two perpendicular sides.*

$$\textit{in direction } AB: \ 3-3=0, \qquad \textit{in direction } AD: \ 4+5-9=0.$$

*The resultant force has zero magnitude.* *(2)*

*(b) Take anticlockwise moments about the centre. The perpendicular half-lengths are 14 cm and 9 cm.*

$$M = 4(14) - 5(14) + 3(9) - 9(14) + 3(9) = -86 \text{ N cm} = -0.86 \text{ N m}.$$

*Total moment $= 0.86$ N m clockwise.* *(3)*

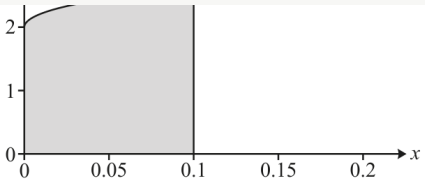
*(c) Zero resultant means no translation, but the non-zero moment produces rotation.*

*The book rotates clockwise about its centre of mass.* *(2)*

# Question 8

## Question 8

The equation of a curve is $y = \sqrt{\sin 4x} + 2\cos 2x$, where $x$ is in radians.
(a) Show that, for small values of $x$, $y \sim 2\sqrt{x} + 2 - 4x^2$. [2]
The diagram shows the region bounded by the curve $y = \sqrt{\sin 4x} + 2\cos 2x$, the axes and the line $x = 0.1$.




**(b) In this question you must show detailed reasoning.**

Use the approximation in part **(a)** to estimate the area of this regi

A child throws a pebble of mass 40 g vertically downwards with a spe
0.8 m above a sandy beach.

(b) In this question you must show detailed reasoning.
Use the approximation in part (a) to estimate the area of this region. [4]

## *Model solution*

*(a) Apply the small-angle approximations to each term. Since* $\sin 4x \sim 4x$*, its square root is* $2\sqrt{x}$*.*

$$y \sim \sqrt{4x} + 2(1 - 2x^2) = 2\sqrt{x} + 2 - 4x^2.$$

$y \sim 2\sqrt{x} + 2 - 4x^2$. ***(2)***

*(b) The area is under the approximate curve from* $0$ *to* $0.1$*, so integrate the polynomial.*

$$\begin{aligned} \textit{Area} &\sim \int_0^{0.1} \left(2\sqrt{x} + 2 - 4x^2\right) \mathrm{d}x \\ &= \left[\frac{4}{3}x^{3/2} + 2x - \frac{4}{3}x^3\right]_0^{0.1} \\ &= \frac{4}{3}(0.1)^{3/2} + 0.2 - \frac{4}{3}(0.1)^3 = 0.240830\ldots \end{aligned}$$

*Estimated area* $= 0.241$ *square units (3 s.f.).* ***(4)***

# Question 9

## Question 9

A child throws a pebble of mass $40$ g vertically downwards with a speed of $6$ m s$^{-1}$ from a point $0.8$ m above a sandy beach.
(a) Calculate the speed at which the pebble hits the beach. [2]
The pebble travels $3$ cm through the sand before coming to rest.
(b) Find the magnitude of the resistance force of the sand on the pebble, assuming it is constant. Give your answer correct to 3 significant figures. [5]

## *Model solution*

*Take downward as positive. During the fall, use* $v^2 = u^2 + 2as$.

$$v^2 = 6^2 + 2(9.8)(0.8) = 51.68.$$

*Impact speed* $= \sqrt{51.68} = 7.19$ *m s*$^{-1}$. *(2)*

*(b) The pebble stops over* $0.03$ *m, so find its acceleration first.*

$$0 = 51.68 + 2a(0.03), \qquad a = -861.333\ldots \text{ m s}^{-2}.$$

*The resistance* $R$ *acts upward, so Newton's second law in the downward direction gives*

$$0.04(9.8) - R = 0.04(-861.333\ldots), \qquad R = 34.845\ldots$$

*Resistance force* $= 34.8$ *N (3 s.f.).* *(5)*

# Question 10

## Question 10

Zac is measuring the growth of a culture of bacteria in a laboratory. The initial area of the culture is $8$ cm$^2$. The area one day later is $8.8$ cm$^2$.
At first, Zac uses a model of the form $A = a + bt$, where $A$ cm$^2$ is the area $t$ days after he begins measuring and $a$ and $b$ are constants.
(a) Find the values of $a$ and $b$ that best model the initial area and the area one day later. [2]
(b) Calculate the value of $t$ for which the model predicts an area of $15$ cm$^2$. [1]
(c) Zac notices the area covered by the culture increases by $10\%$ each day. Explain why this model may not be suitable after the first day. [1]
Zac decides to use a different model for $A$. His new model is $A = Pe^{kt}$, where $P$ and $k$ are constants.
(d) Find the values of $P$ and $k$ that best model the initial area and the area one day later. [3]
(e) Calculate the value of $t$ for which the area reaches $15$ cm$^2$ according to this model. [2]
(f) Explain why this model may not be suitable for large values of $t$. [1]

