Maths with Matt

Physics and maths resources

OCR AS Level Further Mathematics A

H235/Y531 - Pure Core

Monday 16 May 2022

Questions and Fully Worked Model Solutions

This booklet places the **question and the solution together**. The mathematics is set out using proper fractions, powers, roots and exact notation, with extra written explanation so that a student can see *why* each step is being taken.

Solutions have been checked against the OCR Monday 16 May 2022 mark scheme. The aim is a clear **full-mark model response**, not merely a list of final answers.

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Question 1

Question 1

(a) Determine whether the point $(19, -12, 17)$ lies on the line $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$. [3]

Vectors $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 6 \\ 2 \end{pmatrix}$.

(b) (i) Find, in degrees, the angle between $\mathbf{a}$ and $\mathbf{b}$. [3]

(ii) Find a vector which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. [2]

Model solution

(a) A point is on the line only if the same parameter produces every coordinate.

$$4 + 3\lambda = 19 \Rightarrow \lambda = 5, \qquad 7 + 4\lambda = 17 \Rightarrow \lambda = \frac{5}{2}.$$

The parameter values are inconsistent, so $(19, -12, 17)$ does not lie on the line. **(3)**

(b)(i) Use the scalar-product formula to find the angle.

$$\mathbf{a} \cdot \mathbf{b} = 1(-3) + (-2)6 + 2(2) = -11, \quad |\mathbf{a}| = 3, \quad |\mathbf{b}| = 7, \quad \cos\theta = -\frac{11}{21}.$$

$\theta = 122°$ *(to 3 s.f.).* **(3)**

(b)(ii) The cross product is perpendicular to both input vectors.

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -16 \\ -8 \\ 0 \end{pmatrix}.$$

One suitable vector is $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. **(2)**

Question 2

Question 2

Matrices A and B are given by $A = \begin{pmatrix} a & 1 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 5 \\ -1 & 0 \end{pmatrix}$, where a is a constant.

(a) Find the matrices $A + B$, AB, and A^2. [3]

(b) (i) Given that $\det A = 25$, find a. [2]

(ii) You are given instead that $ax + y = -2$, $-x + 3y = -6$ does not have a unique solution. Determine a. [2]

Model solution

(a) Add corresponding entries, then multiply rows by columns.

$$A + B = \begin{pmatrix} a-2 & 6 \\ -2 & 3 \end{pmatrix}, \quad AB = \begin{pmatrix} -2a-1 & 5a \\ -1 & -5 \end{pmatrix}, \quad A^2 = \begin{pmatrix} a^2-1 & a+3 \\ -a-3 & 8 \end{pmatrix}.$$

$A + B = \begin{pmatrix} a-2 & 6 \\ -2 & 3 \end{pmatrix}$, $AB = \begin{pmatrix} -2a-1 & 5a \\ -1 & -5 \end{pmatrix}$, $A^2 = \begin{pmatrix} a^2-1 & a+3 \\ -a-3 & 8 \end{pmatrix}$. *(3)*

(b)(i) Evaluate the determinant.

$$\det A = 3a - 1(-1) = 3a + 1 = 25.$$

$a = 8$. *(2)*

(b)(ii) No unique solution requires a singular coefficient matrix.

$$3a + 1 = 0.$$

$a = -\frac{1}{3}$. *(2)*

Question 3

Question 3

In this question you must show detailed reasoning.
The roots of $5x^3 - 3x^2 - 2x + 9 = 0$ are α, β and γ. Find a cubic equation with integer coefficients whose roots are $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$. [6]

Model solution

Vieta's formulae give the symmetric quantities from which the new cubic can be formed.

$$\alpha + \beta + \gamma = \frac{3}{5}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = -\frac{2}{5}, \quad \alpha\beta\gamma = -\frac{9}{5}.$$

For $u = \alpha\beta$, $v = \beta\gamma$, *and* $w = \gamma\alpha$,

$$u + v + w = -\frac{2}{5}, \quad uv + vw + wu = \alpha\beta\gamma(\alpha + \beta + \gamma) = -\frac{27}{25}, \quad uvw = (\alpha\beta\gamma)^2 = \frac{81}{25}.$$

$$\therefore \quad x^3 + \frac{2}{5}x^2 - \frac{27}{25}x - \frac{81}{25} = 0.$$

$25x^3 + 10x^2 - 27x - 81 = 0.$ *(6)*

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Question 4

Question 4

Prove that $3^n > 10n$ for all integers $n \geq 4$. [5]

Model solution

Use induction because the assertion concerns every integer from a starting value onwards. For $n = 4$,

$$3^4 = 81 > 40 = 10 \times 4.$$

Assume $3^k > 10k$ *for an integer* $k \geq 4$. *Then*

$$3^{k+1} = 3 \cdot 3^k > 30k = 10(k+1) + 10(2k-1) > 10(k+1),$$

since $2k - 1 > 0$. *Thus truth for* k *implies truth for* $k + 1$.

