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Physics and maths resources

OCR A Level Mathematics B (MEI)

H640/03 - Pure Mathematics and Comprehension

Thursday 20 June 2024

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Thursday 20 June 2024 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

Solve the inequality $\frac{x}{5} > 6 - x$. **(2)**

Model solution

Collect the terms involving x on one side; multiplying by 5 is safe because 5 is positive.

$$\frac{x}{5} > 6 - x \quad \Longrightarrow \quad x > 30 - 5x \quad \Longrightarrow \quad 6x > 30.$$

$x > 5$. *(2)*

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Question 2

Question 2

(a) The function $f(x)$ is defined by $f(x) = \sqrt{1+2x}$ for $x \geq -\frac{1}{2}$. Find an expression for $f^{-1}(x)$ and state the domain of this inverse function. **(3)**

(b) Explain why $g(x) = 1 + x^2$, with domain all real numbers, has no inverse function. **(1)**

Model solution

(a) To find an inverse, undo the square root. The outputs of f are non-negative, so this will also give the inverse domain.

$$y = \sqrt{1+2x} \quad \Longrightarrow \quad y^2 = 1 + 2x \quad \Longrightarrow \quad x = \frac{y^2 - 1}{2}.$$

$f^{-1}(x) = \dfrac{x^2-1}{2}$, *with domain* $x \geq 0$. *(3)*

(b) An inverse function requires each output to come from only one input. Here, for example, $g(2) = g(-2) = 5$, *so* g *is not one-to-one.*

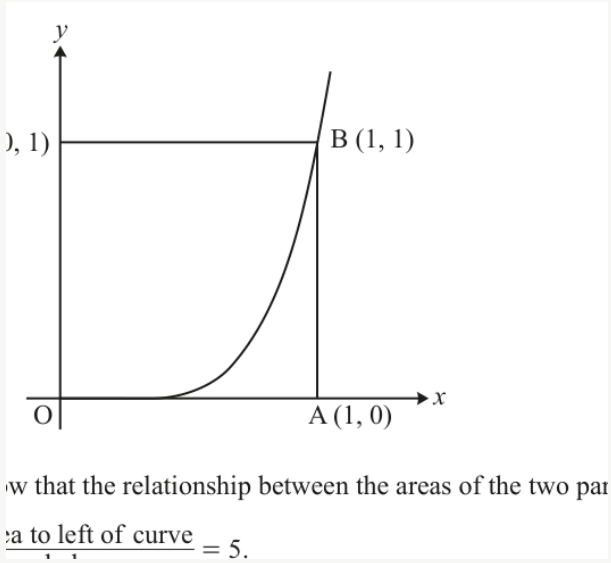
Therefore g *has no inverse function on all real numbers.* *(1)*

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Question 3

Question 3

In this question you must show detailed reasoning.
The diagram shows the curve with equation $y = x^5$ and the square $OABC$ where the points A, B and C have coordinates $(1, 0)$, $(1, 1)$ and $(0, 1)$ respectively. The curve cuts the square into two parts.



w that the relationship between the areas of the two par

a to left of curve $= 5$.

Show that the relationship between the areas of the two parts of the square is

$$\frac{\text{Area to left of curve}}{\text{Area below curve}} = 5.$$

(4)

Model solution

The area below the curve is found by integrating $y = x^5$ from $x = 0$ to $x = 1$. The square has area 1, so the other region is what remains.

$$\textit{Area below curve} = \int_0^1 x^5 \, \mathrm{d}x = \left[\frac{x^6}{6}\right]_0^1 = \frac{1}{6}.$$

$$\textit{Area to left of curve} = 1 - \frac{1}{6} = \frac{5}{6}.$$

$$\frac{\textit{Area to left of curve}}{\textit{Area below curve}} = \frac{5/6}{1/6} = 5.$$

$$\frac{\textit{Area to left of curve}}{\textit{Area below curve}} = 5. \qquad \textit{(4)}$$

Question 4

Question 4

In this question you must show detailed reasoning.