## *Model solution*

*(a) Substitute the two observations into the linear model.*

$$8 = a, \qquad 8.8 = 8 + b.$$

$a = 8,\ b = 0.8.$ **(2)**

*(b) Set $A = 15$ in $A = 8 + 0.8t$.*

$$15 = 8 + 0.8t \quad \Rightarrow \quad t = 8.75.$$

$t = 8.75$ *days.* **(1)**

*(c) A linear model adds the same $0.8$ cm$^2$ every day, whereas a $10\%$ increase gives a larger absolute increase as the area grows.*

*The linear model is unsuitable after the first day.* **(1)**

*(d) Use $t = 0$ and $t = 1$ in the exponential model.*

$$P = 8, \qquad 8.8 = 8e^k \Rightarrow e^k = 1.1 \Rightarrow k = \ln(1.1) = 0.0953\ldots$$

$P = 8, \quad k = \ln(1.1)\ (= 0.0953\ldots).$ **(3)**

*(e) Solve the exponential equation using logarithms.*

$$15 = 8e^{(\ln 1.1)t} \Rightarrow t = \frac{\ln(15/8)}{\ln 1.1} = 6.60\ldots$$

$t = 6.60$ *days.* **(2)**

*(f) The model predicts unlimited exponential growth, but laboratory space and resources are finite.*

*It is unsuitable for large $t$.* **(1)**

# Question 11

## Question 11

The first three terms of a geometric sequence are $5k-2$, $3k-6$, $k+2$, where $k$ is a constant.
(a) Show that $k$ satisfies the equation $k^2-11k+10=0$. [3]
(b) When $k$ takes the smaller of the two possible values, find the sum of the first 20 terms of the sequence. [3]
(c) When $k$ takes the larger of the two possible values, find the sum to infinity of the sequence. [2]

## Model solution

*(a) In a geometric sequence, successive ratios are equal.*

$$\frac{3k-6}{5k-2}=\frac{k+2}{3k-6} \Rightarrow (3k-6)^2=(5k-2)(k+2).$$

$$9k^2-36k+36=5k^2+8k-4 \Rightarrow k^2-11k+10=0.$$

$k^2-11k+10=0.$ *(3)*

*(b) Factor the quadratic:* $(k-1)(k-10)=0$. *For the smaller value* $k=1$, *the sequence is* $3,-3,3,\ldots$, *with ratio* $-1$.

$$S_{20}=\frac{3(1-(-1)^{20})}{1-(-1)}=0.$$

$S_{20}=0.$ *(3)*

*(c) For* $k=10$, $a=48$ *and* $r=\frac{1}{2}$, *so a sum to infinity exists.*

$$S_\infty=\frac{48}{1-1/2}=96.$$

$S_\infty=96.$ *(2)*

# Question 12

## Question 12

In this question the unit vectors **i** and **j** are in the $x$- and $y$-directions respectively.
The velocity **v** m s$^{-1}$ of a particle is given by $\mathbf{v} = 3\mathbf{i} + (6t^2 - 5)\mathbf{j}$. The initial position of the particle is $7\mathbf{j}$ m.
(a) Find an expression for the position vector of the particle at time $t$ s. [4]
(b) Find the Cartesian equation of the path of the particle. [2]

## *Model solution*

*(a) Position is the integral of velocity. Include a constant vector, then use the initial position.*

$$\mathbf{r} = \int \left[3\mathbf{i} + (6t^2 - 5)\mathbf{j}\right] \, \mathrm{d}t = 3t\mathbf{i} + (2t^3 - 5t)\mathbf{j} + \mathbf{c}.$$

*At* $t = 0$, $\mathbf{r} = 7\mathbf{j}$, *so* $\mathbf{c} = 7\mathbf{j}$.

$\mathbf{r} = 3t\mathbf{i} + (2t^3 - 5t + 7)\mathbf{j}$ *m.* **(4)**

*(b) Since* $x = 3t$, *substitute* $t = x/3$ *into* $y = 2t^3 - 5t + 7$.

$y = 2\left(\frac{x}{3}\right)^3 - 5\left(\frac{x}{3}\right) + 7.$ **(2)**

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# Question 13

## Question 13

The curve with equation $y = px + \dfrac{8}{x^2} + q$, where $p$ and $q$ are constants, has a stationary point at $(2, 7)$.

(a) Determine the values of $p$ and $q$. [5]

(b) Find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. [1]

(c) Hence determine the nature of the stationary point at $(2, 7)$. [2]

## *Model solution*

*(a) At a stationary point the first derivative is zero. Differentiate using negative powers, then put $x = 2$.*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p - 16x^{-3}, \qquad 0 = p - 16(2)^{-3},$$

*so $p = 2$. The point lies on the curve, so*

$$7 = 2(2) + \frac{8}{2^2} + q = 6 + q.$$

$p = 2, \quad q = 1.$ *(5)*

*(b) Differentiate the first derivative again.*

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 48x^{-4} = \dfrac{48}{x^4}.$ *(1)*

*(c) At $x = 2$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 48/16 = 3 > 0$, so the curve is concave upwards.*

*The stationary point is a minimum.* *(2)*

# Question 14

## Question 14

A man runs at a constant speed of $4$ m s$^{-1}$ along a straight horizontal road. A woman is standing on a bridge that spans the road. At the instant that the man passes directly below the woman she throws a ball with initial speed $u$ m s$^{-1}$ at $\alpha^\circ$ above the horizontal. The path of the ball is directly above the road. The man catches the ball $2.4$ s after it is thrown. At the instant the man catches it, the ball is $3.6$ m below the level of the point of projection.