Since the base case and inductive step hold, $3^n > 10n$ *for all integers* $n \geq 4$. **(5)**

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Question 5

Question 5

In this question you must show detailed reasoning.
(a) Use an algebraic method to find the square roots of $-16+30i$. [5]
(b) By finding the cube of one of your answers to part (a), determine a cube root of $-\frac{99}{4}+\frac{5}{2}i$. Give your answer in the form $a+bi$. [2]

Model solution

(a) Put a square root equal to $a+bi$, with a and b real, and compare real and imaginary parts.

$$(a+bi)^2 = a^2 - b^2 + 2abi = -16+30i, \quad a^2-b^2=-16, \quad ab=15.$$

Substitute $b=15/a$ into the real-part equation.

$$a^2 - \frac{225}{a^2} = -16 \Rightarrow a^4+16a^2-225=0 \Rightarrow (a^2-9)(a^2+25)=0.$$

Thus $a^2=9$; using $ab=15$ fixes matching signs.

The square roots are $3+5i$ and $-3-5i$. *(5)*

(b) Use the square already found to calculate a cube efficiently.

$$(3+5i)^3 = (-16+30i)(3+5i) = -198+10i,$$

$$-\frac{99}{4}+\frac{5}{2}i = \frac{-198+10i}{8} = \left(\frac{3+5i}{2}\right)^3.$$

A cube root is $\frac{3}{2}+\frac{5}{2}i$. *(2)*

Question 6

Question 6

The matrix $A = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$ represents the transformation T, which is a reflection in a certain straight line. The mirror line passes through the origin, O.
(a) Explain why there must be a line of invariant points for T. State its geometric significance. [2]
(b) By considering the line of invariant points for T, determine the equation of the mirror line in the form $y = mx + c$. [4]
The coordinates of P are $(1, 5)$.
(c) By considering the image of P under the transformation T, or otherwise, determine the point on the mirror line closest to P. [3]
(d) The line $y = ax + 2$ is an invariant line for T. Determine a. [2]

Model solution

(a) A reflection fixes every point on its mirror line. Thus its line of invariant points is the mirror line.

The invariant line is the mirror line. **(2)**

(b) An invariant point satisfies $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$\frac{1}{13}\begin{pmatrix} 5x + 12y \\ 12x - 5y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 12y = 8x, \quad 12x = 18y.$$

$y = \frac{2}{3}x + 0$. **(4)**

(c) The midpoint of a point and its reflection is on the mirror line and is closest to the original point.

$$P' = A\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}, \qquad \textit{midpoint} = \left(\frac{1+5}{2}, \frac{5-1}{2}\right) = (3, 2).$$

The point is $(3, 2)$. **(3)**

(d) Invariant lines other than the mirror line are perpendicular to it, so their gradient is the negative reciprocal of $2/3$.

$a = -\frac{3}{2}$. **(2)**

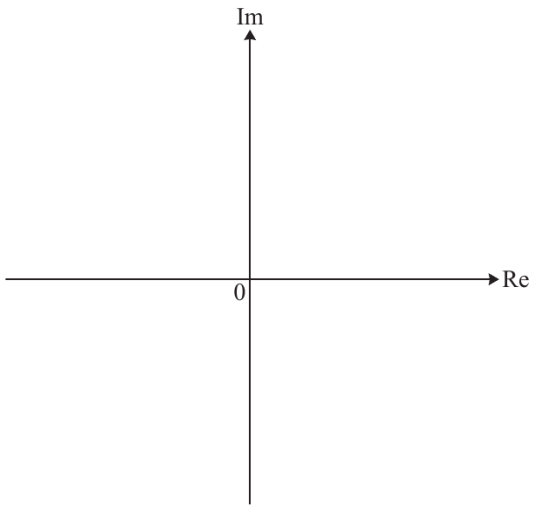
Question 7

Question 7

In this question you must show detailed reasoning.
Two loci, C_1 and C_2, are defined as follows:

$$C_1 : \arg(z+2-i) = \frac{\pi}{4} \qquad \text{and} \qquad C_2 : \arg(z-2-\sqrt{3}-2i) = \frac{2\pi}{3}.$$