Determine the exact value of $\dfrac{1}{\sqrt{2}+1}+\dfrac{1}{\sqrt{3}+\sqrt{2}}+\dfrac{1}{2+\sqrt{3}}$. **(2)**

Model solution

Rationalise each denominator. Consecutive terms then cancel, which is the useful structure in this sum.

$$\frac{1}{\sqrt{2}+1}=\frac{\sqrt{2}-1}{2-1}=\sqrt{2}-1,$$

$$\frac{1}{\sqrt{3}+\sqrt{2}}=\frac{\sqrt{3}-\sqrt{2}}{3-2}=\sqrt{3}-\sqrt{2}, \qquad \frac{1}{2+\sqrt{3}}=\frac{2-\sqrt{3}}{4-3}=2-\sqrt{3}.$$

$$(\sqrt{2}-1)+(\sqrt{3}-\sqrt{2})+(2-\sqrt{3})=1.$$

The exact value is 1. *(2)*

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Question 5

Question 5

In this question you must show detailed reasoning.
Using the substitution $u = x + 1$, find the value of the positive integer c such that

$$\int_{c}^{c+4} \frac{x}{(x+1)^2}\,\mathrm{d}x = \ln 3 - \frac{1}{3}.$$

(6)

Model solution

The stated substitution turns the denominator into u^2, while $x = u - 1$. The new limits are $c + 1$ and $c + 5$.

$$u = x + 1, \qquad \mathrm{d}u = \mathrm{d}x,$$

$$\int_{c}^{c+4} \frac{x}{(x+1)^2}\,\mathrm{d}x = \int_{c+1}^{c+5} \frac{u-1}{u^2}\,\mathrm{d}u = \int_{c+1}^{c+5} \left(\frac{1}{u} - \frac{1}{u^2}\right)\mathrm{d}u.$$

This is now a sum of powers of u, so it can be integrated directly.

$$\int \left(\frac{1}{u} - \frac{1}{u^2}\right)\mathrm{d}u = \ln u + \frac{1}{u}.$$

$$\left[\ln u + \frac{1}{u}\right]_{c+1}^{c+5} = \ln\left(\frac{c+5}{c+1}\right) + \frac{1}{c+5} - \frac{1}{c+1}.$$

Match the logarithmic and fractional parts with the given value.

$$\ln\left(\frac{c+5}{c+1}\right) = \ln 3, \qquad \frac{1}{c+5} - \frac{1}{c+1} = -\frac{1}{3}.$$

$$\frac{c+5}{c+1} = 3 \Longrightarrow c = 1, \qquad \frac{1}{6} - \frac{1}{2} = -\frac{1}{3}.$$

$c = 1.$ *(6)*

Question 6

Question 6

In this question you must show detailed reasoning.
Solve the equation $\tan x - 3\cot x = 2$ for values of x in the interval $0° \leq x \leq 360°$. **(5)**

Model solution

Multiply by $\tan x$***, using*** $\tan x \cot x = 1$***, to form a quadratic in*** $\tan x$.

$$\tan^2 x - 3 = 2\tan x \implies \tan^2 x - 2\tan x - 3 = 0.$$

$$(\tan x - 3)(\tan x + 1) = 0 \implies \tan x = 3 \quad \textit{or} \quad \tan x = -1.$$

$$x = 71.6°,\ 251.6°,\ 135°,\ 315°.$$

$x = 71.6°,\ 135°,\ 251.6°,\ 315°.$ ***(5)***

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Question 7

Question 7

Prove that $\sin 8\theta \tan 4\theta + \cos 8\theta = 1$. **(3)**

Model solution

Use the double-angle identity on $\sin 8\theta$*, then replace* $\tan 4\theta$ *by a sine-over-cosine ratio.*

$$\sin 8\theta \tan 4\theta + \cos 8\theta = 2\sin 4\theta \cos 4\theta \left(\frac{\sin 4\theta}{\cos 4\theta}\right) + \cos 8\theta = 2\sin^2 4\theta + \cos 8\theta.$$

$$2\sin^2 4\theta + \cos 8\theta = 2\sin^2 4\theta + 1 - 2\sin^2 4\theta = 1.$$

$\sin 8\theta \tan 4\theta + \cos 8\theta = 1.$ ***(3)***

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Question 8

Question 8

In this question you must show detailed reasoning.