(a) Explain what it means that the ball is modelled as a particle. [1]

(b) Find the vertical component of the ball's initial velocity. [2]

(c) Find each of the following.

- The value of $u$
- The value of $\alpha$

[4]

## *Model solution*

*(a) A particle model ignores the ball's size and shape, treating its mass as concentrated at a point.*

*The ball is treated as a point mass; its size and shape are neglected.* **(1)**

*(b) Take upwards as positive. Apply vertical constant-acceleration motion over $2.4$ s.*

$$-3.6 = u_y(2.4) + \frac{1}{2}(-9.8)(2.4)^2,$$

$$u_y = 10.26 \text{ m s}^{-1}.$$

*Vertical component* $= 10.26$ *m s*$^{-1}$. **(2)**

*(c) The ball must stay directly above the man, so its horizontal velocity equals $4$ m s$^{-1}$. Combine perpendicular components.*

$$u = \sqrt{4^2 + 10.26^2} = 11.01\ldots, \qquad \tan\alpha = \frac{10.26}{4}.$$

$u = 11.0$ *m s*$^{-1}$ *(3 s.f.),* $\alpha = 68.7^\circ$ *(3 s.f.).* **(4)**

# Question 15

## Question 15

The circle $x^2 + y^2 + 2x - 14y + 25 = 0$ has its centre at the point $C$. The line $7y = x + 25$ intersects the circle at points $A$ and $B$.
Prove that triangle $ABC$ is a right-angled triangle. [9]

## *Model solution*

*Complete the square to find the centre, then find the two intersection points. Perpendicular radii from the centre will establish the right angle.*

$$(x+1)^2 + (y-7)^2 = 25,$$

*so* $C = (-1, 7)$. *From* $x = 7y - 25$,

$$(7y - 24)^2 + (y-7)^2 = 25 \Rightarrow 50y^2 - 350y + 600 = 0.$$

$$y^2 - 7y + 12 = 0 \Rightarrow y = 3 \text{ or } 4.$$

*Thus the intersections are* $(-4, 3)$ *and* $(3, 4)$. *Let* $A = (-4, 3)$, $B = (3, 4)$. *Then*

$$m_{AC} = \frac{3-7}{-4-(-1)} = \frac{4}{3}, \qquad m_{BC} = \frac{4-7}{3-(-1)} = -\frac{3}{4}.$$

$$m_{AC} m_{BC} = -1.$$

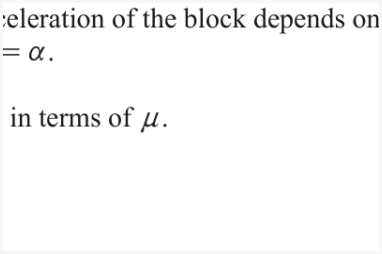
*Therefore* $AC$ *and* $BC$ *are perpendicular, so the angle* $ACB$ *is a right angle.*

*Triangle* $ABC$ *is right-angled at* $C$. **(9)**

# Question 16

## Question 16

A block of mass $m$ kg rests on rough horizontal ground. The coefficient of friction between the block and the ground is $\mu$. A force of magnitude $T$ N is applied at an angle $\theta$ radians above the horizontal as shown in the diagram and the block slides without tilting or lifting.




(a) Show that the acceleration of the block is given by $\frac{T\cos\theta}{m} - \mu g + \frac{\mu T\sin\theta}{m}$. [4]

For a fixed value of $T$, the acceleration of the block depends on the value of $\theta$. The acceleration has its greatest value when $\theta = \alpha$.

(b) Find an expression for $\alpha$ in terms of $\mu$. [3]

## Model solution

*(a) The vertical acceleration is zero. The upward component of $T$ reduces the normal reaction, and hence reduces the friction.*

$$R + T\sin\theta = mg \Rightarrow R = mg - T\sin\theta,$$

$$F = \mu R = \mu(mg - T\sin\theta).$$

*Resolving horizontally in the direction of motion,*

$$T\cos\theta - F = ma.$$

*Hence*

$$a = \frac{T\cos\theta}{m} - \mu g + \frac{\mu T\sin\theta}{m}.$$

$$a = \frac{T\cos\theta}{m} - \mu g + \frac{\mu T\sin\theta}{m}. \qquad \textit{(4)}$$

*(b) Differentiate because the maximum is specified as $\theta$ varies while $T$ is fixed.*

$$\frac{\mathrm{d}a}{\mathrm{d}\theta} = -\frac{T\sin\theta}{m} + \frac{\mu T\cos\theta}{m}.$$

*At the maximum this is zero, so $\mu\cos\alpha = \sin\alpha$.*

$$\alpha = \tan^{-1}\mu. \qquad \textit{(3)}$$