By considering the representations of C_1 and C_2 on an Argand diagram, determine the locus $C_1 \cap C_2$. [7]



Model solution

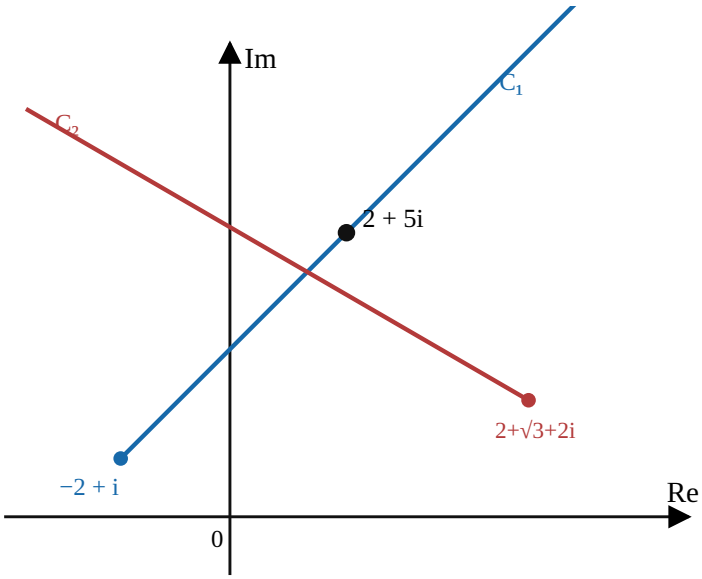
Each constant-argument locus is a half-line beginning where its bracket is zero. Its argument gives the gradient.

$C_1 :\ y-1 = x-(-2), \quad y = x+3, \qquad C_2 :\ y-2 = -\sqrt{3}\big(x-(2+\sqrt{3})\big), \quad y = -\sqrt{3}x+2\sqrt{3}+5.$

The gradients are $\tan(\pi/4) = 1$ *and* $\tan(2\pi/3) = -\sqrt{3}$*. Solving for their intersection,*

$$x+3 = -\sqrt{3}x+2\sqrt{3}+5 \Rightarrow (1+\sqrt{3})x = 2(1+\sqrt{3}) \Rightarrow x = 2, \quad y = 5.$$

The point is forwards along each half-line, so it satisfies both argument conditions.



$C_1 \cap C_2 = \{2 + 5i\}$. *(7)*

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Question 8

Question 8

The line segment AB is a diameter of a sphere, S. The point C is any point on the surface of S.
(a) Explain why $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$ for all possible positions of C. [3]
You are now given that $A = (11, 12, -14)$ and $B = (9, 13, 6)$.
(b) Given that $C = (2p, p, 1)$, where p is a constant, determine the coordinates of the possible positions of C. [6]

Model solution

(a) The plane containing A, B, C cuts the sphere in a circle with AB as a diameter. Therefore $\angle ACB = 90°$, by the angle-in-a-semicircle theorem, so the two vectors are perpendicular and their scalar product is zero. If $C = A$ or $C = B$, one vector is zero, so the result still holds.

$\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$ for every possible C. ***(3)***

(b) Use this perpendicularity after forming the two vectors.

$$\overrightarrow{AC} = \begin{pmatrix} 2p-11 \\ p-12 \\ 15 \end{pmatrix}, \qquad \overrightarrow{BC} = \begin{pmatrix} 2p-9 \\ p-13 \\ -5 \end{pmatrix}.$$

$$(2p-11)(2p-9) + (p-12)(p-13) + 15(-5) = 0 \Rightarrow 5p^2 - 65p + 180 = 0.$$

$$p^2 - 13p + 36 = 0 \Rightarrow (p-4)(p-9) = 0.$$

Substitute both permitted values of p into $(2p, p, 1)$.

The possible positions of C are $(8, 4, 1)$ and $(18, 9, 1)$. ***(6)***