(a) Express $\cos x + \sqrt{3}\sin x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give R and α in exact form. **(4)**

(b) Hence solve $\cos x = \sqrt{3}(1 - \sin x)$ for $-\pi \leq x \leq \pi$. Give the roots in exact form. **(4)**

Model solution

(a) Expand the required form and match the coefficients of $\sin x$ *and* $\cos x$.

$$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha,$$

$$R\cos\alpha = \sqrt{3}, \qquad R\sin\alpha = 1.$$

$$R^2 = 3 + 1 = 4 \Longrightarrow R = 2, qquad \tan\alpha = \frac{1}{\sqrt{3}} \Longrightarrow \alpha = \frac{\pi}{6}.$$

$\cos x + \sqrt{3}\sin x = 2\sin\left(x + \frac{\pi}{6}\right)$, *so* $R = 2$, $\alpha = \frac{\pi}{6}$. *(4)*

(b) Bring the sine term to the left so that part (a) applies.

$$\cos x + \sqrt{3}\sin x = \sqrt{3} \Longrightarrow 2\sin\left(x + \frac{\pi}{6}\right) = \sqrt{3}.$$

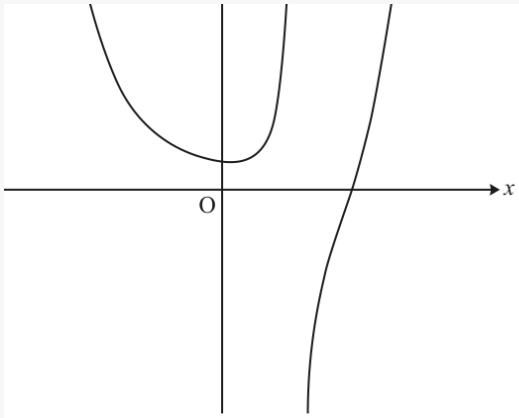
$$\sin\left(x + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \textit{so} \quad x + \frac{\pi}{6} = \frac{\pi}{3} \textit{ or } \frac{2\pi}{3}.$$

$x = \frac{\pi}{6}$ *or* $x = \frac{\pi}{2}$. *(4)*

Question 9

Question 9

This question is about $f(x) = 0$, where $f(x) = x^4 - x - \dfrac{1}{3x-2}$.



(a) Show, by calculation, that the equation f(x) = 0 has a root between $x = 1$ and $x = 2$.

(a) Show, by calculation, that $f(x) = 0$ has a root between $x = 1$ and $x = 2$. **(2)**
(b) A spreadsheet gives $f(1.5) = 3.1625$, $f(1.25) = 0.619977679$ and $f(1.125) = -0.250466087$. Write down a suitable next value of x. **(1)**
(c) Determine a root of $f(x) = 0$, correct to 1 decimal place. **(1)**
(d) A spreadsheet gives $f(0) = 0.5$, $f(1) = -1$, and then increasingly large alternating values as its x-values approach $0.666\ldots$.
(i) Explain why rows 2 and 3 make it look as if there is a root between 0 and 1. **(1)**
(ii) Explain why this process will not find a root between 0 and 1. **(1)**

Model solution

(a) A sign change for a continuous part of the function indicates a root between the two inputs.

$$f(1) = 1 - 1 - 1 = -1, \qquad f(2) = 16 - 2 - \frac{1}{4} = \frac{55}{4} > 0.$$

There is a change of sign, so a root lies between 1 and 2. **(2)**

(b) Bisection next tests a number between the two latest inputs which have opposite signs.

For example, $x = 1.1875$. **(1)**

(c) Continuing bisection gives, for example, $f(1.15) \approx -0.09$, *so the root rounds as required.*

$x = 1.2$ *(to 1 decimal place).* **(1)**

(d)(i) The two function values have opposite signs.

Since $f(0) > 0$ *and* $f(1) < 0$, *there is an apparent sign change.* **(1)**

(d)(ii) The sign change occurs across a discontinuity, not through a zero.

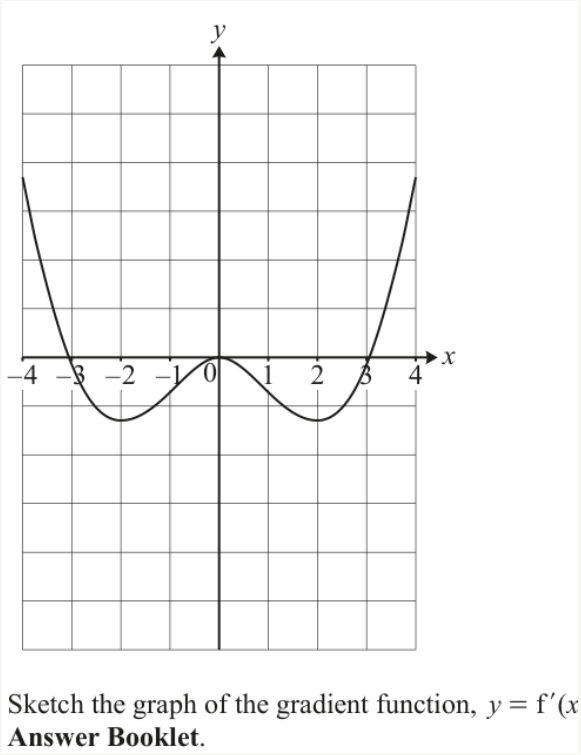
f is undefined at $x = \frac{2}{3}$, where there is a vertical asymptote; the spreadsheet is approaching this asymptote. **(1)**

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Question 10

Question 10

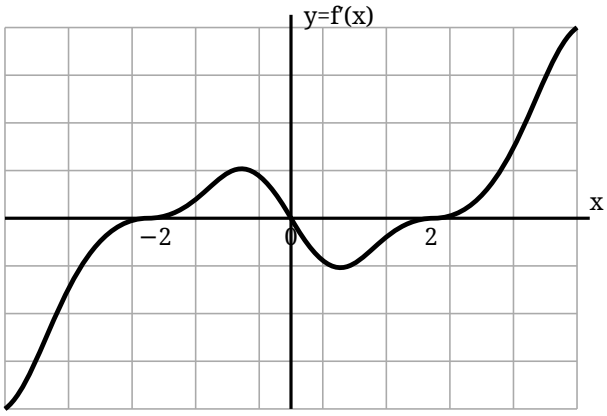
The diagram shows the curve $y = f(x)$.



Sketch the graph of the gradient function $y = f'(x)$ on the copy of the diagram. **(3)**

Model solution

The gradient is zero at each turning point of f, namely at x close to -2, 0 and 2. The curve is symmetric about the y-axis, so its derivative has rotational symmetry about the origin. Its sign follows whether f is rising or falling.



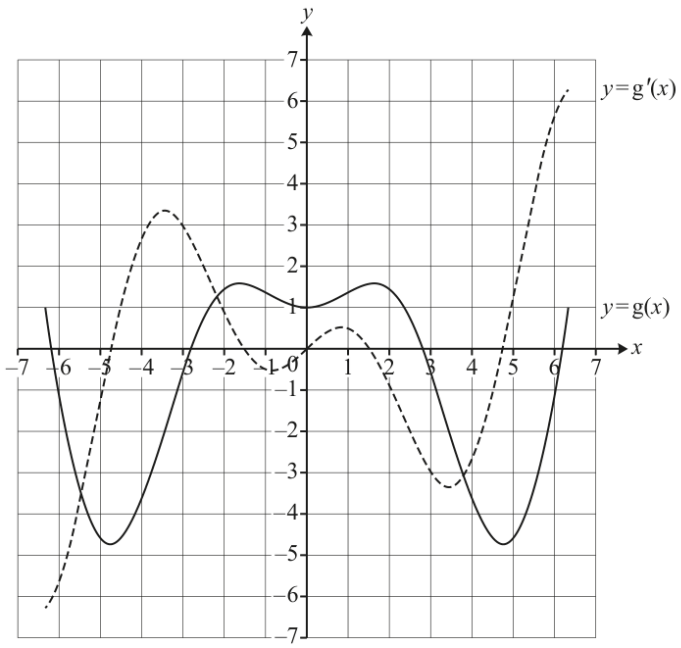
A cubic-shaped sketch crossing the x-axis at $x \approx -2, 0, 2$, with rotational symmetry about the origin. *(3)*

Question 11

Question 11

Fig. 11.1 shows $y = g(x)$, where $g(x) = x \sin x + \cos x$, and its gradient function for $-2\pi \leq x \leq 2\pi$.

Fig. 11.1



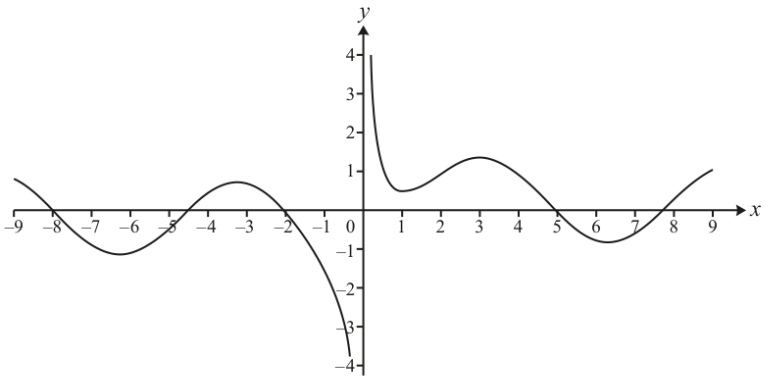
(a) Show that the x-coordinates of the points on the curve $y = g(x)$ where the gradient is 1 satisfy the equation $\frac{1}{x} - \cos x = 0$.

(**a**) Show that the x-coordinates of points on $y = g(x)$ where the gradient is 1 satisfy $\frac{1}{x} - \cos x = 0$.

(**3**)

Fig. 11.2 shows part of $y = \frac{1}{x} - \cos x$.

Fig. 11.2



(b) Use the Newton-Raphson method with a suitable starting value to find the smallest positive x-coordinate of a point on the curve $y = x\sin x + \cos x$ where the gradient is 1.

You should write down at least the following.

- The iteration you use
- The starting value
- The solution correct to **4** decimal places

(c) Explain why $x_1 = 3$ is **not** a suitable starting value for the Newton-Raphson method in part **(b)**.

(**b**) Use Newton–Raphson, with a suitable starting value, to find the smallest positive x-coordinate of a point on $y = x\sin x + \cos x$ where the gradient is 1. State the iteration, starting value, and solution to 4 decimal places. **(4)**

(**c**) Explain why $x_1 = 3$ is not suitable in part (b). **(1)**

Model solution

(a) Differentiate using the product rule. Setting the gradient equal to 1 then rearranges to the supplied equation.

$$g'(x) = \sin x + x\cos x - \sin x = x\cos x.$$

$$x\cos x = 1 \implies \cos x = \frac{1}{x} \implies \frac{1}{x} - \cos x = 0.$$

The required equation is $\dfrac{1}{x} - \cos x = 0$. *(3)*

(b) Put $F(x) = \dfrac{1}{x} - \cos x$. *Newton--Raphson uses* $x_{n+1} = x_n - \dfrac{F(x_n)}{F'(x_n)}$, *and Fig. 11.2 suggests a starting value near* 5.

$$F'(x) = -\frac{1}{x^2} + \sin x, \qquad x_{n+1} = x_n - \frac{\frac{1}{x_n} - \cos x_n}{-\frac{1}{x_n^2} + \sin x_n}.$$

$$x_1 = 5 \implies x = 4.9172\ldots$$

Smallest positive x*-coordinate:* 4.9172. *(4)*

(c) Newton--Raphson divides by $F'(x)$. *At* $x = 3$, *the graph is close to a turning point, so this derivative is close to zero and the next step is unreliable.*

$x_1 = 3$ *gives a gradient close to zero, so the iteration takes a large step away from the required root.* **(1)**

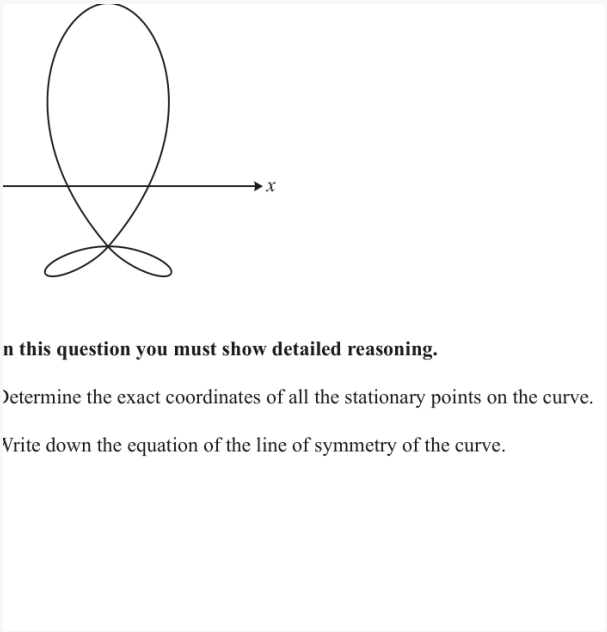
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Question 12

Question 12

The diagram shows the curve with parametric equations

$$x = 2 + 2\sin\theta, \qquad y = 2\cos\theta + \cos 2\theta, \qquad 0 \leq \theta \leq 2\pi.$$



(**a**) In this question you must show detailed reasoning. Determine the exact coordinates of all stationary points on the curve. **(8)**
(**b**) Write down the equation of the line of symmetry of the curve. **(1)**

Model solution

(a) A stationary point has $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. *Here* $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$, *so solve* $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$ *where this denominator is non-zero.*

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -2\sin\theta - 2\sin 2\theta = -2\sin\theta(1 + 2\cos\theta).$$

$$\sin\theta = 0 \quad \textit{or} \quad \cos\theta = -\frac{1}{2}.$$

$$\theta = 0,\ \frac{2\pi}{3},\ \pi,\ \frac{4\pi}{3}.$$

Substitute each parameter value into both parametric equations.

θ	x	y
0	2	3
$\frac{2\pi}{3}$	$2+\sqrt{3}$	$-\frac{3}{2}$
π	2	-1
$\frac{4\pi}{3}$	$2-\sqrt{3}$	$-\frac{3}{2}$

Stationary points: $(2, 3)$, $(2+\sqrt{3}, -\frac{3}{2})$, $(2, -1)$, $(2-\sqrt{3}, -\frac{3}{2})$. *(8)*

(b) The coordinates occur in left--right pairs about $x = 2$.

Line of symmetry: $x = 2$. **(1)**

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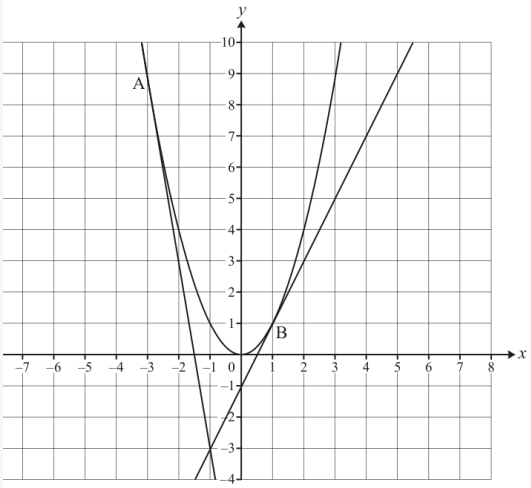
Question 13

Question 13

Substitute appropriate values of t_1 and t_2 to verify that t_1t_2 gives the correct y-coordinate of the intersection of the tangents at points A and B in Fig. C1. **(1)**

. C1 shows the curve $y = x^$ together with tangents to the curve at points A $(-3, 9)$ and 1, 1). The tangents cross at the point $(-1, -3)$. This has x-coordinate -1, which is the mean of x-coordinates of points A and B.

. C1



Model solution

Fig. C1 gives $t_1 = -3$ and $t_2 = 1$, and labels the intersection with $y = -3$.

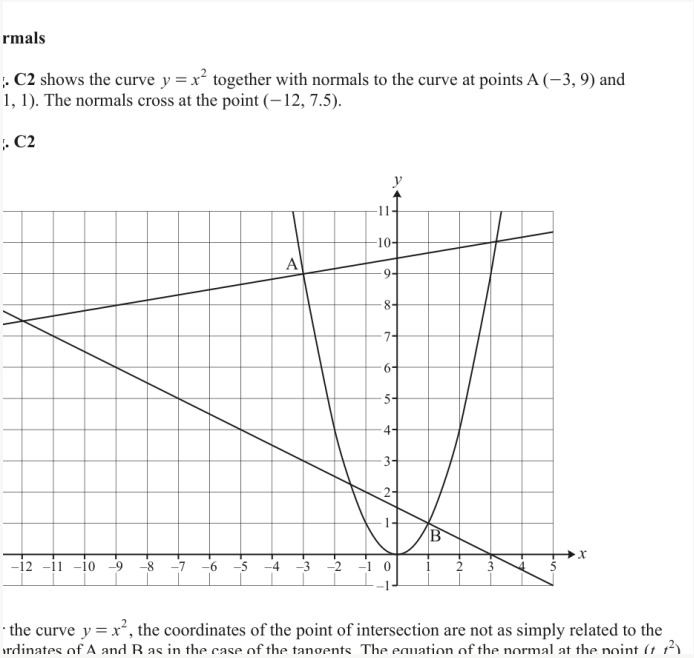
$$t_1t_2 = (-3)(1) = -3.$$

This agrees with the y-coordinate -3. **(1)**

Question 14

Question 14

Substitute appropriate values of t_1 and t_2 to verify that $t_1^2+t_2^2+t_1t_2+\frac{1}{2}$ gives the correct y-coordinate of the intersection of the normals at points A and B in Fig. C2. **(1)**



Model solution

Use the parameters of the two labelled points, $t_1 = -3$ *and* $t_2 = 1$.

$$t_1^2 + t_2^2 + t_1t_2 + \frac{1}{2} = 9 + 1 - 3 + \frac{1}{2} = \frac{15}{2} = 7.5.$$

This agrees with the y*-coordinate* 7.5. *(1)*

Question 15

Question 15

(**a**) Show that, for $y = ax^2 + bx + c$, the tangent at the point with x-coordinate t is

$$y = (2at + b)x - at^2 + c.$$

(3)

(**b**) Hence show that the tangents at two points P and Q cross at a point whose x-coordinate is the mean of the x-coordinates of P and Q. **(3)**

Model solution

(a) Differentiate for the gradient, then use point--gradient form through $(t, at^2 + bt + c)$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b,$$

$$y - (at^2 + bt + c) = (2at + b)(x - t).$$

$$y = (2at + b)x - at^2 + c.$$

$y = (2at + b)x - at^2 + c.$ ***(3)***

(b) Use the result twice, with $t = x_P$ *and* $t = x_Q$*, then equate the two tangent equations.*

$$(2ax_P + b)x - ax_P^2 + c = (2ax_Q + b)x - ax_Q^2 + c.$$

$$2a(x_P - x_Q)x = a(x_P - x_Q)(x_P + x_Q).$$

For distinct points P, Q*,* $x = \dfrac{x_P + x_Q}{2}$*, the mean of their* x*-coordinates.* ***(3)***

Question 16

Question 16

Show that

$$a\left(\frac{x_P + x_Q}{2}\right)^2 + b\left(\frac{x_P + x_Q}{2}\right) + c - a\left(\frac{x_P - x_Q}{2}\right)^2$$

is equivalent to

$$ax_Px_Q + b\left(\frac{x_P + x_Q}{2}\right) + c.$$

(2)

Model solution

The terms with b and c already agree, so simplify the difference of the two squared terms involving a.

$$a\left\{\frac{(x_P + x_Q)^2 - (x_P - x_Q)^2}{4}\right\} = a\left\{\frac{4x_Px_Q}{4}\right\} = ax_Px_Q.$$

The expression is $ax_Px_Q + b\left(\frac{x_P + x_Q}{2}\right) + c.$ *(2)*

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Question 17

Question 17

Show that, for $y = x^2$, the normal at (t, t^2) is

$$y = -\frac{1}{2t}x + t^2 + \frac{1}{2}.$$

(3)

Model solution

The normal is perpendicular to the tangent, so its gradient is the negative reciprocal of the tangent gradient.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x, \qquad m_{normal} = -\frac{1}{2t}.$$

$$y - t^2 = -\frac{1}{2t}(x - t) = -\frac{x}{2t} + \frac{1}{2}.$$

$y = -\frac{1}{2t}x + t^2 + \frac{1}{2}.$ *(3)*

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Question 18

Question 18

A student investigates intersection points of tangents to $y = 6x^2 - 7x + 1$, at pairs of points with x-coordinates differing by 5. Find the equation of the curve that all the intersection points lie on. **(2)**

Model solution

The article states that for $y = ax^2 + bx + c$, a fixed separation h gives the translated curve $y = ax^2 + bx + c - \frac{ah^2}{4}$. Here $a = 6$ and $h = 5$.

$$y = 6x^2 - 7x + 1 - \frac{6 \times 5^2}{4} = 6x^2 - 7x - \frac{73}{2}.$$

$y = 6x^2 - 7x - \frac{73}{2}$. *(2)*

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